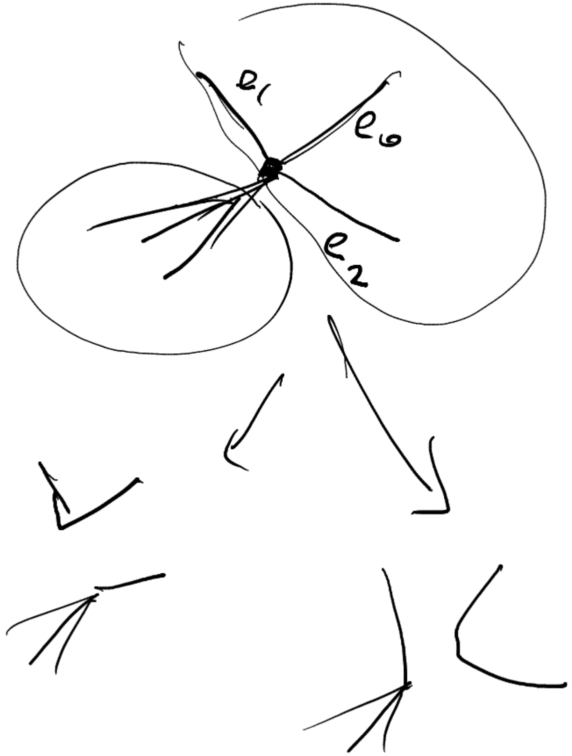
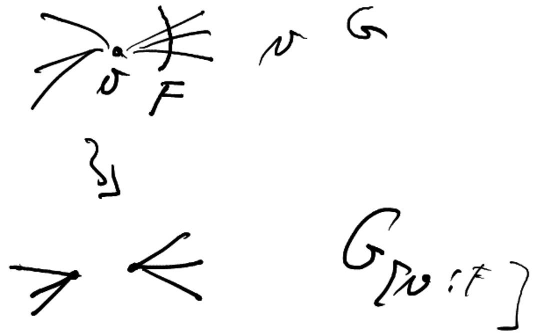
Splitting lemma

Lemma 32 (Splitting lemma, Fleischner/Mader). *Let G be a connected bridgeless graph, v a vertex with $\deg v \geq 4$, and e_0, e_1, e_2 three of its incident edges. Suppose that $G_{[v:e_0,e_1,e_2]}$ is connected. (This in particular holds whenever G is 2-connected.) Then at least one of $G_{[v:e_0,e_1]}$, $G_{[v:e_0,e_2]}$, is bridgeless connected.*

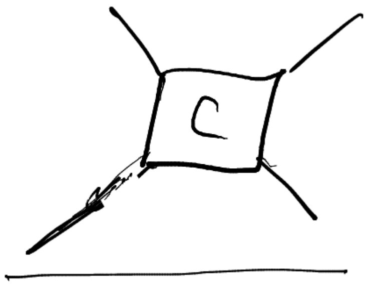
Proof. Let the edge e_i connect v with v_i. Let $G' = G - \{e_0, e_1, e_2\}$ and consider decomposition of G' into edge 2-connected blocks. Next contract each block to a vertex, what we get is a forest, say, F. Let u and u_i be the vertices of F corresponding to v and v_i ($i = 0, 1, 2$). As G is connected and bridgeless, the same is true for $F + \{uu_i : i = 0, 1, 2\}$. (In particular, the only leaves of F are among u, u_0, u_1, u_2.)



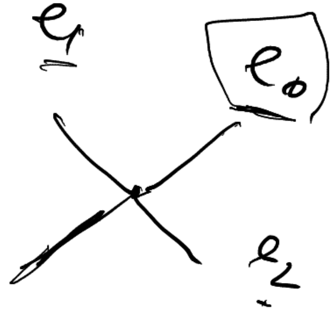


(Důsledek Spl. lemma)

