Snarks with a 4-cut – Isaacs' dot product

Let G, H be graphs, ab, cd edges of G, e an edge of H, let x, y be the other two neighbours of one end of e, u, v the other two neighbours of the other end. To form the *Isaacs' dot product* $G \cdot H$ of G and H we delete edges ab and cd from G, e with its end-vertices from H, and add edges ax, by, cu, dv.

Theorem 25 (Isaacs, 1975)**.** *If G and H are snarks then so is $G \cdot H$. If both G and H are cyclically 4-edge-connected and if the vertices a, b, c, d are all different, then $G \cdot H$ is also cyclically 4-edge-connected.*

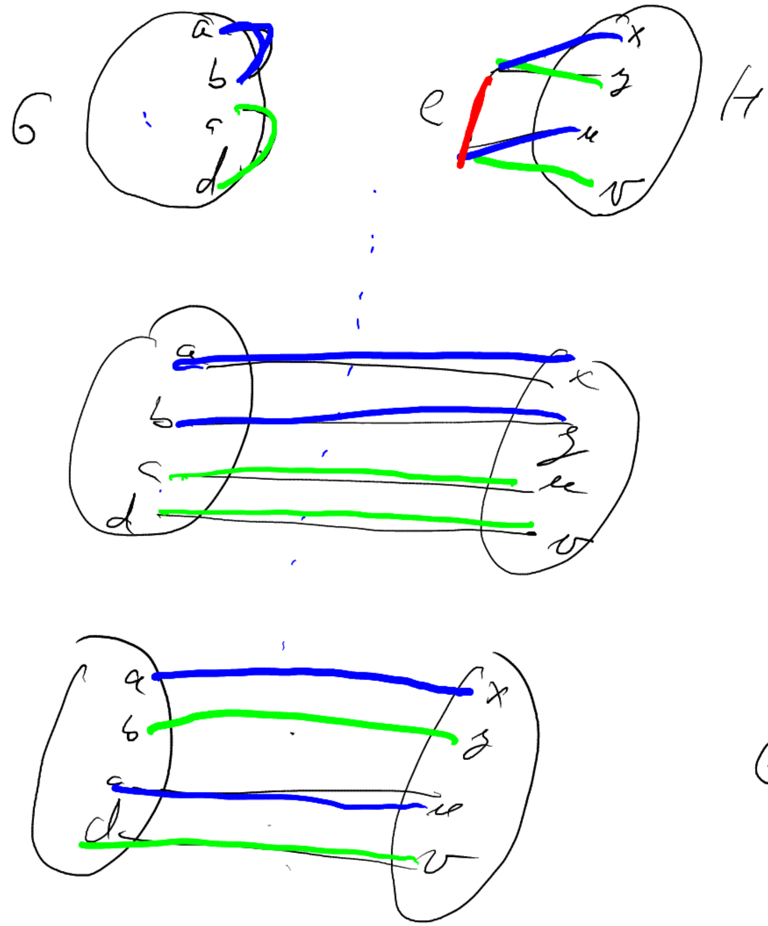
Proof. Suppose we have an edge 3-coloring f of $G \cdot H$. We distinguish two cases.

(1) $f(ax) = f(by)$ (2) $f(ax) \neq f(by)$

. . .

