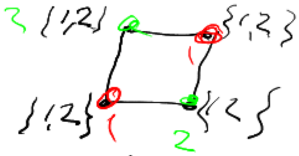
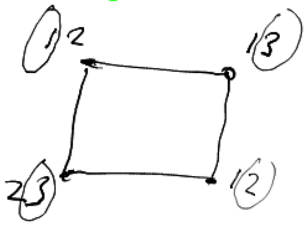
Choosability



Df

$L: V(G) \to$ set of colors ... list assignment



$v \in V(G)$ ... $L(v) = L_v$ ... list of v

L-coloring of G is any proper coloring $c: V(G) \to$ colors s.t. $\forall v$ $c(v) \in L(v)$. ... no monochr. edge.

Choosability of G, $\chi_\ell(G) = ch(G)$, is the minim. k s.t. if $(\forall v \in V(G) \; |L(v)| \geq k) \Longrightarrow$ G has an L-coloring

$\chi_\ell(G) \geq \chi(G)$

$\forall v$ $L(v) = \{1, 2, \dots, k\}$

L-coloring $=$ k-coloring

if $\chi(G) > k$ then G has L-coloring for this L

no $\to$ $\chi_\ell(G) > k$.

$\chi_\ell(G) \leq d+1$ if G is d-degenerated $\to$ $\forall H \subseteq G$ $\delta(H) \leq d$

$\leq d$

Pf by induction

$\chi(G) \leq d+1$ ... well-known

$v_0 \in G$: $\deg(v_0) \leq d$

fix $L: V(G) \to$ colors s.t. $\forall v$ $|L(v)| \geq d+1$

Use the statement for H (smaller than G, d-degener.) & restr. of L to $V(H)$

$\Longrightarrow \exists$ L-col. of H

It remains to find $c(v_0) \in L(v_0)$, $|L(v_0)| \geq d+1$

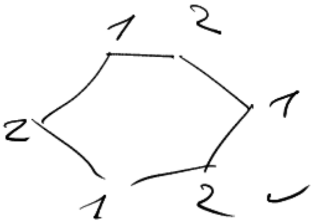
choose $c(v_0) \in \underbrace{L(v_0)}_{\geq d+1} - \underbrace{\{c(u) : u \in V(H),\ uv_0 \in E(G)\}}_{\leq d}$

"?" $\chi_\ell(C_n) = \chi(C_n)$ (obs: $\chi_\ell(C_n) \geq \chi(C_n)$)

- C_n is 2-degenerated $\to \chi_\ell(C_n) \leq 3 = \chi(C_n)$ if n is odd
- let n be even, show: $\chi_\ell(C_n) = 2$ $\quad |L(v)| = 2$

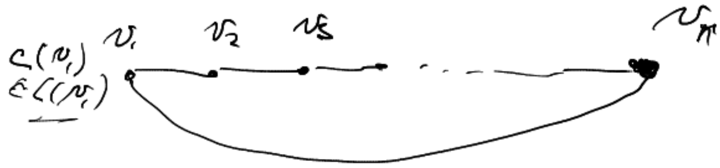
consider a list assignment $L : V(C_n) \to$ colors, WMA all lists are of size 2

1) $L(v)$ is the same $\forall v$, wlog $L(v) = \{1,2\}$



2) $V(C_n) = \{v_1, \dots, v_n\}$

wma: $L(v_1) \neq L(v_n)$, choose $\boxed{c(v_1) \in L(v_1) \setminus L(v_n)}$.



$c(v_2) \in \underbrace{L(v_2)}_{2} \setminus \{c(v_1)\}$

$c(v_3) \in L(v_3) \setminus \{c(v_2)\}$

$\vdots$

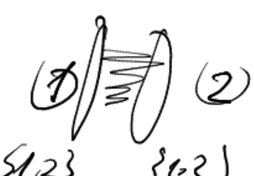
$c(v_n) \in \underbrace{L(v_n)}_{\text{size } 2} \setminus \{c(v_{n-1}), c(v_1)\}$