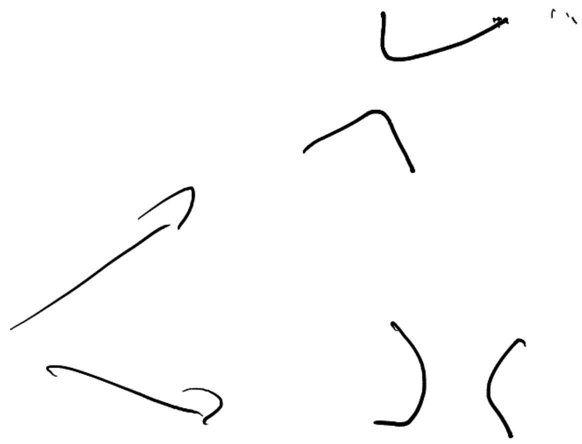
G, 2-souvislý, kubický, $G \cong C_4$



G/C



2-souvislý



2-souvislý

Also splitting, say, e_0, e_1 away from v corresponds to adding $F + \{uu_2, u_0u_1\}$ – for such graphs we need to check edge 2-connectivity. We have just two possibilities:

F is disconnected. As $G_{[v:e_0,e_1,e_2]}$ is connected and G bridgeless, the component containing u contains also (exactly) one u_i. Moreover, this component is a path connecting u with u_i. The other important vertices (say u_j, u_k, where $\{i, j, k\} = \{0, 1, 2\}$) are in the other component, this component is a $u_j - u_k$ path. In this case, splitting away e_i, e_j or e_i, e_k preserves 2-connectivity. Easily, on of these includes the desired cases (as $0 \in \{i, j, k\}$). See the first case in Figure **??**.

F is connected. Let T be the minimal subtree containing u_0, u_1, u_2. Let $w \in T$ be such that F is T plus a $w - u$ path. There is (at least one) i such that w is in a $u_i -$

u_j and in a $u_i - u_k$ path (again, $\{i, j, k\} = \{0, 1, 2\}$). Again, splitting away e_i, e_j or e_i, e_k preserves 2-connectivity, and at least one of these is what we search for. See the second case in Figure **??**. $\square$

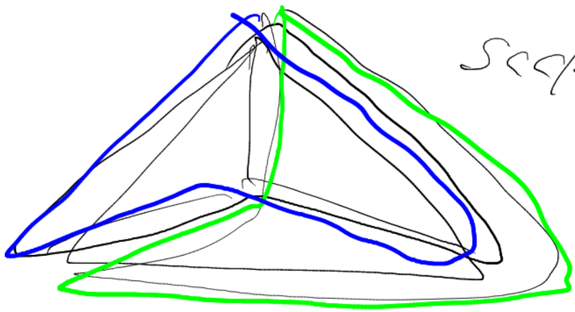
Shortest cycle cover problem

We briefly remark a related problem: the *shortest cycle cover problem.* Given a bridgeless graph G we care about a collection of cycles that covers every edge of G *at least once.* We denote by $scc(G)$ the minimal total length of such collection. Jaeger's 8-flow gives easily a 4-cover by 7 cycles; it follows that $scc(G) \leq 4m$. This can be certainly improved; the best known general result is $scc(G) \leq \frac{5}{3}m$ (Jamshy and Tarsi). (Better results are known for some classes of graphs, in particular for cubic graphs.) It is conjectured that $\boxed{scc(G) \leq \frac{7}{5}m}$ and this would, if true, imply the CDC conjecture.

cycly $C_1 - C_k$

$\bigcup C_i = E(G)$

$\min \sum |C_i| := scc(G)$



$scc(K_4) \leq 8$

$\frac{8}{6}$ 6

$1.\overline{3}$

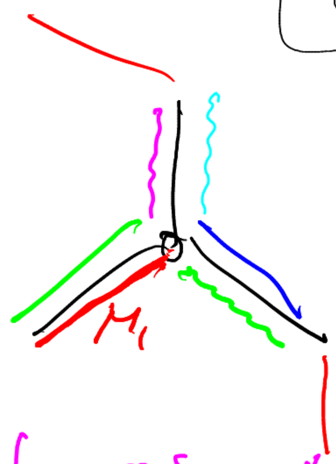
9 мос

Berge-Fulkerson conjecture

Conjecture 3 (Berge, Fulkerson). *If G is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$.*

Notes:

- true in 3-edge-colorable graphs
- true for the Petersen graph
- corollary: five matchings that cover all edges
- open: constant number of matchings that cover all edges



G-x-y má perf. pár.

