(More about tree decomp.)

1) algo — given G, k ... ask $tw(G) \le k$?

... NP-complete (Arnberg, Corneil, Proskurowski 1987)

$\underline{\forall k}$ ex. a linear-time algo to det. tw (and find a tree dec.)

$\underbrace{k^{ck^3}} \cdot n$ (better — approx.)

Open Problem: P for planar graphs

2) $\mathcal{G}_k := \{G : tw(G) \le k\}$ ... closed on minors ($H \le_m G \in \mathcal{G}_k \Rightarrow H \in \mathcal{G}_k$)

MCC ... minor closed class

$\longrightarrow$ every MCC is $\boxed{Forb_{\le_m}(\mathcal{F})}$ for some $\mathcal{F}$

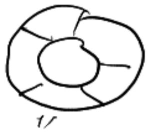
$\boxed{\text{Graph Minor Theorem}}$ $\Rightarrow$ we can take $\mathcal{F}$ finite

$\boxed{\text{GMT}}$ $\le_m$ is WQO on finite graphs

$\mathcal{G}_0$ = edge-less graph $= Forb(K_2)$

$\mathcal{G}_1$ = forests $= Forb(K_3)$

$\mathcal{G}_2$ = ser.-par. gr. $Forb(K_4)$

$\mathcal{G}_3$: $Forb(K_5$,  ,  , $K_{2,2,2})$

$\quad = C_5 \square K_2$, $\quad = W_8$, $\quad = $ octahedron

$\mathcal{G}_4$ ... > 70 graphs

Def WQO ... well-quasi-ordering: relation $\le$ ... transitive & reflexive (quasi order)

$\forall a_1, a_2, a_3 \ldots$ $\boxed{\exists i < j : a_i \le a_j}$ good pair

$\Longleftrightarrow$ no infinite antichain & no inf. decr. seq.

($\Leftarrow$ omitted)

(3) Look for nice descr. of $\mathrm{Forb}_{\leq_m}(H)$ for ~~previous~~ various H

Thm (Robertson, Seymour, 2003)

$\forall n$ .... structural description of $\mathrm{Forb}_{\leq}(K_n)$ using tw

(4) $tw(\mathrm{Grid}_n) = n$ $\quad (\geq n-1, \leq n)$

 $n=4$ $\quad \boxed{G \geq_m H \Rightarrow tw(G) \geq tw(H)}$

Grid_n is planar

$\not\geq_m K_5, K_{3,3}$ (Kuratowski)

$\mathrm{Forb}_{\leq_m}(H)$ can have unbdd tw

Thm (Grid theorem) $\forall r \; \exists k \; \forall G : tw(G) \geq k \Rightarrow G \geq_m \mathrm{Grid}_r$

Coroll. $\boxed{tw \text{ is bounded on } \mathrm{Forb}_{\leq_m}(H)} \iff H$ is planar

Proof $\Rightarrow$, $\neg \Leftarrow \neg$: (H not planar) $\mathrm{Grid}_n \in \mathrm{Forb}_{\leq_m}(H)$ (because minor of planar gr. is planar)

$tw(\mathrm{Grid}_n) \geq n \; (-1)$

$\neg \Rightarrow \neg$: $\forall k \; \exists G_k \in \mathrm{Forb}_{\leq_m}(H) : tw(G_k) \geq k$

$\forall r \; \exists k \; \forall G : tw(G) \geq k \Rightarrow G \geq_m \mathrm{Grid}_r$

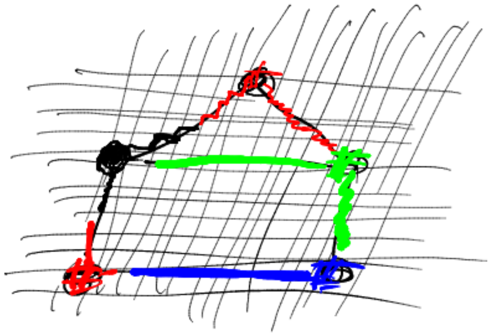
$\forall r \; \exists k$ $\quad G_k \geq_m \mathrm{Grid}_r \geq_m H$