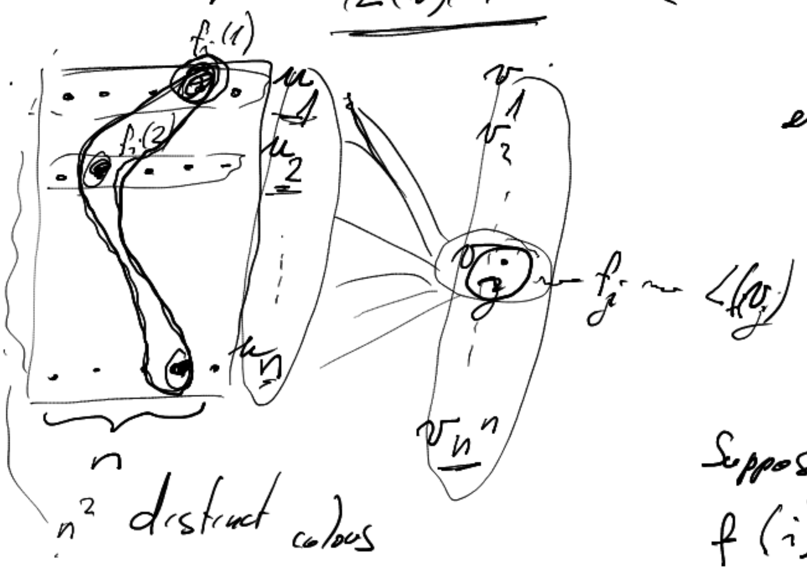
$= L(v_n) \setminus \{c(v_{n-1})\}$ $\quad c(v_1) \notin L(v_n)$

Lemma $\chi_\ell(K_{n,n^n}) > n$

Proof We need to present a $L : V(K_{n,n^n}) \to$ colors s.t.

$\forall v \ |L(v)| \geq n$ & no L-coloring exists.





n^2 distinct colors

enumerate all mappings $[n] \to [n]$ as $f_1, f_2, \dots, f_{n^n}$

$L(u_i) = \{(i,1), (i,2), \dots, (i,n)\}$

$L(v_i) = \{(j, f_i(j)) : j = 1, \dots, n\}$

Suppose there is L-col. c.

$f(i) = c(u_i)$, $f : [n] \to [n]$

So $\exists j : f = f_j$ $\quad c(v_j)$ has no valid option

$c(v_j) \neq c(u_1) = (1, f_j(1)) \in L(v_j)$, ...

$\blacksquare$

planar graphs

colorings .... 4CT G is planar $\Rightarrow \chi(G) \leq 4$

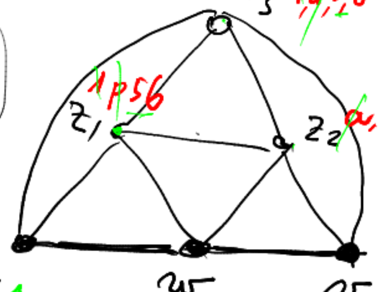
not true with 3

much easier with 5

choosability $\chi_\ell(G) \leq 5$ with almost easy proof

not true with 4

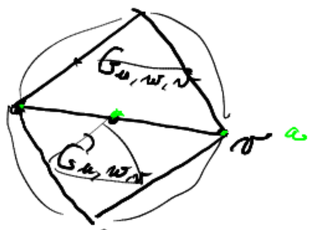
$G_{u,w,v}$



$L_{1,p,a}$

"$\Rightarrow$" If vertices u, w, v have colors 1, p, a then the col. cannot be extended

$G_{u,v}$



[$L_{1,a} : V(G_{u,v}) \to$ col.

$L_{1,a}(u) = 1$, $L_{1,a}(v) = a$

$L_{1,a}(w) = \{1, a, p, q\}$

"$\Rightarrow$" no $L_{1,a}$-col. of $G_{u,v}$

if $c(w) = p$ .. previous case $L_{1,a}$ on "top" $\sim L_{1,p,a}$

q ... symmetric sol. in $G'_{u,w,v}$ $L_{1,a}$ on "bottom" $\sim L_{1,q,a}$

G = 16 copies of $G_{u,v}$, sharing vertices u, v

$\to L(u) = \{1, 2, 3, 4\}$ $\forall i \in L(u)\ \forall j \in L(v)$

$\to L(v) = \{a, b, c, d\}$ one copy of $G_{u,v}$ has list $L_{i,j}$

$G_{u,v} \setminus \{u, v\}$

Theorem (C. Thomassen) If G is planar then $\chi_\ell(G) \le 5$.