G není snark

H není snark ◻

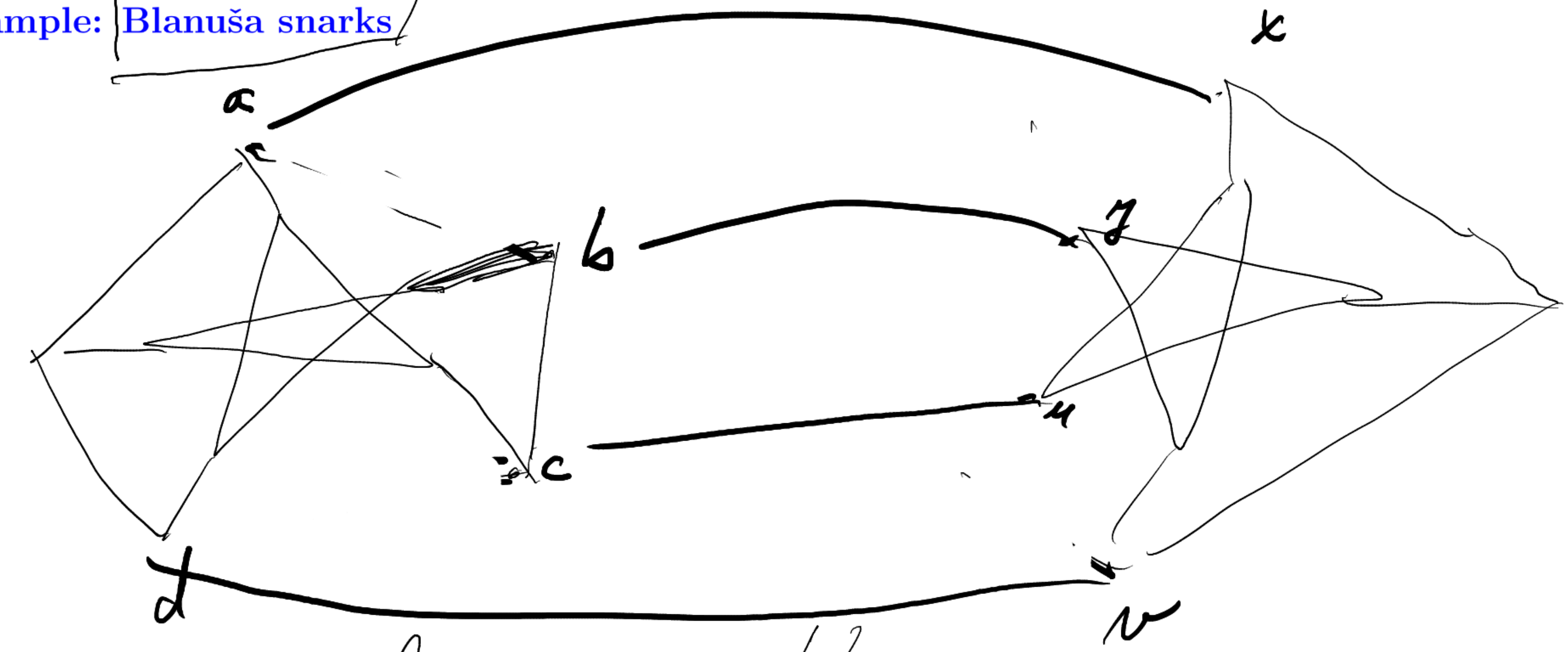


G.H snark

⇑

G,H snarky

Example: Blanuša snarks

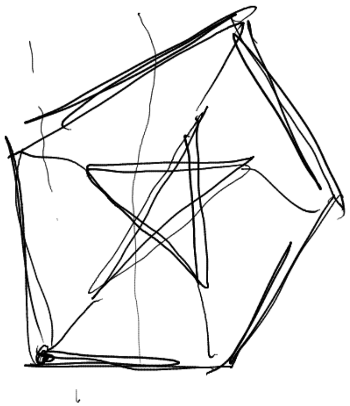


3-reg.-graf s 18 vrcholy

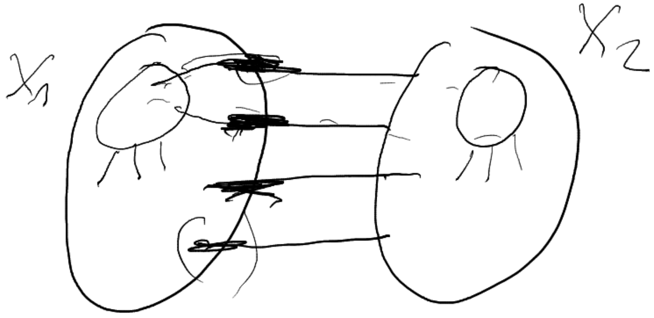
G je k-souvislý $\iff \forall U \subsetneq V(G),\ U \neq \emptyset \quad |\delta(U)| \geq k$

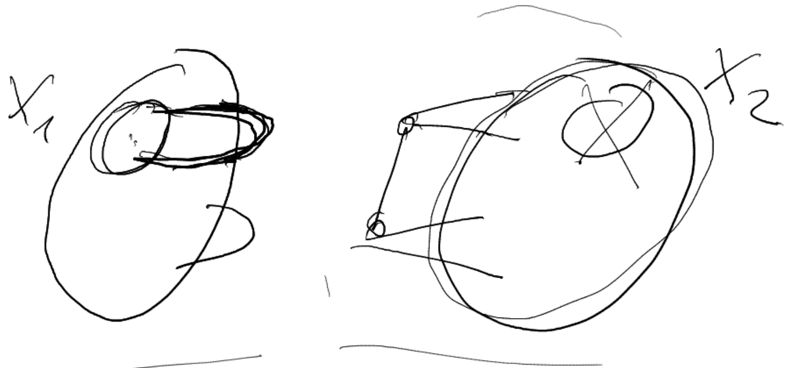
G je interně k-souvislý $\iff \forall U \quad |\delta(U)| \geq k$ nebo $|U| = 1$
nebo $|\overline{U}| = 1$