3-obarvení

M_1, M_2, M_3 perf. pár. $E(G) = M_1 \dot\cup M_2 \dot\cup M_3$

BF: $M_1, M_2, M_3, M_1, M_2, M_3$

2

$\forall G \; \exists k \; \exists M_1, \ldots, M_k$

cubic bridgeless

$E(G) = M_1 \cup \ldots \cup M_k$

$k = |E(G)|$

Petersen thm: G je 3-reg. bez mostů, $e \in E(G)$

$\Rightarrow$ G má perf. párování obs. e

Tutte / Berge?

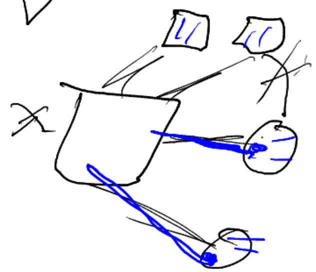
G má perf. párování

$\Updownarrow$

$\forall X \subseteq V(G)$ #lichých komp. v $G \setminus X \leq |X|$

$\Uparrow$ těžko ověřit

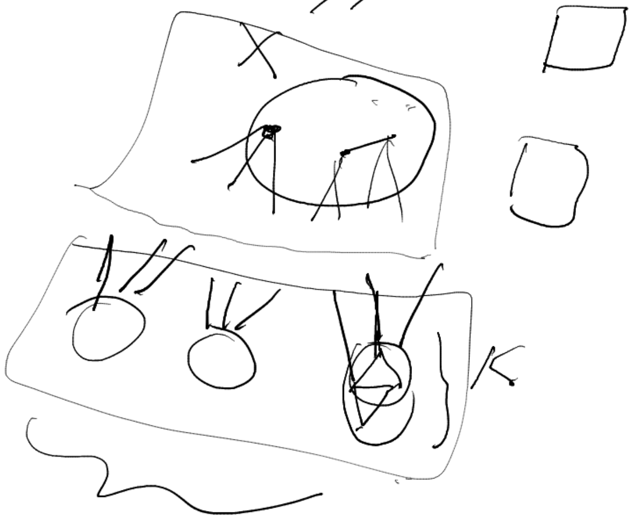
$\Downarrow$ ekviv.