$\boxed{\not\geq_m H}$ $\quad \lightning$

Assume H is planar

$\Downarrow$

$\exists r \; H \leq_m \mathrm{Grid}_r$

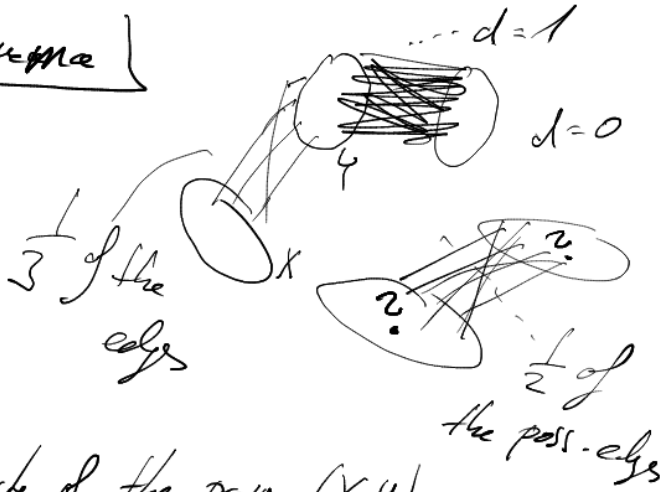


Szemerédi's regularity lemma

$G=(V,E)$ ... simple graph

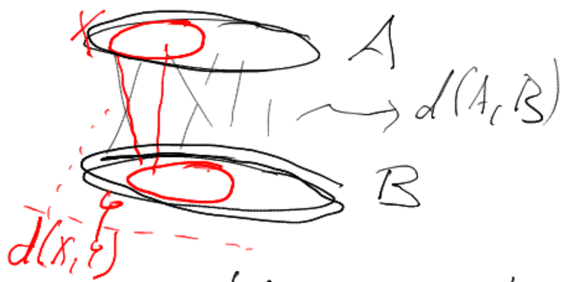
$X, Y \subseteq V \qquad X \cap Y = \emptyset$

$\|X,Y\|$ = # of X-Y edges in G



$$d(X,Y) = \frac{\|X,Y\|}{\underbrace{|X|\cdot|Y|}} \quad \text{..... density of the pair } (X,Y)$$

$\in [0,1]$



Df $\varepsilon > 0$, $A, B \subseteq V$, A, B disjoint

We say (A,B) is an <u>ε-regular pair</u>

$\Updownarrow$

$\forall X \subseteq A \ \forall Y \subseteq B: \ \underbrace{|X| \geq \varepsilon|A|}, \ \underbrace{|Y| \geq \varepsilon|B|} \Rightarrow \big|d(X,Y) - d(A,B)\big| \leq \varepsilon$

$V = V_0 \cup V_1 \cup \dots \cup V_k$ (disjoint sets) are <u>ε-regular partition</u>

$\Updownarrow$

1) $|V_0| \leq \varepsilon|V|$

2) $|V_1| = \dots = |V_k|$

3) all but $\leq \varepsilon k^2$ of the pairs (V_i, V_j) (with $1 \leq i < j \leq k$)

are ε-regular

Theorem (Regularity Lemma)

$\forall \varepsilon > 0 \ \forall m \in \mathbb{N}_{\geq 1} \ \exists M \in \mathbb{N} \ \forall G: \ |G| \geq m \Rightarrow \exists$ ε-regular part. of G into $k+1$ parts

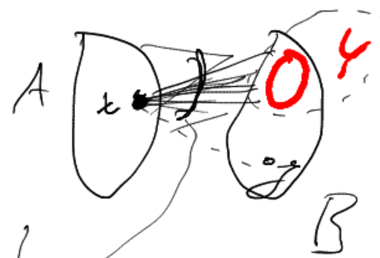
with $m \leq k \leq M$.

