Large graphs

1 Removal lemma

G has $\geq \varepsilon n^2$ triangles $\Longleftarrow$ obvious

Thm $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall G: |G| = n$ & G is ε-far from Δ-free

$\forall F \subseteq E(G),\ |F| < \varepsilon n^2 \Rightarrow G - F \supseteq \Delta$

$\Rightarrow G$ has $\geq \delta n^3$ triangles

Proof $\delta = \dfrac{\varepsilon^3}{2^9 M^3}$; $M := M\left(\frac{\varepsilon}{4}, \lceil \frac{4}{\varepsilon} \rceil\right)$ from reg. lemma $M(\varepsilon, m)$

$k > \frac{4}{\varepsilon} \Rightarrow \frac{1}{k} < \frac{\varepsilon}{4}$

Given G — $n < M \Rightarrow \delta n^3 < 1$ ✓

— $n \geq M$; $m := \lceil \frac{4}{\varepsilon} \rceil$

$\left(1 - \frac{\varepsilon}{4}\right) n \leq ck \leq n$ $\quad c \leq \frac{n}{k}$

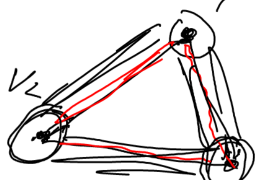
$(V_0, V_1, \ldots, V_k)$ $\frac{\varepsilon}{4}$-reg. part., $m \leq k \leq M$, $c = |V_1| = \ldots = |V_k|$ $\quad |V_0| \leq \frac{\varepsilon}{4} n$

$G' = G - F$; $F = \{e \text{ inc. with } V_0\} \cup \{e \text{ inside some } V_i\} \cup \{e \text{ in irreg. pairs}\} \cup \{e \text{ in reg. pairs of dens.} < \frac{1}{2}\varepsilon\}$

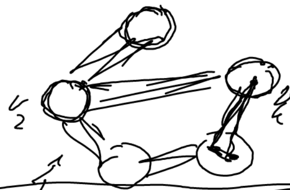
$\leq |V_0| \cdot n \leq \frac{1}{4} \varepsilon n^2$ $\qquad \leq k \binom{c}{2} \leq \frac{1}{8} \varepsilon n^2$ $\qquad \leq \frac{1}{4} \varepsilon k^2 c^2$ $\qquad \leq \frac{1}{2} \varepsilon c^2 \binom{k}{2}$

$\leq k \cdot \frac{n^2}{2k^2} = \frac{n^2}{2k} \leq \frac{n^2}{2} \cdot \frac{\varepsilon}{4}$ $\qquad \leq \frac{1}{4} \varepsilon n^2$ $\qquad < \frac{1}{4} \varepsilon n^2$

$|F| < \varepsilon n^2 \Rightarrow G - F \supseteq \Delta$



$\Rightarrow$ reg Δ's

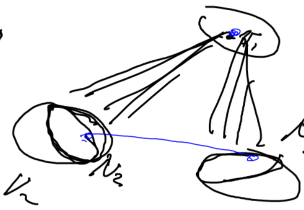


typical $v_1 \in V_1$ ... $\boxed{\geq \frac{1}{4} \varepsilon c}$ neighb. in V_2 & in V_3 $\quad \geq \frac{1}{4} \varepsilon c$ neighb.

empty OR $\frac{\varepsilon}{4}$-reg. with density $\geq \frac{\varepsilon}{2}$

# of typical $\geq |V_1| - 2 \cdot \frac{\varepsilon}{4} \cdot |V_1| \geq \frac{1}{2} c$

Fix typ. $v_1 \in V_1$; $N_i = N(v_1) \cap V_i$ $\quad (i = 2, 3)$ $\qquad \boxed{|N_i| \geq \frac{1}{4} \varepsilon c}$



$\Rightarrow$ # N_2-N_3 edges $\geq \left(\underset{\text{dens.}}{\frac{1}{2}\varepsilon} - \underset{\text{reg.}}{\frac{1}{4}\varepsilon}\right)\left(\frac{1}{4}\varepsilon c\right)^2 = \frac{1}{4^3} \varepsilon^3 c^2$

# of triangles cont. v_1

triangles $\geq \frac{1}{2} c \cdot \frac{1}{2^6} \varepsilon^3 c^2 = \frac{1}{2^7} \varepsilon^3 c^3 \geq \frac{1}{2^7} \varepsilon^3 \cdot \left(\left(1 - \frac{\varepsilon}{4}\right) \frac{n}{M}\right)^3 \geq \delta n^3 \cdot 4 \left(1 - \frac{\varepsilon}{4}\right)^3$

$$\left(\delta = \frac{\varepsilon^3}{2^9 M^3}\right)$$

$\geq 4 \left(\frac{3}{4}\right)^3 = \frac{27}{16} \geq 1$

$\geq \delta n^3$ ✓

(Property testing)

algo to decide
- G has P
- G is ε-far from having P (we need to delete $\geq \varepsilon n^2$ edges to have P)

with Pr(error) < ε

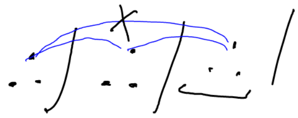
(G HUGE, fast algo)