$\Downarrow$

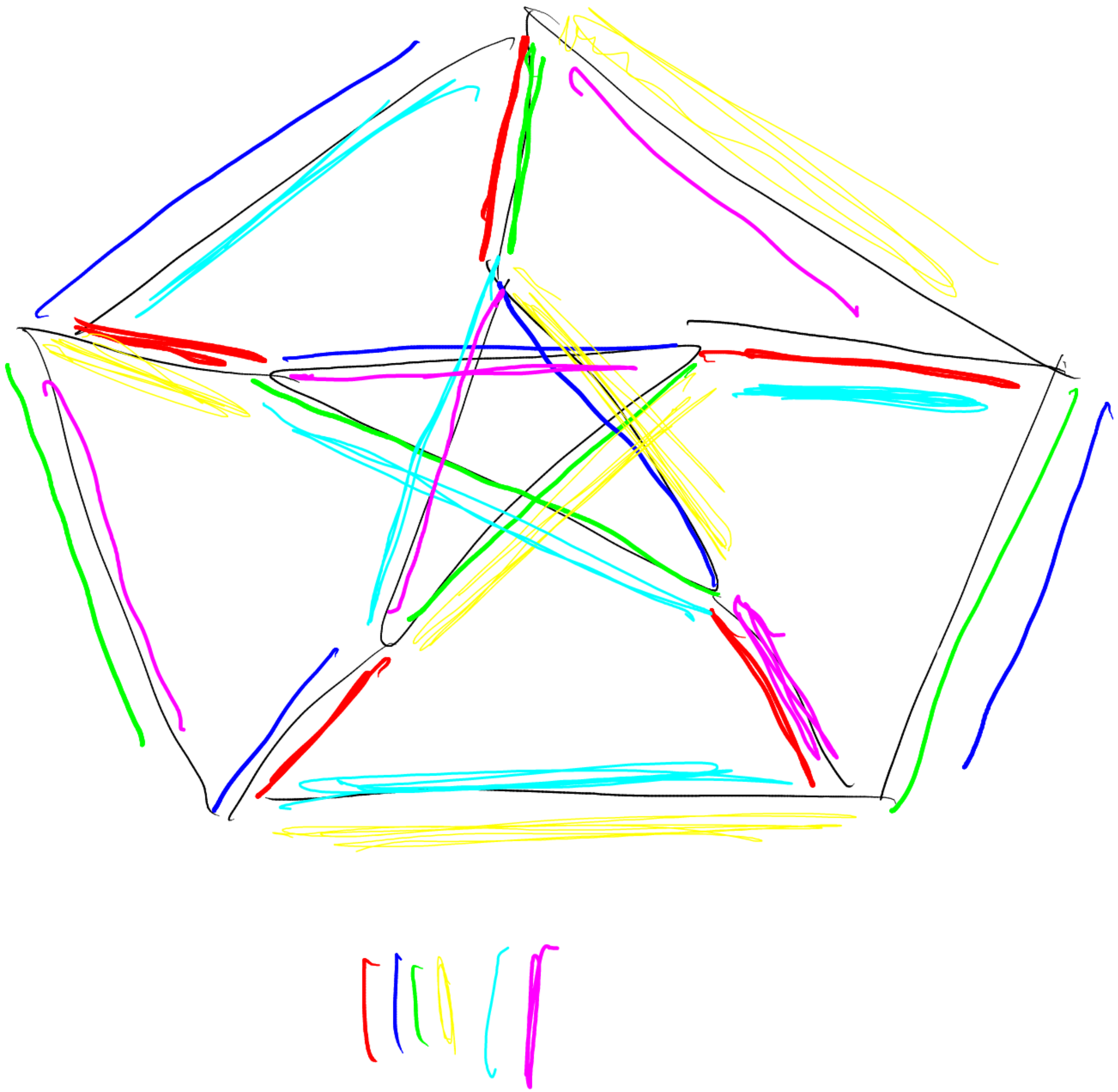
Petersenova věta

G 3-reg. 2-souv. (hranově)



$\sum_{v \in K} \deg(v) = 3|K|$ liché

$= 2E(K) + \underbrace{\deg(K)}_{\geq 3 \Leftarrow \text{liché} \neq 1}$



Matching polytope and applications

We will look at various sets of edges geometrically. That is, we consider $\mathbb{R}^{E(G)}$ as a euclidean vector space (which it is) and study various polytopes in it. For a set $M \subseteq E(G)$, we define c_M – the *characteristic vector of* M – by $c_M(e) = 1$ if $e \in M$, and $c_M(e) = 0$ otherwise.

Definition 33. *The* matching polytope *of a (multi)graph* G *is defined by*

$$MP(G) = \operatorname{conv}\{c_M : M \text{ is a matching in } G\}.$$

It is not hard to see that all points c_M (for a matching M) are in fact vertices of $MP(G)$. Note that we consider non-perfect matchings too, so the zero vector is a vertex of every matching polytope.

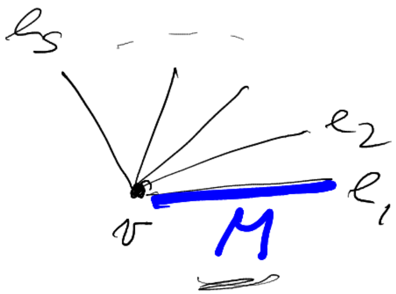
For many application it is desirable to obtain description of the matching polytope as

$$c_M = \sum_{i=1}^{k} \alpha_i c_{M_i}$$

$\alpha_i \geq 0$

$\sum \alpha_i = 1$

$M_1, \dots, M_k \neq M$



	e_1	e_2	—	e_s				
c_M	1	0	—	0	0	1	1	0
c_{M_i}	1	0	0	—	0			

$\forall v$: v je pokryt $M \to v$ je pokryt M_i $\forall i$ & ve stejné hraně

c_v	0	0	—	0
c_{M_i}	0	—	0	

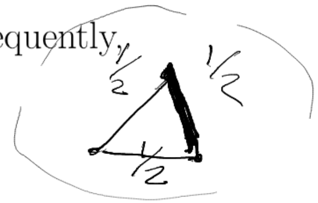
$\Rightarrow$ na $\forall\, \delta(v)$ je $M = M_1 = \dots = M_k$

$\Rightarrow M = M_1 = \dots = M_k$

an intersection of halfspaces. An application for a problem related to Berge–Fulkerson conjecture will follow shortly, for an (original) application in combinatorial optimization consider the task to find a maximal matching in a graph with weighted edges. This is the same as solving a linear program over the matching polytope, and we can do this using ellipsoid method. (We only need to provide an efficient representation of the matching polytope, for details see XXX.)

