Minors

Df $H \leq_m G$: $\exists\, G' \subseteq G$ & H can be obtained from G' by edge contractions

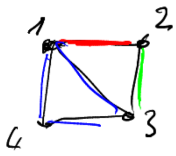
Df **Model** of H in G is a ftion $\varphi : V(H) \to$ subgraphs of G

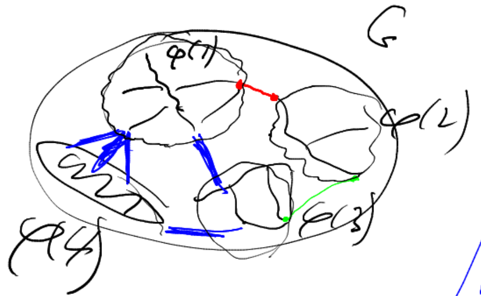
s.t. $\forall v$ $\varphi(v)$ is a connected subgraph of G

$\forall u \neq v$ $\varphi(u) \cap \varphi(v) = \emptyset$

$\forall uv \in E(H)$ $\exists$ edge in G connecting $\varphi(u)$ with $\varphi(v)$

$H = K_4 - e$





Observation

$H \leq_m G$

$\Updownarrow$

H has a model in G

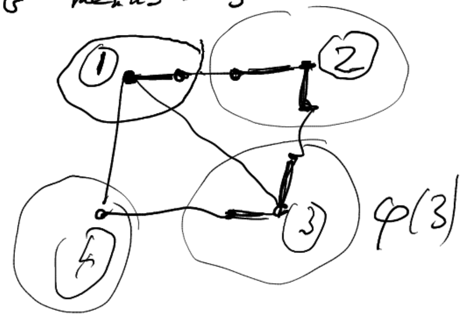
Proof $\Uparrow$ contract edges in $\varphi(1), \varphi(2), \ldots$
throw away some edges to get H

$\Downarrow$ sim.

Observation 1) $H \leq_t G \Rightarrow H \leq_m G$

2) if $\Delta(H) \leq 3 \Rightarrow$ the converse is also true

Pf 1) $H \leq_t G$ means: $\exists\, H' \subseteq G$ s.t. H' is a subdivision of H



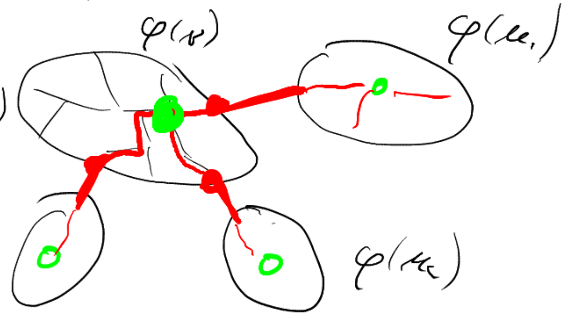
$\varphi(v) = v +$ half of each connecting path from v to u $(\forall uv \in E(H))$

2) $\Delta(H) \leq 3$ & H has a model in G

$\varphi(v)$ is connected $\to$ has a sp. tree

$\forall v$

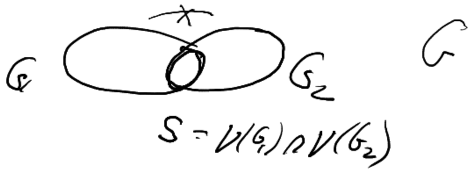
$u_1, u_2, u_3 \in N_H(v)$



$\Longrightarrow H \leq_t G$

G has a subdiv. of H

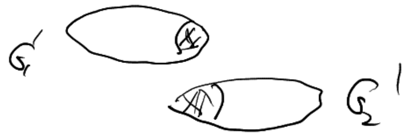
[Clique-sums]



Df $G = G_1 \cup G_2$

$S = V(G_1) \cap V(G_2)$

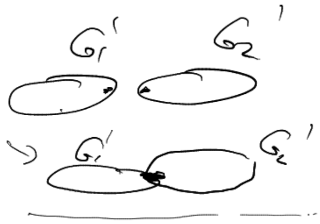
$G_i' = G_i$ + all edges in S



$k = |S|$ We say G is a k-sum of G_1' & G_2'.

ex. 0-sum $\Longleftrightarrow$ disj. union

1-sum $\Longleftrightarrow$ union overlapping at a vertex $\to$



Thm $G \in Forb(H) \Longleftrightarrow \exists$ 3-connected $G_1, \dots, G_m \in Forb(H)$:

