Tree decompositions

→ trees are nice!

G ... a graph; T ... a tree; $\mathcal{V} = (V_t)_{t\in T}$ ... coll. of $\boxed{V_t \subseteq V(G)}$

nodes $t \in V(T)$

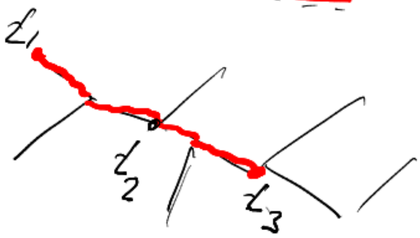
bags corr. to t

Df $(T, \mathcal{V})$ is a tree-decomposition of G :⟺

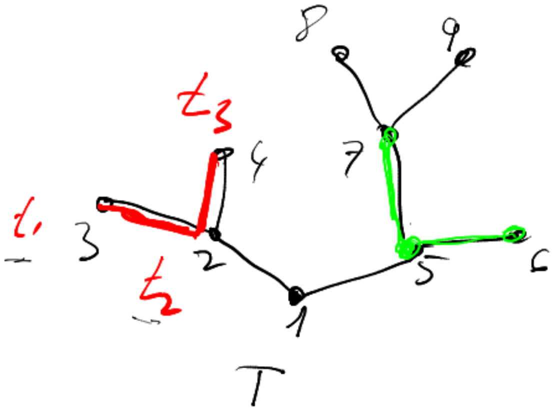
(T1) $V(G) = \bigcup_{t\in T} V_t$ (∀ vertex is covered)

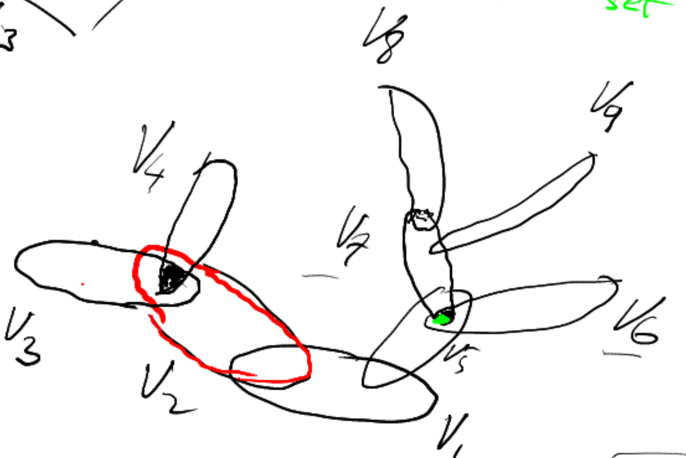
(T2) $\forall e\in E(G)\ \exists t\in T$: both ends of e are in V_t (∀ edge is covered)

(T3) $\forall t_1, t_2, t_3 \in T : t_2 \in t_1 T t_3 \implies V_{t_1}\cap V_{t_3} \subseteq V_{t_2}$



T3 ⟺ $\forall v\in V(G)$ $\{t\in T : v\in V_t\}$ is a connected set





Df width of $(T,\mathcal{V})$ is $\max\{|V_t| - 1,\ t\in T\}$

It is denoted $tw(T,\mathcal{V})$

$tw(G) := \min\{tw(T,\mathcal{V}) : (T,\mathcal{V})$ t.d. of G$\}$

☺ ∀ tree has $tw \le 1$.

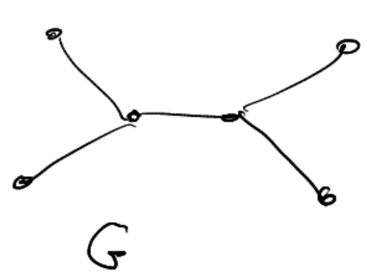
$V_a = \{v\}$ ...