Ex. $P = \{G : G \text{ is } K_3\text{-free}\}$

Algo: $\varepsilon \rightsquigarrow \delta$; choose s big enough (but still $O(1)$)

given G — sample $\frac{s}{\delta}$ triples of vertices ($3 \cdot \frac{s}{\delta}$ vertices)

check if we see ≥ 1 K_3



1) $G \not\supseteq K_3$ ⟹ can't see any, OK

2) $G \supseteq K_3$, even G is ε-far from being K_3-free --- by R.L. G has $\geq \delta n^3$ K_3's

P(error) = P(no K_3 sampled) $\leq (1-6\delta)^{\frac{s}{\delta}} \leq e^{-6s} \leq \varepsilon$ (s big enough)

3) "G in between" ⟹ we don't care! :)

$\geq \frac{6\delta n^3}{\cancel{n^3}}$

$\leq 1-6\delta$

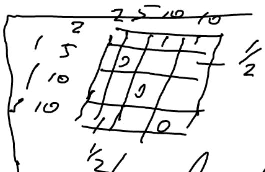
Pr(random triple is not a K_3)

$1-x \leq e^{-x}$

"dense case"

② Homomorphisms

F  G



Df $\varphi : V(F) \to V(G)$ s.t. $uv \in E(F) \Rightarrow \varphi(u)\varphi(v) \in E(G)$

extending to graphons $W : [0,1]^2 \to [0,1]$ sym.

$hom(F, W) = \int\int\cdots\int \prod_{ij \in E(F)} W(x_i, x_j)\, dx_1 \ldots dx_k$

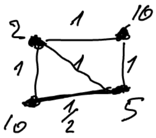
$x_1 \ldots x_k$ $\quad x_i = \varphi(i)$

$k = |V(F)|$

G weighted: nodes $\alpha_v(G)$ .... $\alpha_\varphi := \prod_{u \in V(F)} \alpha_{\varphi(u)}(G)$

edges $\beta_{uv}(G)$ $\quad hom_\varphi(F, G) := \prod_{uv \in E(F)} \beta_{\varphi(u)\varphi(v)}(G)$

[$hom(F, G) := \sum_{\varphi : hom\, F \to G} \alpha_\varphi\, hom_\varphi(F, G)$



$\alpha = \beta \equiv 1$ ... just # of hom

$G = K_k$ --- # of k-colorings (chrom. pol)

$hom(C_k, G) = Tr(A^k) = \sum_{i=1}^n \lambda_i^k$

$hom(P_k, G)$ = # of walks with $k-1$ steps

$hom(S_k, G) = \sum_{v \in V(G)} d_G^k(v)$ $\quad v \longleftarrow$  S_5

ex: # of ind. sets in F = $hom(F, G)$ $\quad G =$ 

[F is Eulerian] = hom(F, G)

(exercise)

$\sum_{I \subseteq V(F),\ I\ indep.} p^{|I|}$

$n^{\underline{k}} = n(n-1)(n-2)\ldots(n-k+1)$

$t(F, G) = \dfrac{hom(F, G)}{n^k}$

$t_{inj}(F, G) = \dfrac{inj(F, G)}{n^{\underline{k}}}$ ----- # of inj. hom

$t_{ind}(F, G) = \dfrac{ind(F, G)}{n^{\underline{k}}}$ --- # of hom to isom. graphs

hom

$A_e = \{\varphi : \varphi(e) \in E(G)\}$ $\quad e \notin E(F)$

$\left|\bigcap_{e \in I} A_e\right| = inj(F + I, G)$

$|\bigcup A_e|$

$inj(F, G) = \sum_{F' \supseteq F} ind(F', G)$ ...... same V, more E

$ind(F, G) = \sum_{F' \supseteq F,\ V' = V} (-1)^{|E(F')| - |E(F)|}\, inj(F', G)$

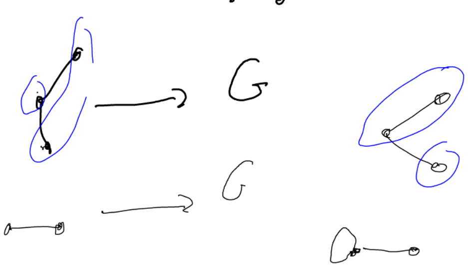
(incl.-excl.)