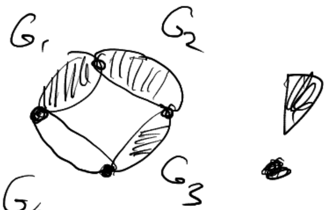
G can be constr. by (≤ 2)-sum of $G_1, G_2, \dots, G_m$

H is 3-connected

$Forb_{\leq_m}$

0-sum or 1-sum or 2-sum

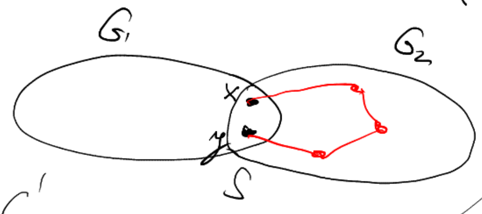
NO!



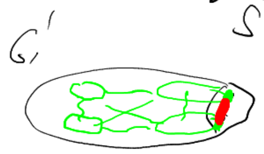
$\Rightarrow$

Pf by ind. on $|V(G)|$

- G is 3-conn. ... nothing to prove
- $S \subseteq V(G)$ min. cut, $|S| \leq 2$



$H \not\leq_m G \Rightarrow H \not\leq G_1', G_2'$



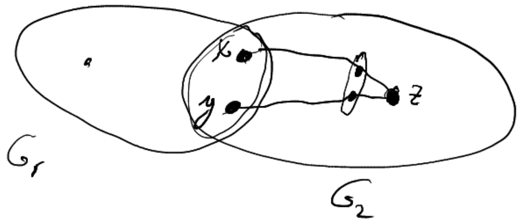
$|S| = 0$ $H \leq_m G_1 = G_1' \subseteq G$ $\Rightarrow H \leq_m G$ ↯

$|S| = 1$ —//—

$|S| = 2$ assume $H \leq G_1'$

$\exists$ path P from x to y in G_2

$G_1' \leq_m G_1 + P \subseteq G$ so if $H \leq_m G_1'$ then $H \leq_m G$ ↯



$\exists z \in V(G_2) \setminus V(G_1)$

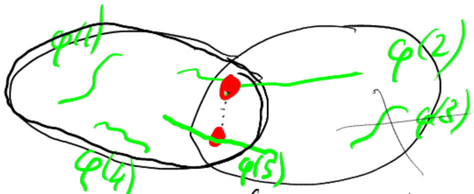
Menger! 2-paths $z \to \{x, y\}$

S was a min. cut $\Rightarrow$ no 1-cut, G is 2-conn.

$\Leftarrow$ It is enough to assume $n=2$. i.e.

G is a clique-sum of $G_1', G_2' \in \mathrm{Forb}(H)$

Want! $G \in \mathrm{Forb}(H)$. $\quad G = G_1 \cup G_2$



Assume $H \leq_m G$

... there is a model φ of H in G

$S = V(G_1) \cap V(G_2)$

$\boxed{\text{Case 1}}$ $\exists u : \varphi(u) \subseteq V(G_1') \setminus S$

$\exists v : \varphi(v) \subseteq V(G_2') \setminus S$

There are ≤ 2 vertices $x : \varphi(x) \cap S \neq \emptyset$

$\Rightarrow$ These vertices are an ≤ 2-cut in H, sep. u from v

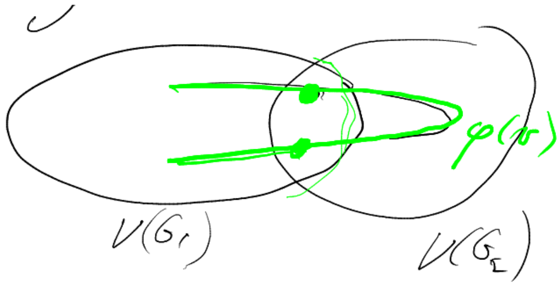
$\boxed{\text{Case 2}}$ $\forall v \in V(H)$ $\varphi(v) \cap V(G_1) \neq \emptyset$

WMA $\Rightarrow \varphi'(v) = \varphi(v) \cap V(G_1)$ ... a model of H in G_1'

$\Rightarrow \varphi'(v)$ are pairwise disjoint ✓

$\Rightarrow$ Is $\varphi'(v)$ connected?

$\Rightarrow uv \in E(H)$, are $\varphi'(u)\ \varphi'(v)$ adjacent?



minors

$\boxed{\text{Thm}}$ 1) $\mathrm{Forb}(K_2)$ = 0-sums of copies of K_1 = graphs with no edges

2) $\mathrm{Forb}(K_3) = (\leq 1)$-sums of copies of K_1, K_2 = forests

3) $\mathrm{Forb}(K_4) = (\leq 2)$-sums — " — K_1, K_2, K_3 ~ series-parallel graphs (equiv. if G is 2-conn.)