For each $f \in MP(G)$ and $v \in V(G)$ we have $\sum_{e\in\delta(v)} f(e) \le 1$ (we sum over all edges incident with one vertex), as this inequality holds for all vectors c_M. This, however does not describe $MP(G)$ completely (Exercise!).

Next, we observe that for each vertex set X of odd size, each matching uses at most $(|X|-1)/2$ edges induced by X. Consequently,

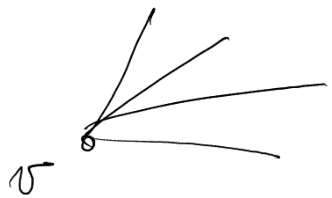


vrch. popis $\rightsquigarrow$ stěn. popis

$\max \langle w, c_M \rangle : c \in MP(G)$

$\to$ max. je ve vrcholu ... $c = c_M$

pro nějaké M



$\forall M$ ... podm.

$|M \cap \delta(v)| \le 1$

$\Rightarrow c_M(\delta(v)) \le 1$

$c_M(\delta(v)) = \underbrace{\sum_{e\in\delta(v)} c_M(e)}$

$f(\delta(v)) \le 1$ ✓

for each such X we have inequality

$$\sum_{e \in E(G[X])} f(e) \leq \frac{|X|-1}{2},$$

satisfied for each $f = c_M$ and so for each $f \in MP(G)$. This is already enough to describe the matching polytope.

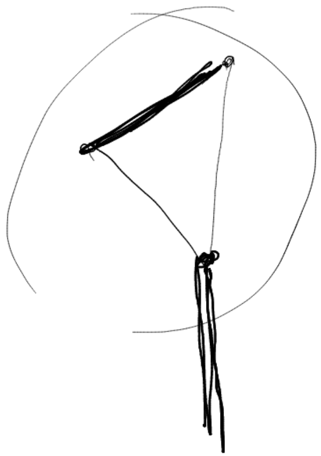
$$\boxed{f(S) := \sum_{e \in S} f(e)}$$

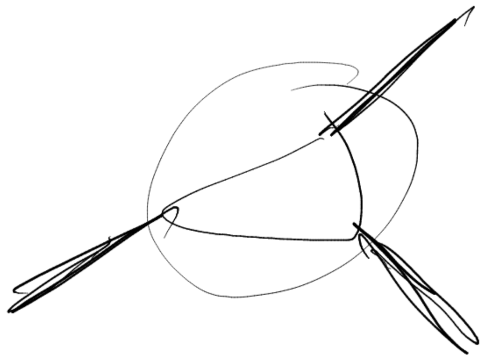
Theorem 34 (Edmonds)**.** *For every graph G we have* $\boxed{MP(G)} =$ $\subseteq$ zřejmé. $\supseteq$ později

$$\{f \in \mathbb{R}^{E(G)} : \quad f(\delta(v)) \leq 1 \ \forall v \in V(G)$$
$$f(E(G[X])) \leq \tfrac{|X|-1}{2}$$
$$\forall X \subseteq E(G) \textit{ of odd size}\}.$$

Theorem 35. *Let $PMP(G)$ be the polytope of perfect matchings that is $PMP = \text{conv}\{c_M : M \textit{ is a perfect matching in } G\}$.*

$$
\begin{aligned}
&Then\ PMP(G) = \\
&\{f \in \mathbb{R}^{E(G)} : f(\delta(v)) = 1\ \forall v \in V(G) \\
&\qquad\qquad f(\delta(X)) \geq 1 \\
&\qquad\qquad\quad \forall X \subseteq E(G)\ of\ odd\ size\}\,.
\end{aligned}
$$





r-graphs

We say a graph G is an r-*graph*, if G is r-regular, and for every odd set of vertices X the size of the edge-cut $\delta(X)$ is at least r. For example, a 3-graph is the same as a bridgeless cubic graph (Exercise!).

Application 1 Every r-graph has a uniform cover by perfect matchings. That is, there is a list of perfect matchings such that each edge is in the same number of them. (Easily, this number must be $1/r$.)