G je cyklicky k-souvislý $\iff \forall U \quad |\delta(U)| \geq k$ nebo $G[U]$ je les
nebo $G[\overline{U}]$ je les



2-souvislý
3-souvislý
není 4-souvislý je interně/cykl. 4-souvislý
cykl. 5-souvislý





↯ $\exists X \subseteq V(G \cdot M) : |X| > 1$

$|V \setminus X| > 1$

$|\delta(X)| \le 3$

$X = X_1 \circ X_2$

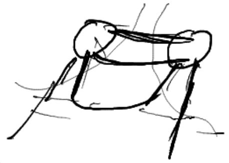
1) $X_2 = \emptyset$ $\quad |\delta(X)| \ne |\delta(X_1)|$

$|X_1| \ne |X|$

$X_1 = V(G) \ldots \delta(X) = 4$

2) $X_1 = \emptyset$ ✓

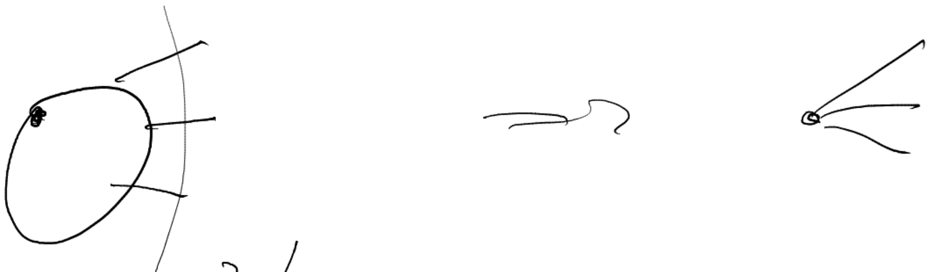
3) $X_1, X_2 \ne \emptyset$ $\quad \delta(X) = \delta(X_1) \text{ XOR } \delta(X_2)$



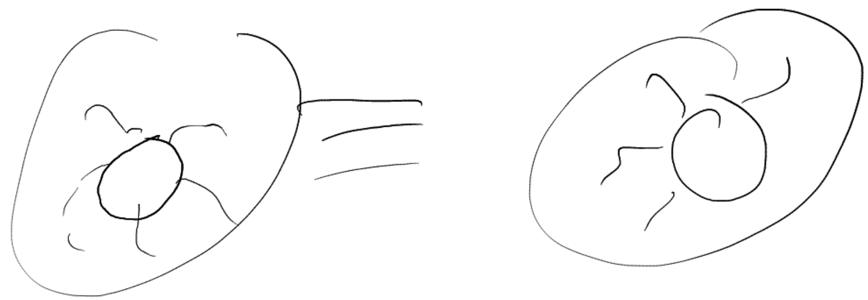
$\underbrace{|\delta(X)|}_{\le 3} = \underbrace{|\delta(X_1)|}_{\ge 4} + \underbrace{|\delta(X_2)|}_{\ge 4} - \underbrace{2|E(X_1, X_2)|}_{\ge 2}$

$\delta(X_1)$ mod malo $-2\#$ sąsiad

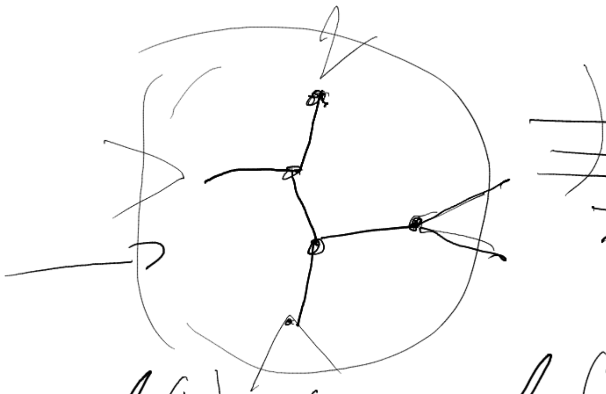
↯



≤ 3 hrany



vyhledat E-souv.



