Combinatorics and Graph Theory III – 2020/21
Take-home variant of the exam

1. Let G be a graph with n vertices such that the edges of G are covered by matchings $M_1, M_2, \ldots, M_n$. Suppose there is no $1 \le i, j \le n$ and pairwise distinct vertices a, b, c, d for which $ab \in M_i$, $bc \in M_j$, $cd \in M_i$. Show that $\|G\| = o(n^2)$.

2. Let G be 2-degenerated, $v_0 \in V(G)$, L a list assignments such that $|L(v)| \ge 3$ for $v \in V(G) \setminus \{v_0\}$ and $|L(v_0)| = 1$. Show that G is L-colorable.

3. Show that if a graph has tree-width more than $3k$ then it has a bramble of order at least $k+1$. Don't use the duality theorem, the purpose of this problem is to give an easier proof of a weaker version. You may proceed as you wish, but a suggested approach is as follows:

   - Let $W \subseteq V(G)$ be a set of size $2k+1$. We say that W is *k-breakable* if there is $X \subseteq V(G)$ of size at most k such that every component of $G - X$ has at most k vertices of W. Show that if $tw(G) \le k-1$ then every W of size $2k+1$ is k-breakable.
   - Let $W \subseteq V(G)$ be a set of size $2k+1$. Put

     $$\mathcal{B} = \{X \subseteq V(G) :\ G[X] \text{ is connected and } |X \cap W| \ge k+1\}.$$

     Show that $\mathcal{B}$ is a bramble and that if W is not k-breakable, then the order of $\mathcal{B}$ is at least $k+1$.
   - Let G be a graph such that every set of $2k+1$ vertices is k-breakable. Show that $tw(G) \le 3k$.