Proof. Let G be the graph and let $\boxed{f(e) = 1/r}$ for each edge of G. We will show that f is in the perfect matching polytope $PMP(G)$. Obviously the sum around each vertex equals 1. Now for each odd set X the size of $\delta(X)$ is at least r, which gives the other condition □

✓

$|\delta(X)| \geq r \quad \forall X$ liché velikosti

X liché v G kubickém

$\Rightarrow |\delta(X)|$ liché

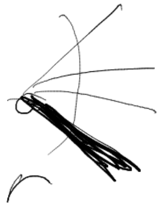
≥ 3

$f \in PMP(G)$

$\Leftrightarrow \exists \alpha_1, \dots, \alpha_k \in [0,1]$

$\sum \alpha_i = 1$

$f = \sum \alpha_i \chi_{M_i}$



1) $f(\delta(v)) = 1$ ✓

$r \cdot \frac{1}{r}$

2) $f(\delta(X)) = \frac{1}{r} |\delta(X)| \geq \frac{1}{r} \cdot r = 1$

$f \in PMP(G)$

$\Rightarrow \exists\, \alpha_1, \dots, \alpha_k \in [0,1]$

$\sum \alpha_i = 1$

$f = \sum \alpha_i \, c_{\mu_i}$

$\mathbb{Q}$ sow.

~~mnoi rěš. α~~

Δ ~~je~~ $c_{\mu_1}, \dots, c_{\mu_k}$ afinní nezávislé

$\Rightarrow \alpha_1, \dots, \alpha_k$ dostanu Gauss. eliminací

$\boxed{\Rightarrow \ \alpha_i \in \mathbb{Q}}$

$\Downarrow$

$\alpha_i = \dfrac{K_i}{N}$

$\exists \ K_i \times \mu_i \quad (i = 1, \dots, k)$

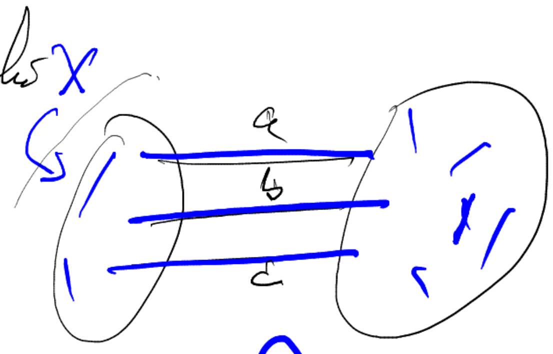
$\forall e$ je v $\dfrac{N}{v}$ z nich.

Corollaries of Application 1

1) Every bridgeless cubic graph has a uniform cover by perfect matchings.

2) Every bridgeless cubic graph has a perfect matching. (This of course has easier proofs.) It also has a perfect matching using any given edge. (This, too, can be proved by an application of Tutte's theorem, but it's always good to have another proof technique.)

3) Every bridgeless cubic graph has a perfect matching that contains no odd cut of size 3. Indeed, every matching that is a part of the uniform cover works. Consequently, every such graph has a 2-factor that does not contain a triangle.



5-edge-connected factor

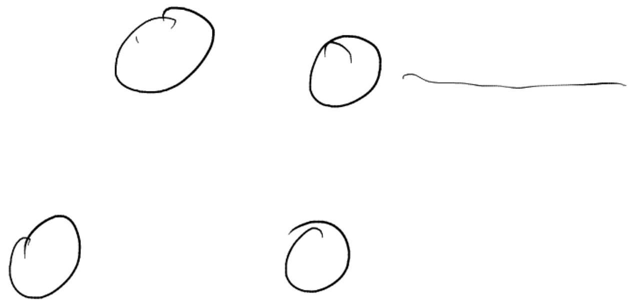
A more complicated result of Kaiser and Škrekovski says that every graph contains a 2-factor that intersects every 3-cut and every 4-cut. As a corollary we get the following result that is often useful for dealing with properties of flows and cycles in graphs.

Theorem 36 (Kaiser, Škrekovski)**.** *Let G be a 3-edge-connected graph. Then G contains a cycle C such that the graph G/C (where each component of C is contracted to a vertex) is 5-edge-connected.*

(The proof is essentially a cut-uncrossing argument.)

$\forall G$ 3-hr. souvr.

PP ?