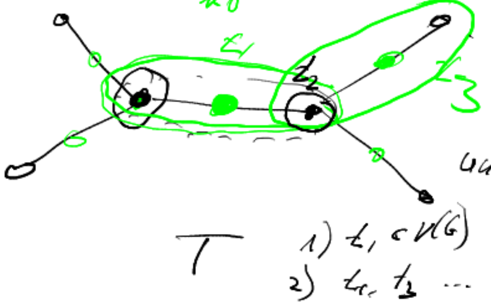
$V(T) = \{a_v : v\in V(G)\} \cup \{b_e : e\in E(G)\}$

$\boxed{V(T) = V(G)\cup E(G)}$

$t\in V(G)$ ... $V_t = \{t\}$ ✓ T1

$uv = t\in E(G)$ ... $V_t = \{u, v\}$ ⟹ T2





$V_{t_1}\cap V_{t_3} = \emptyset$ unless

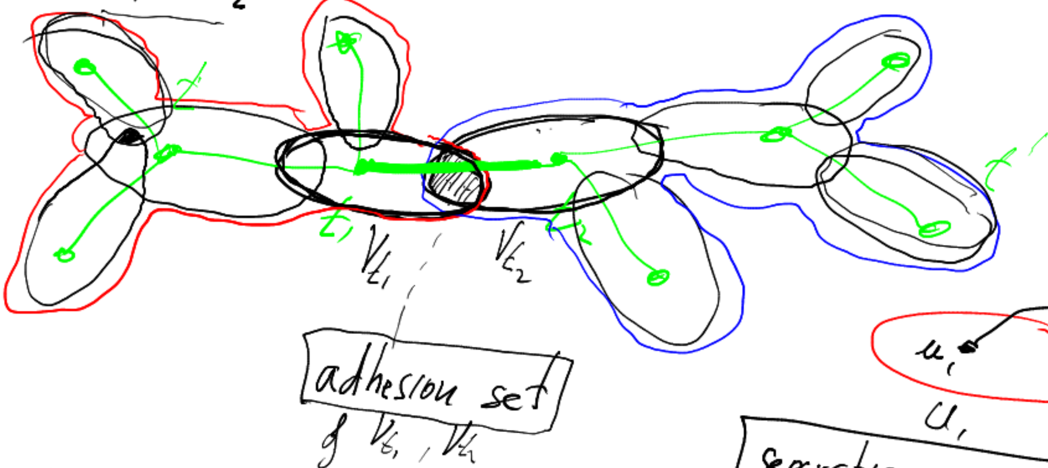
1) $t_1 \in V(G)$, t_3 ... inc. edge ✓

2) t_1, t_3 ... adj. edges ✓

Lemma $(T,\mathcal{V})$ ... t.d. of G; $t_1 t_2 \in E(T)$, T_1, T_2 ... comp. of $T - t_1 t_2$ s.t. $t_1 \in T_1$, $t_2 \in T_2$.

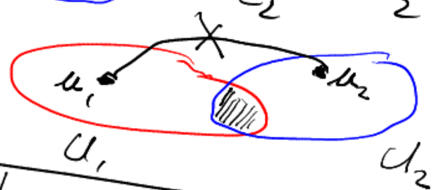
$$\boxed{U_1 = \bigcup_{t \in T_1} V_t} \qquad U_2 = \bigcup_{t' \in T_2} V_{t'}$$

Then $V_{t_1} \cap V_{t_2}$ separates U_1 from U_2



$V_{t_1} \subseteq U_1$

$V_{t_2} \subseteq U_2$



$\boxed{\text{Separation } U_1, U_2 \text{ is induced by edge } t_1 t_2 \text{ of } T.}$

Proof

(1) $v \in U_1 \cap U_2$

... $\exists t \in T_1, t' \in T_2$: $v \in V_t$ & $v \in V_{t'}$

$t_1, t_2 \in tTt' \Rightarrow$ by (T3) $\underbrace{V_t \cap V_{t'}}_{v \in} \subseteq V_{t_1}$ & $\subseteq V_{t_2}$