≥ 4

3-reg. graf

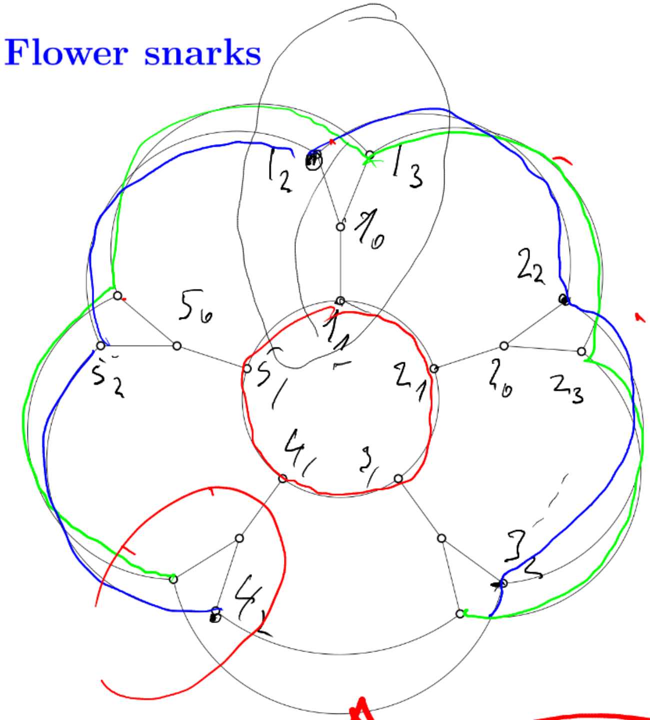
les

$X \subseteq V(G)$ $\quad G[X] \Rightarrow \deg(X) \geq 4$

$\deg(X) \geq 2 \cdot \#$ listů

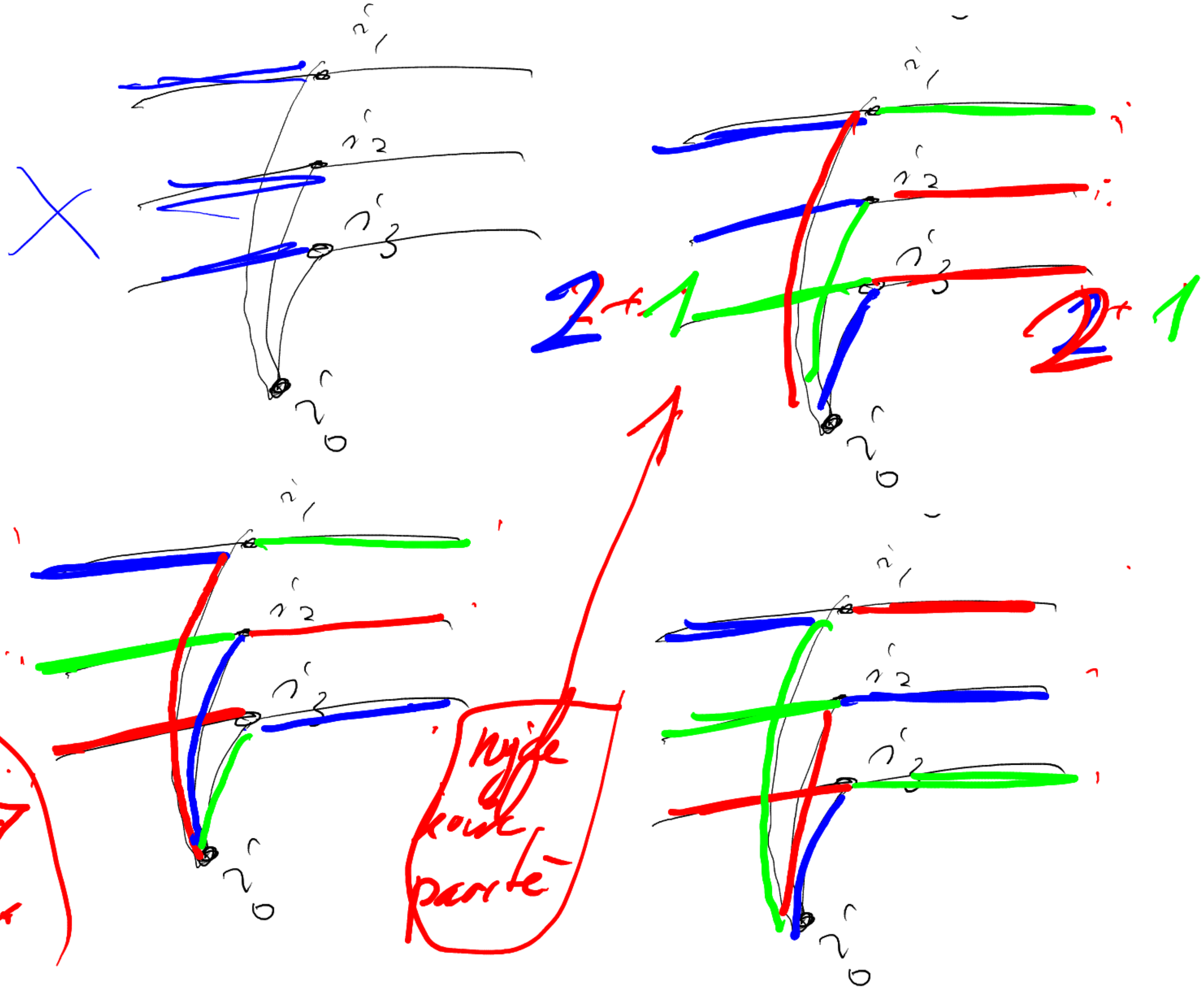
& $\forall$ les má ≥ 2 listy

Flower snarks



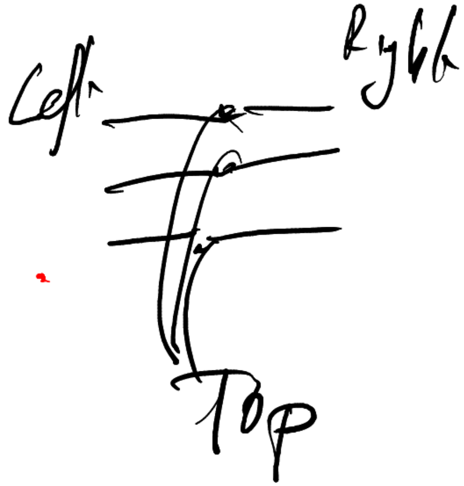
J_5

Nejde
kvůli záměně cykl. pořadí



Let n be odd. To describe a graph J_n, we start with three copies of C_n, we denote its vertices by i_1, i_2, i_3 for $i = 1, \ldots, n$. Replace edges $n_2 1_2$ and $n_3 1_3$ by $n_2 1_3$ and $n_3 1_2$. Finally, for each i we add a new vertex i and join it by an edge to i_1, i_2, i_3. On Figure **??** we can see J_5 (this particular graph is sometimes called the flower snark). and J_3 — is just a Y-Δ transformation of Pt (equivalently, it is Pt$\equiv K_4$).