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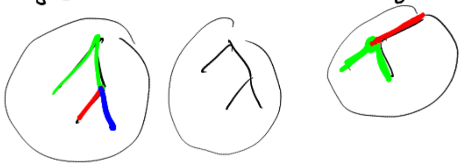
pf 1) ✓

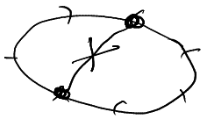
2) forests = (≤ 1)-sums of K_1, K_2



3) If $G \in Forb(K_4)$ & G is 3-conn. $\Rightarrow$ what is G?

G is small $\boxed{K_1, K_2, K_3}$

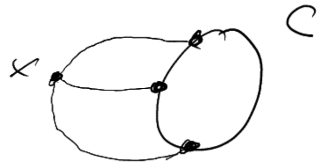
G has ≥ 4 vertices $\Rightarrow$ G has a cycle



C – shortest cycle. Note: C is induced cycle

If $G = C$ ... G is not 3-conn.

So $\exists x \in G \setminus C$

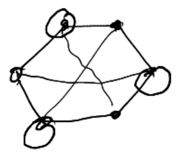


by 3-conn. $\exists$ 3 disj. paths x to C $\Rightarrow$ $G \geq_m K_4$ ↯

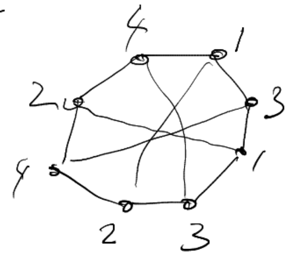
$G \in Forb(K_4) \xRightarrow{(Thm)} \exists G_1, \dots, G_m \in \underline{Forb(K_4)} \text{ \& } \underline{3\text{-conn.}}$

s.t. G is (≤ 2)-sum of $G_1 \dots G_m$.

$\forall i$: G_i is a copy of K_1, K_2 or K_3.



W_6 $\cong$ $K_{3,3}$



W_8 ... Wagner graph / Möbius ladder

Thm (Wagner 1937) $Forb_m(K_5)$ = ≤ 3-sum of planar graphs & copies of W_8

no proof in this

Corollary For $k \leq 4$

$\boxed{\text{Hadwiger conj: true for all } k}$

$G \ngeq_m K_{k+1} \Rightarrow \chi(G) \leq k$

(known for $k \leq 6$
($k=6$: Robertson, Seymour, Thomas 1993))

$k=1$ G has no edges $\chi(G) \leq 1$.

$k=2$ G is a forest ≤ 2

$k=3$ (≤ 2)-sum of K_1, K_2, K_3 ≤ 3

$k=4$ (≤ 3)-sum of P & W_8 ≤ 4

How to force a clique minor?

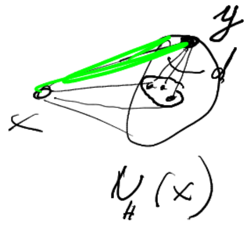
Proposition G has avg. deg. $\geq 2^{r-2} \Rightarrow G \geq_m K_r$
$(\forall r \in \mathbb{N})$.

Proof By ind. $r \leq 2$ true.

Assume $r \geq 3$, $d_\emptyset(G) \geq 2^{r-2} \Rightarrow \underset{\varepsilon(G)}{:=} \dfrac{|E(G)|}{|V(G)|} \geq 2^{r-3}$

H be a minimal minor of G with $\varepsilon(H) \geq 2^{r-3}$

$x \in V(H)$.
$\deg(x) \neq 0$



$H' = H/xy$ has $\varepsilon(H') < 2^{r-3}$

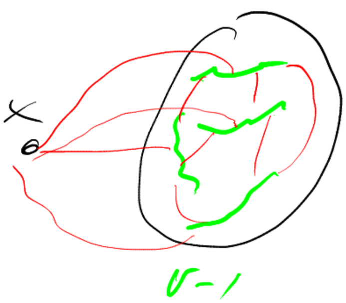
$|E(H')| = |E(H)| - d - 1$

$|V(H')| = |V(H)| - 1$

$\Rightarrow d \geq 2^{r-3}$

$H[N_H(x)]$ is a graph with $\delta \geq 2^{r-3}$

$\Rightarrow d_\emptyset \geq 2^{r-3} \Rightarrow$ it has K_{r-1} minor



$\Rightarrow H \geq_m K_r$

G true