$\Rightarrow v \in V_{t_1} \cap V_{t_2}$

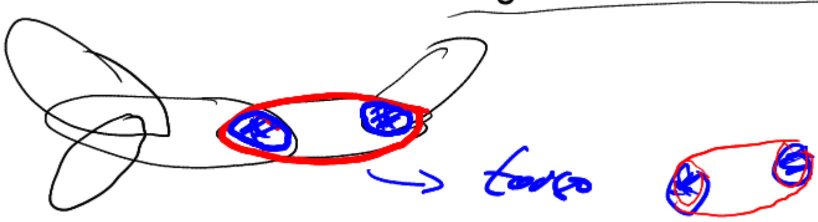
(2) $u_1 \in U_1 \setminus U_2$ & $u_1 u_2 \in E(G)$
$u_2 \in U_2 \setminus U_1$ $\Downarrow$(T2)
$\exists t \in T$ $u_1, u_2 \in V_t$ $\Rightarrow$ $t \in T_1$ or $t \in T_2$
$\Downarrow$ $u_1, u_2 \in U_1$ $\quad$ $\Downarrow$ $u_1, u_2 \in U_2$ ↯

→ parts of $(T,\mathcal{V})$ ..... $G[V_t]$

(T2): $\forall$edge is in some part

→ $(T,\mathcal{V})$ decomposes G into parts $G[V_t]$

→ torso of $(T,\mathcal{V})$ is supergraph of a part $G[V_t]$ obtained by making every adhesion set complete



Lemma $\forall H \subseteq G$ the pair $(T, (V_t \cap V(H))_{t \in T})$ is a tree decomp. of H. ✓

Conseq., $tw(H) \le tw(G)$.

Pf (T1) $\forall v \in V(H) \subseteq V(G) \Rightarrow \exists t \in T \quad v \in V_t$

$\Rightarrow v \in V_t \cap V(H)$

(T2) $\forall uv \in E(H) \subseteq E(G) \Rightarrow \exists t \in T \quad uv \in G[V_t]$

$\Rightarrow uv \in H[V_t \cap V(H)]$

(T3) also trivial

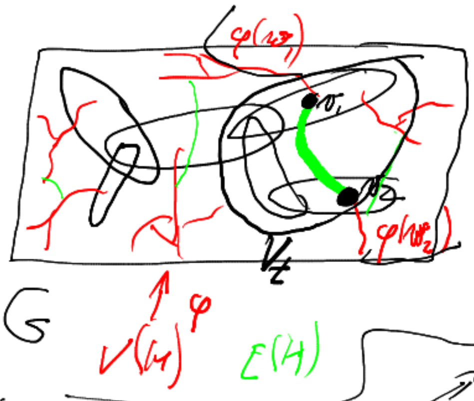
Lemma $H \le_m G$, let $\varphi : V(H) \to$ conn. subgraphs of G be a model of H.

Then (T, W_t) is a t.d. of H, where $W_t := \{w \in H : \varphi(w)$ intersects $V_t\}$

Conseq., $tw(H) \le tw(G)$. ✓

Proof φ maps $V(H)$ to disjoint subgraphs $\Rightarrow |W_t| \le |V_t|$

$\Rightarrow tw(T, W) \le tw(T, V) \Rightarrow tw(H) \le tw(G)$.



(T1) $w \in V(H)$, $\varphi(w) \subseteq G$

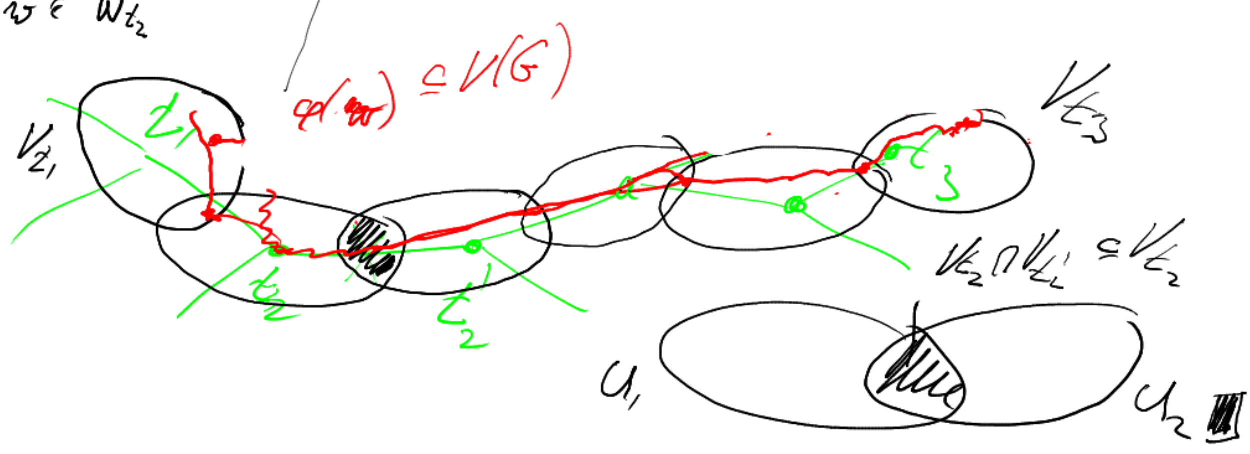
$v \in \varphi(w) \quad \exists t \in T \quad v \in V_t \Rightarrow w \in W_t$