Theorem 26 (Isaacs, 1975)**.** *If n is odd then J_n is a snark. If $n \geq 7$ then J_n is cyclically 6-edge-connected.*

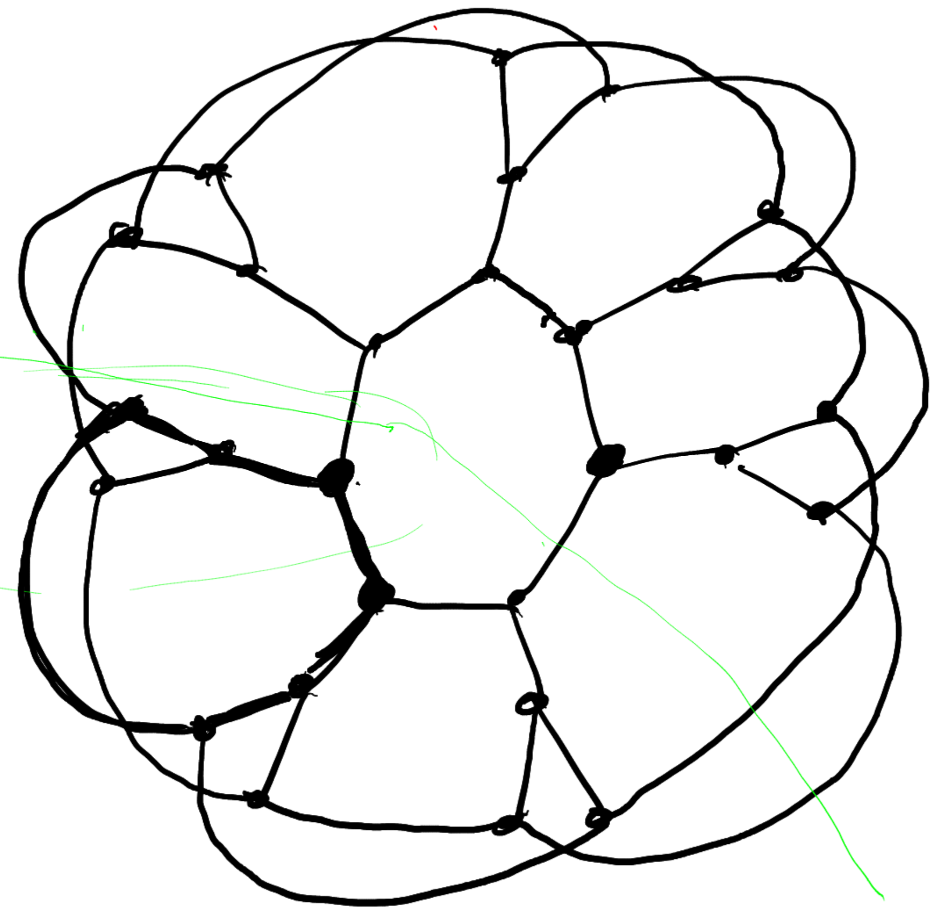


Proof. Suppose J_n can be edge-colored using three colors. Let B_i denote the subgraph induced by vertices i, i_1, i_2, i_3 and the incident edges (see Fig. **??**). We divide the edges of this subgraph into three triples, Left, Right, and Top. (Of course the Right edges of B_i are the Left edges of B_{i+1}.) Clearly not all edges of of L can be of the same color, as then it is not possible to color T. Thus there are two possibilities.