$C_5, C_6, \ldots$

NO C_3, C_4

Application 2

Theorem 37 (Kaiser, Král̆, Norine). *Every bridgeless cubic graph G has perfect matchings M_1, M_2 such that $|M_1 \cup M_2| \geq \frac{3}{5}|E(G)|$.*

(1) *Proof.* First use Application 1, namely the third corollary: Let M be a perfect matching that contains no odd cut of size 3. Define $f(e) = 1/5$ for $e \in M$ and $f(e) = 2/5$ elsewhere.

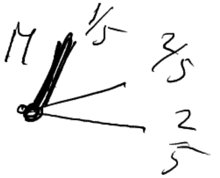
(2) We check that f is in PMP_G. The sum around each vertex is 1. If X is an odd-size vertex set, then $|\delta(X)|$ is odd, therefore either 3, or at least 5. In the latter case, $\sum_{e \in \delta(X)} f(e) \geq 5 \cdot \frac{1}{5} = 1$, which we need. In the former case, we know by the choice of M that exactly one of the edges in $\delta(X)$ is in M, therefore $\sum_{e \in \delta(X)} f(e) = \frac{1}{5} + \frac{2}{5} + \frac{2}{5} = 1$.

(3) As f is in the perfect matching polytope, f is a convex combination of c_{M_i} for some

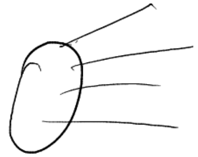
M --- p.p. $\quad M \not\supseteq \{e,f,c\}$ --- 3-řez

$f(e) = \begin{cases} 1/5 & e \in M \\ 2/5 & e \notin M \end{cases}$

$f(\delta(v)) = 1$



$f(\delta(X)) \geq 1$?



liché X

$|\delta(X)| = \begin{cases} 1 & \times \\ 3 & \checkmark \\ 5+ & \checkmark \end{cases}$

liché X 1/5

$|M \cap \delta(X)|$ liché ✓

$\neq 3$ ✓

perfect matchings M_i. Put $S = E(G) \setminus M$. By definition of f, we have $f(S) = \frac{2}{5}|S|$, hence $c_{M_i}(S) \geq \frac{2}{5}|S|$ for some M_i involved in the convex combination for f. Now $|M \cup M_i| = |E(G)| \cdot (\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{5}) = \frac{3}{5}|E(G)|$. $\square$

The above may be generalized as follows. For a graph G define $m_i(G)$ to be the maximum fraction of edges that can be covered by a union of i perfect matchings – that is

$$m_i(G) := \max\{\frac{|M_1 \cup \cdots \cup M_i|}{|E(G)|} : M_i \text{ are perfect ma}$$

So we found that $m_2(G) \geq 3/5$ for every 3-graph G, and this bound is attained for the Petersen graph. [KKN] did further find that $m_3(G) \geq 27/35$ for a 3-graph G. If Berge-Fulkerson conjecture is true, we have $m_5(G) = 1$.

Exercises: 1. Prove that a 3-graph is the same as a bridgeless cubic graph. **2.** Find up-

$f = \sum \alpha_i c_{M_i}$

$f(S) = \sum_{e \in S} f(e) = \frac{2}{5}|S|$

$\sum_{e \in S} \sum_i \alpha_i c_{M_i}(e)$

$\sum_i \alpha_i \underbrace{c_{M_i}(S)}$

$c(G)$ konv. komb. $c_{M_i}(S)$

per and lower bounds for $m_3(G)$ when G is a cubic bridgeless graph. (Note that $m_3(G) \geq 27/35$ is the best known so-far.)

3. Find some bounds on $m_i(G)$ for a general i, and use this to estimate number of perfect matchings needed to cover all edges of a graph G.

Now we give the postponed proof of Theorem 35.

Proof. Let P_G be the polytope defined by the inequalities (**??**). Easily $PMP_G \subseteq P_G$, as all vertices of the perfect matching polytope (i.e., all c_M for a perfect matching M) satisfy the inequalities (**??**). For the other inclusion, we proceed by contradiction: we take the graph G with smallest $|V(G)|+|E(G)|$, and one vertex f of P_G such that $f \notin PMP_G$.