(T2) $w_1 w_2 \in V(H)$ ... $\exists v_1 \in \varphi(w_1)$, $v_2 \in \varphi(w_2)$: $v_1 v_2 \in E(G)$

$\exists t \in T$: $v_1, v_2 \in V_t \Rightarrow w_1, w_2 \in W_t$

(T3) $t_2 \in t_1 T t_3 \quad w \in W_{t_1} \cap W_{t_3}$ | $\varphi(w)$ intersects V_{t_1} & V_{t_3}. Note: $\varphi(w)$ is connected. Thus $\varphi(w)$ intersects $V_{t_2} \Rightarrow w \in W_{t_2}$

we want: $w \in W_{t_2}$

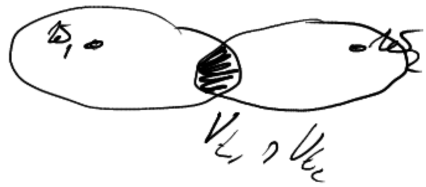


Lemma Any set $W \subseteq V(G)$ that is not $\subseteq G[V_t]$ contains 2 vertices separated by an adhesion set of $(T, \mathcal{V})$.

Proof orient edges of T: $t_1 t_2 \in E(T)$ ... def. U_1, U_2 as above

$V_{t_1} \cap V_{t_2}$ sep. U_1 from U_2.

ex. $w_1 \in U_1 \setminus U_2$ $w_2 \in U_2 \setminus U_1$, $w_1, w_2 \in W$

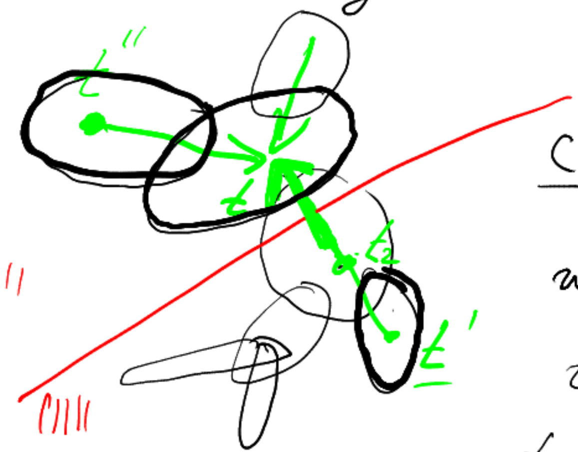


✓

OR

$W \in U_2$ $t_1 \to t_2$ $W \subseteq U_1$ ... $t_2 \to t_1$

Follow dir. edges — sink t



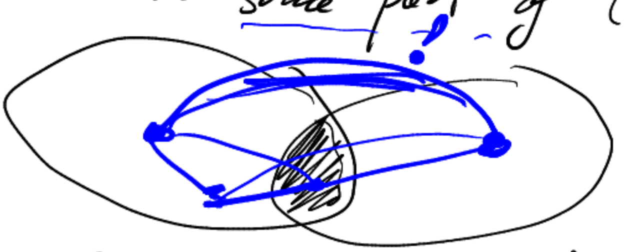
Claim $W \subseteq V_t$

$w \in W$ ..... $\exists t' \in T$: $w \in V_{t'}$

tt_2 sep. t from t' is directed towards t $\Rightarrow$ $w \in V_{t''}$

by T_3 : $w \in V_t$ ■

Corollary Every complete subgraph of G is contained in some part of $(T, \mathcal{V})$.



Corollary $tw(G) \geq \omega(G) - 1$

in partic. $tw(K_n) \geq n-2$