(1) Edges of L use one color twice. Say, they use colors 1, 1, and 2 in some order. It is easy to check that then edges of R use colors 2, 3, and 3, in some order. In the next block we will use 1, 1, 2 on the right, and so on. As n is odd, we get a contradiction.

(2) Edges of L use all three colors. Again, it is simple to explore the two possibilities how to extend the coloring on R: both are obtained from the coloring of L by a cyclic shift (i.e., a permutation formed by one 3-cycle). In between the blocks B_n and B_1 we introduced a transposition by the construction of the graph. Thus if there is an edge 3-coloring, then we can write an identity as a composition of 3-cycles and one transposition, which is a contradiction.

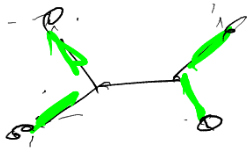
TODO: cyclic connectivity?



spousta 6-řezů

of snarks

Superposition construction (Kochol)



- G: a graph with all degrees 1 or 3
- a flow on G: a nowhere-zero $\mathbb{Z}_2^2$-flow where we ignore Kirchhoff's condition at degree 1 vertices.

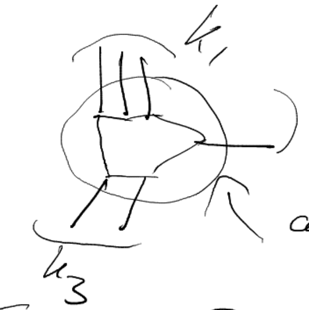
$\Leftrightarrow$ dobarvení hranové 3-obarvení (pomocí barev $(0,1)$, $(1,0)$, $(1,1)$)

- Observation: let E_1 be the edges incident to degree 1 vertices, let φ be a flow. Then $\varphi(E_1) = 0$.

$\varphi(S) = \sum_{e \in S} \varphi(e)$

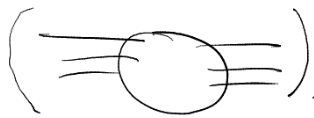
(slepíme vš. vrcholy st. 1 do nového vrcholu $*$ ...
φ je tok všude až na $*$
$\Rightarrow$ je to tok &
Kirchh. zákon v $*$: $\varphi(E_1)=0$)

- (k_1, k_2, k_3)-supervertex: a graph as above, with E_1 split into three nonempty subsets of sizes k_1, k_2, k_3.



- (k_1, k_2)-superedge: a graph as above, with E_1 split into two nonempty subsets of sizes k_1, k_2.

- proper superedge: a superedge, where the sum over each of the two parts is nonzero.



$\forall$ tok φ:
$\varphi(S_1) \neq 0$

$(\varphi(S_1) + \varphi(S_2) = 0$
$\Downarrow$
$\varphi(S_1) = \varphi(S_2))$

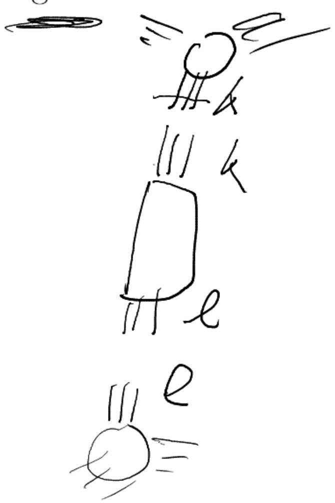
Given:

- A snark G,
- a list of *supervertices* G_v for $v \in V(G)$
- a list of *proper superedges* G_e for $e \in E(G)$

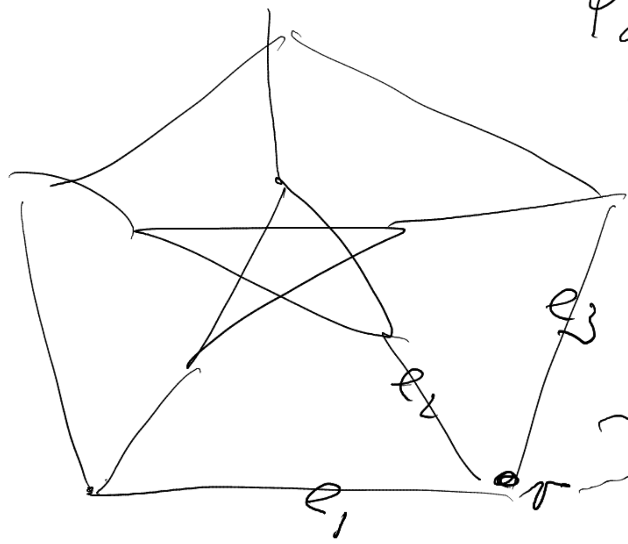
Conclusion: The superposition is a snark.