$hom(F, G) = \sum_{P\ part.\ of\ F} inj(F/P, G)$

$\leadsto$ $inj(F, G) = \sum c_P\, hom(F/P, G)$

some constants

in general: Möbius inversion formula



(3) Matrix norms

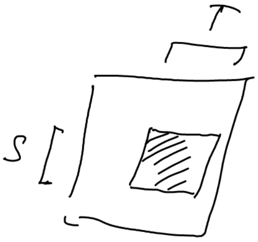
$A \in \mathbb{R}^{n \times n}$ --- $\|A\|_1 = \frac{1}{n^2} \sum_{i,j=1}^{n} |A_{ij}|$ ℓ_1-norm (usually w/o the $\frac{1}{n^2}$ factor)

$\|A\|_2 = \left(\frac{1}{n^2} \sum A_{ij}^2 \right)^{1/2}$ ℓ_2-norm / Frobenius norm $\left(= \frac{\sqrt{Tr(AA^T)}}{n} \right)$

$\|A\|_\infty = \max |A_{ij}|$ ℓ_∞-norm

cut norm

$\|A\|_\square = \frac{1}{n^2} \max_{S,T \subseteq [n]} \left| \sum_{i \in S, j \in T} A_{ij} \right|$



$\Rightarrow$ $\|A\|_\square \le \|A\|_1 \le \|A\|_2 \le \|A\|_\infty$

comparing graphs

$\boxed{V(G) = V(G')}$

• $d_1(G,G') = \frac{2|E(G) \triangle E(G')|}{n^2} = \|A_G - A_{G'}\|_1 \in [0,1]$ (sym. diff.)

edit distance

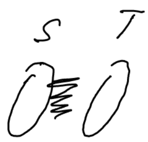
• $d_\square(G,G') = \|A_G - A_{G'}\|_\square = \max_{S,T} \frac{|e_G(S,T) - e_{G'}(S,T)|}{n^2}$ → not $|S|\cdot|T|$!

cut distance

ex.: G, G' ... indep. rand. graphs, $p = \frac{1}{2}$

whp $d_1(G,G') \doteq \frac{1}{2}$

$d_\square(G,G') = O\left(\frac{1}{\sqrt{n}}\right)$



$\boxed{|V(G)| = |V(G')|}$

$\hat{\delta}_\square(G,G') = \min_{\hat{G},\hat{G}'} d_\square(\hat{G},\hat{G}')$ $\hat{G},\hat{G}'$ ..... relabeled copies

$\boxed{G(\ell) = \text{replace the vertex by } \ell \text{ clones}}$

general G, G'

$\delta_\square(G,G') = \lim_{k \to \infty} \hat{\delta}_\square(G(kn'), G'(kn))$

$|G| = n$, $|G'| = n'$

$G(n')$, $G'(n)$ have the same # of vert.

(extending to functions)

$W : [0,1]^2 \to \mathbb{R}$

$$\|W\|_\square = \sup_{S,T \subseteq [0,1]} \left| \iint_{S \times T} W(x,y)\, dx\, dy \right|$$

cut norm $\rightsquigarrow d_\square(U,W) = \|U - W\|_\square$

$$\|W\|_\square \le \|W\|_1 \le \|W\|_2 \le \|W\|_\infty \le 1$$

$$\|W\|_2 \le \|W\|_1^{1/2} \quad \text{(Cauchy)}$$

cut distance $\delta_\square(U,W) := \inf_{\varphi} d_\square(U, W^\varphi) \longrightarrow W^\varphi(x,y) = W(\varphi(x), \varphi(y))$

φ invertible measure-preserving

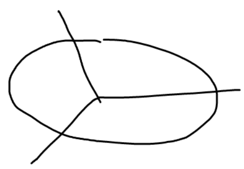
Lemma H, H' graphs $\longrightarrow \delta_\square(H,H') = \delta_\square(W_H, W_{H'})$

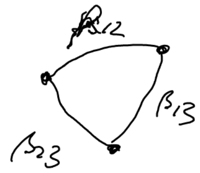
Reg. lemma variant

$\forall \varepsilon > 0\ \exists S(\varepsilon) \in \mathbb{N}\ \forall G$ has equitable part. P with k parts $\left(\frac{1}{\varepsilon} \le k \le S(\varepsilon)\right)$

$$\delta_\square(G, G/P) \le 2\varepsilon$$

P





Weak R.L.

$\forall k\ \forall G\ \exists$ part. P with k parts s.t.

$$\delta_\square(G, G/P) \le \frac{2}{\sqrt{\log k}}$$

Convergence of graphs

<u>Df</u> Seq. of graphs (G_n) is <u>convergent</u> iff $\forall F$ graph seq. $t(F, G_n)$ has a limit.

TFAE

<u>Thm</u> 1) (G_n) is convergent

2) (G_n) is Cauchy in $\delta_\square$

3) $\exists$ graphon W s.t. $\forall F$ $t(F, G_n) \longrightarrow t(F, W)$

4) $\exists$ graphon W s.t. $\delta_\square(W, W_{G_n}) \to 0$

$W: [0,1]^2 \to [0,1]$
symm.
measurable