We have $0 < f(e) < 1$ **for each edge** e **of** G**.** If $f(e) = 0$ for some edge e, we let $G' = G - e$ and f' to be the restriction of f to $E(G')$. It is easy to check that $f' \in P_{G'}$, and as G' is smaller than G, we have $P_{G'} = PMP_{G'}$ and f' is a convex combination of characteristic vectors of perfect matchings of G'. When we take these matchings as perfect matchings of G (by extending the characteristic vector by a 0 in the coordinate indexed by e), we get $f \in PMP_G$, a contradiction.

On the other hand, if $f(e) = 1$ for some edge $e = uv$, then we put $G' = G - u - v$. Again, we let $f' = f|_{E(G')}$ and we check that $f' \in P_{G'} = PMP_{G'}$. By extending all the perfect matchings that occur in the convex combination for f' by the edge e we get perfect matchings whose convex combination is f, again a contradiction.

G has no vertices of degree ≤ 1. G certainly does not have isolated vertices (by inequality (**??**)), and if v is a vertex incident only with an edge e, then $f(e) = 1$, which we already disproved. Consequently, $|E(G)| \geq |V(G)|$.

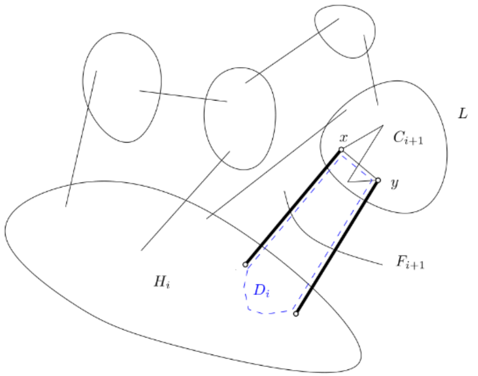
Case 1. $|E(G)| = |V(G)|$ G is 2-regular, thus a disjoint union of circuits. None of these is odd (otherwise we let X be the set of vertices of an odd circuit and get a contradiction with inequality (**??**)). For even circuits it is easy to ... (Exercise!).

Case 2. $|E(G)| > |V(G)|$ As f is a vertex of a polytope in $\mathbb{R}^{E(G)}$, at least $|E(G)|$ of the inequalities are satisfied with an equality. (Exercise!) Thus, one of them must be (*) $\sum_{e\in\delta(X)} f(e) = 1$ for some $X \subseteq V(G)$, such that $1 < |X| < |V(G)|$ and $|X|$ is odd. As $|X|$ is odd, every perfect matching of G contains an edge of $\delta(X)$. This together with (*) implies that each of the sought-for matchings involved in the representation of f contain exactly one edge of $\delta(X)$. This suggest that we may want to treat X as a single vertex: if there is a representation for f, then this change of the graph will transform them in matchings.

To put this formally, we let $G_1 = G/X$ – all vertices of X are identified to a single vertex, we keep possible multiedges) – and $G_2 = G/\bar{X}$ (where $\bar{X} = V(G) \setminus X$). Again, let f_i be the restriction of f to the edge-set of G_i ($i = 1, 2$). It is easy to check that $f_i \in P_{G_i}$, which implies (Exercise!) that there are perfect matchings $(M_{i,k})_{k=1}^N$ of G_i such that

$$f_i = \frac{1}{N} \sum_{k=1}^{N} c_{M_{i,k}} \, . \tag{2}$$

Recall that each $M_{i,k}$ contains exactly one of the edges of $\delta(X)$ (we abuse the notation slightly, we identify the edges of $\delta(X)$ in G, and the corresponding edges of G_1, G_2). Moreover, if e is one of these edges, then the number of perfect matchings $M_{i,k}$ of G_i for which $e \in M_{i,k}$ is $Nf_i(e)$ (just look at the e-th coordinate of (2)). However, $Nf_1(e) = Nf_2(e) =$



$Nf(e)$ (recall f_i was defined as a restriction of f to $E(G_i)$). Consequently, we may pair up the matchings of G_1 and of G_2 to agree on the edges of $\delta(X)$, indeed we may assume that $M_{1,k}$ and $M_{2,k}$ contain the same edge from the cut Z. We put $M_k = M_{1,k} \cup M_{2,k}$. It is easy to check that f is the average of c_{M_k}, which finishes the proof. $\square$

Theorem 38 (Seymour)**.** *Every bridgeless graph G has a 6-NZF.*

Proof. Equivalently, we will show it has NZ