Corollary: There is a family of cyclically 6-edge-connected snarks.

Corollary: There is a family of cyclically 5-edge-connected snarks with arbitrarily hig girth.



ψ je tok v G $\Leftarrow$

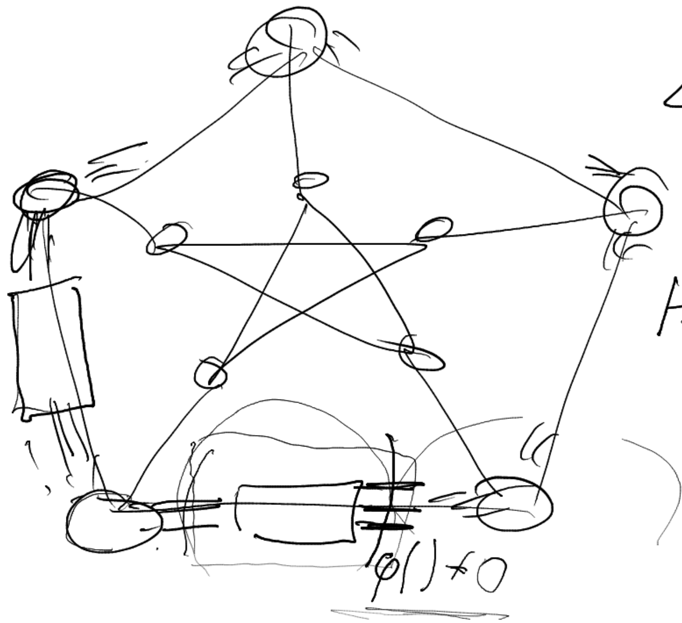


G_v, G_e

G_v konce A_1, A_2, A_3

$\psi(e_i) = \varphi(A_i) \neq 0$

$\psi(e_1) + \psi(e_2) + \psi(e_3) = 0$



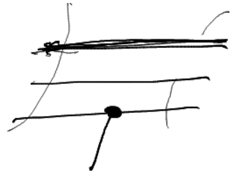
$\Leftarrow$ H není snark, φ … tok v H

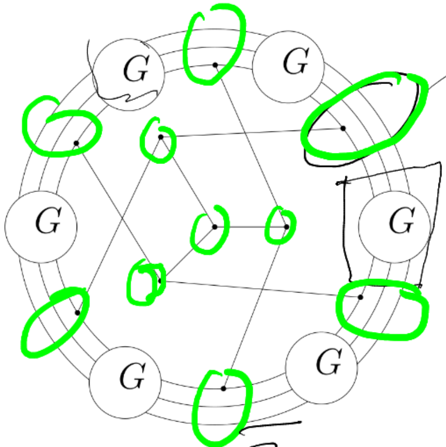
$e \in E(G) \rightsquigarrow G_e$

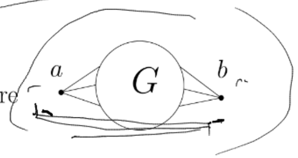
konce S_1, S_2

$\psi(e) := \varphi(S_1) = \varphi(S_2) \neq 0$
$\in \mathbb{Z}_2^2 \setminus \{(0,0)\}$

supervrchol



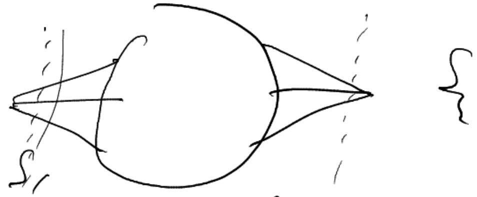


where 

is the Flower snark depicted

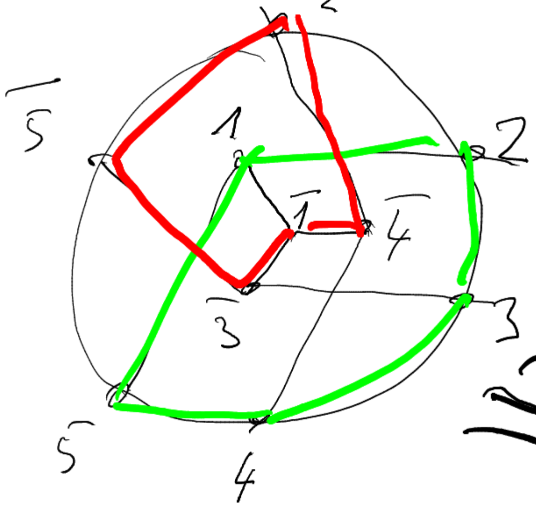
$abc \notin E(J_n)$

superhrana



Pokud tato superhrana není proper tak ∃ toku φ t.ž. $\varphi(S_1)=0$, $\varphi(S_2)=0$