→ part. into single vertices .... obv. ε-reg. ..... need for M

→ part. with $k=1$ $\quad V_1 = V$ ..... need for m

→ triv. true for sparse graphs $\frac{|E|}{|V|}$ small

Lemma (A,B) is an ε-regular pair (in some G) of density d.



$Y \subseteq B$, $|Y| \geq \varepsilon |B|$

Take random bip. graph A, B & prob. of every edge $= p$

$d = p$ w.h.p.

Then all but $< \varepsilon |A|$ vertices in A have each $\geq (d-\varepsilon)|Y|$ neighbours in Y.

$\boxed{\text{Proof}}$ $X = \{ x \in A : \| \{x\}, Y \| < (d-\varepsilon) \cdot |Y| \}$

If $|X| < \varepsilon |A|$, we are done.

Owise: $|X| \geq \varepsilon |A|$, $|Y| \geq \varepsilon |B| \longrightarrow \big| d(X,Y) - d(A,B) \big| \leq \varepsilon$

$$d(A,B) = d$$

$$d(X,Y) = \frac{\|X,Y\|}{|X| \cdot |Y|} = \frac{\sum_{x \in X} \|\{x\},Y\|}{|X| \cdot |Y|} < \frac{(d-\varepsilon) \cdot |Y| \cdot |X|}{|X| \cdot |Y|} = d - \varepsilon \quad ⚡$$

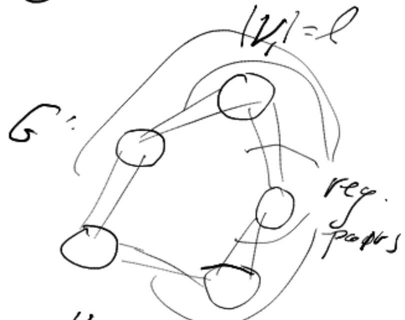
Given G & its ε-regular part. $V_0 \cup V_1 \cup \dots \cup V_k$; $d \in [0,1]$

we define R ..... regularity graph of G with ε, ℓ, d

$\ell = |V_1| = \dots = |V_k|$.

vertices $\{V_1, \dots, V_k\}$

edges V_iV_j ..... (V_i, V_j) is ε-regular & $d(V_i, V_j) \geq d$.

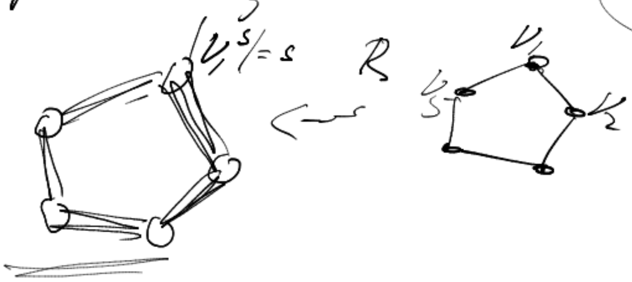


replace

$V_i \rightsquigarrow \underline{V_i^s}$ ... set of s vertices

$V_iV_j \rightsquigarrow$ complete bip. graph V_i^s to V_j^s

$\underline{R_s}$ .... resulting graph



if $R = K_k$

then R_s is complete k-part. graph.

(Blow-up Lemma; Komlós-Sárközy, Szemerédi, 1997)

<u>Lemma</u> $\forall d \in (0,1]$, $\forall \Delta \geq 1$ $\exists \varepsilon_0 > 0$:

$\forall G$ graph $\forall H$ graph $\Delta(H) \leq \Delta$ $\forall s \in \mathbb{N}$

If R ... regul. graph of G with $\varepsilon \leq \varepsilon_0$, $\ell \geq \frac{2s}{d^{\Delta}}$, d

then

$$H \subseteq R_s \implies H \subseteq G.$$

Turán thm: many edges of G $\implies$ $\exists$ compl. subgr. of G

Erdős-Stone — " — $\implies$ $H \subseteq G$

H bipartite ---- separate problem