Even stronger: if G is plane gr., F ... outer face of G, W = vertices inc. w. F

P is a path of ≤ 2 vertices in the border of F

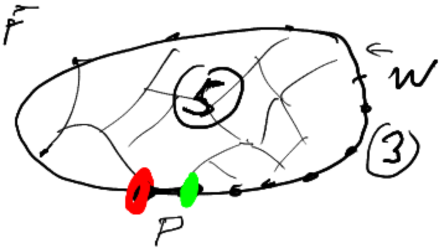
L : list assign. ; $\forall v \in V(G) \setminus W$: $|L(v)| \ge 5$

$\forall v \in W \setminus V(P)$: $|L(v)| \ge 3$

$\forall v \in V(P)$ $|L(v)| = 1$

& if $|V(P)| = 2$ then there is no coedr. — weaker assumptions

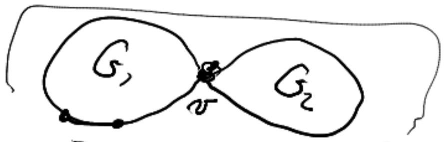
P is not forced to be monochr.



$\boxed{\text{Then } G \text{ is } L\text{-colorable}}$.

Proof By induction on $|V(G)|$.

WMA G is connected, even 2-connected.

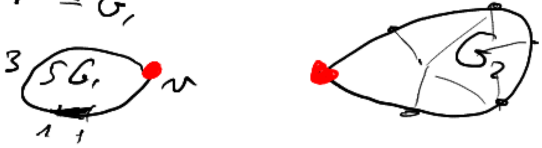


$P \subseteq G_1$

Ind. assumpt: (G_1, F, P) can be colored from L (restr. to $V(G_1)$)

→ L-coloring c_1 of G_1

$c_1(v)$ = red



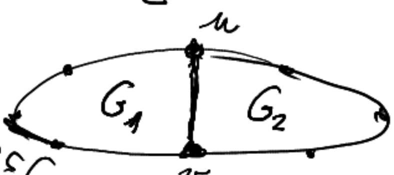
$P_2 = \{v\}$ $L(v) = \{red\}$

$L_2(u) = L(u)$ $u \in G_2 - v$

ind. assumpt. → ex. c_2 : L_2-col. of G_2

$c := c_1 \cup c_2$

→ no chord in the outer cycle

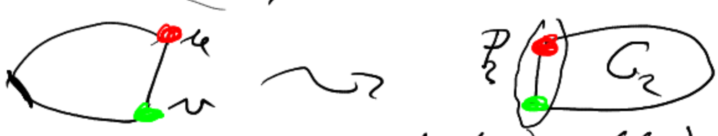


$P \subseteq G_1$

again, first color (G_1, L_1, F, P)

L_1 = L restr. to G_1

→ c_1 ... L-col. of G_1 obtained by ind.



$L_2(w) = L(w)$ $w \ne u, v$

$L_2(u) = c_1(u)$

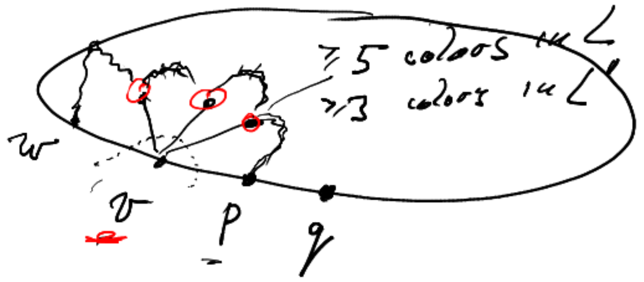
$L_2(v) = c_1(v)$

$P_2 = \{u, v\}$

ind. ass. on (G_2, L_2, F, P_2)

→ WMA $|V(P)| = 2$

(otherwise we precolor 1 or 2 vertices of W



$V(P) = \{p, q\}$

$\{a,b\} \subseteq L(v) \setminus L(p)$

$L'(x) = L(x) \setminus \{a,b\}$ for $x \in N(v) \setminus W$

$L'(x) = L(x)$ otherwise

G'

$(G-v, L', F', \{p,q\})$ satisfies assumptions (2)

& is smaller than G → it has c' : L'-colouring

We need to extend c' to c — L-col. of G

$c(v) \in L(v)$ s.t. 1) $c(v) \neq c(x)$ $x \in N(v) \setminus W$

2) $c(v) \neq c(p)$

3) $c(v) \neq c(w)$

1) & 2) are satisfied if $c(v) \in \{a,b\}$

so we choose $c(v) \in \{a,b\} \setminus \{c(w)\}$