Pak ale φ je tok v Flower snarku ⟹ ↯



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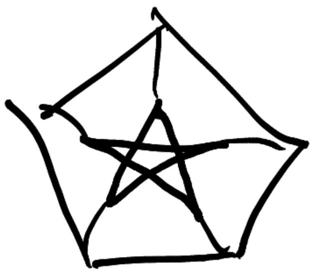
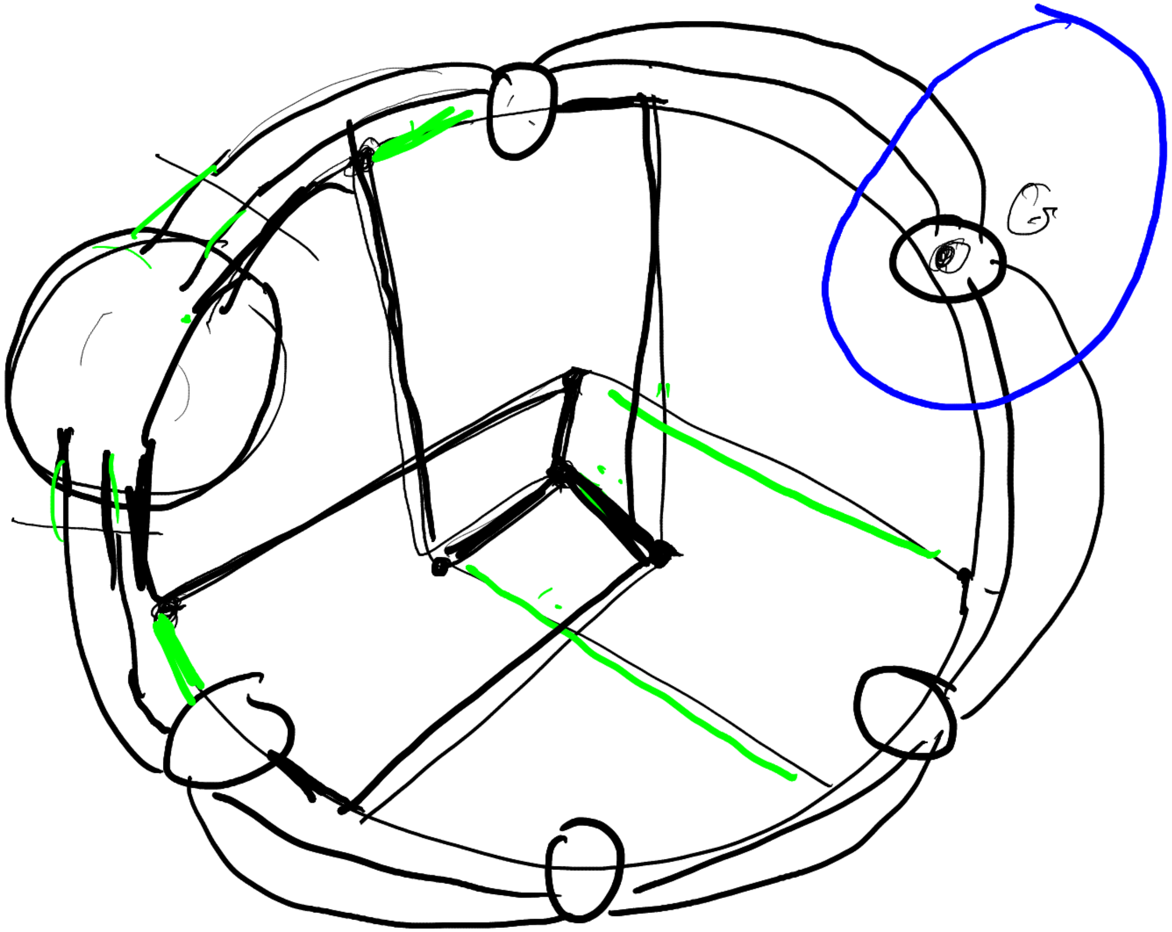
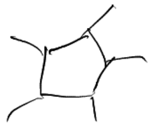




$H \not\cong C_5$ ✓



$X \supseteq$ Kreis C $\quad V \setminus X \supseteq$ Kreis

$|\delta(X)| \leq 5$

1) $X \subseteq G$ $\longrightarrow$ Flower Snark / je ggf. 6-Schnitt!