Another proof of Seymour's 6-flow theorem

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For every $k \geq 1$ we define $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. If $G = (V, E)$ is an oriented graph and $v \in V$ then we let $\delta^+(v)$ ($\delta^-(v)$) denote the set of edges with tail (head) v and we put $\delta(v) = \delta^-(v) \cup \delta^+(v)$. For a function $\phi : E \to \mathbb{Z}_k$, the *boundary* of ϕ is the function $\partial\phi : V \to \mathbb{Z}_k$ given by the rule

$$\partial\phi(v) = \sum_{e \in \delta^+(v)} \phi(e) - \sum_{e \in \delta^-(v)} \phi(e).$$

Every such function satisfies the *zero-sum rule*: $\sum_{v \in V} \partial\phi(v) = 0$ since (after expanding) each edge e contributes $\phi(e) - \phi(e) = 0$ to the total. We call ϕ *nowhere-zero* if if $\phi^{-1}(0) = \emptyset$ and we call it a *flow* or a $\mathbb{Z}_k$-*flow* if $\partial\phi = 0$. Our goal here is to prove the following.

Theorem 1 (Seymour)**.** *Every oriented 2-edge-connected graph has a nowhere-zero $\mathbb{Z}_6$-flow.*

Our proof relies upon the following lemma. Here $V_t(G)$ denotes the set of vertices of degree t in the graph G, and we call G *sub-cubic* if $V_t(G) = 0$ for all $t > 3$.

Lemma 2. *Let G be an orientation of a 2-edge-connected sub-cubic graph and let $\mu : V(G) \to \mathbb{Z}_3$. Let $u \in V_2(G)$ be a distinguished root vertex, for $k = 2, 3$ let $\phi_k^u : \delta(u) \to \mathbb{Z}_k$, and suppose*

i. $\sum_{v \in V} \mu(v) = 0$, *and*

ii. $supp(\mu) \subseteq V_2(G)$, *and*

iii. $\partial\phi_3^u(u) = \mu(u)$, *and*

iv. $\partial\phi_2^u(u) = 0$ *if* $\mu(u) = 0$.

Then there exist flows $\phi_k : E \to \mathbb{Z}_k$ for $k = 2, 3$ satisfying

1. $\phi_k|_{\delta(u)} = \phi_k^u$ *for* $k = 2, 3$, *and*
2. $\partial\phi_3 = \mu$, *and*
3. $supp(\partial\phi_2) \subseteq supp(\mu)$, *and*
4. $(\phi_2(e), \phi_3(e)) \neq (0, 0)$ *for every* $e \in E(G) \setminus \delta(u)$.

Proof. We proceed by induction on $|E(G)|$. Our base cases will be when G is an orientation of a cycle of length two or three. If G is a cycle of length two with vertex set $\{u, v\}$ then the functions ϕ_2^u, ϕ_3^u already satisfy the conclusion (to see this, note that the zero-sum rule gives $\partial\phi_k^u(v) = -\partial\phi_k^u(u)$ and assumption (i) implies $\mu(v) = -\mu(u)$). Next suppose G is a cycle of length three with vertex set $\{u, v, v'\}$. If $\mu(v) = 0$ or $\mu(v') = 0$ then the result follows by contracting vv' and applying the previous argument. Otherwise, we extend ϕ_2^u to the function $\phi_2 : E(G) \to \mathbb{Z}_2$ by defining $\phi_2(vv') = 1$. Similarly, we extend ϕ_3^u to $\phi_3 : E(G) \to \mathbb{Z}_3$ by choosing $\phi_3(vv')$ so that $\partial\phi_3 = \mu$ (this is possible by assumptions (i) and (iii) and the zero-sum rule). The resulting functions yield the result.

Next suppose that G has an edge-cut of size two which separates the vertices into X_1, X_2 where $|X_1|, |X_2| \geq 2$. Assume (without loss) that $u \in X_1$ and for $i = 1, 2$ let G_i be the graph obtained from G by identifying X_i to a new vertex x_i (and deleting any resulting loops). For $i = 1, 2$ define $\mu^i : V(G_i) \to \mathbb{Z}_3$ by the rule

$$\mu^i(v) = \begin{cases} \mu(v) & \text{if } v \neq x_i \\ \sum_{x \in X_i} \mu(x) & \text{if } v = x_i \end{cases}$$

Apply induction to G_2 together with μ^2, ϕ_2^u, ϕ_3^u to obtain ϕ_2^2, ϕ_3^2. Now for $k = 2, 3$ let $\psi_k^{x_1} : \delta_{G_1}(x_1) \to \mathbb{Z}_k$ be obtained by restricting ϕ_k to these edges. Apply induction to the graph G_1 with the root vertex x_1 and $\mu_1, \phi_2^{x_1}$, $\phi_3^{x_1}$ to obtain ϕ_2^1 and ϕ_3^1. Now merging the functions ϕ_k^1 and ϕ_k^2 for $k = 2, 3$ yields the desired solution.

Next suppose there exists a vertex $v \in \mathit{supp}(\mu) \setminus \{u\}$ and let w_1, w_2 be its neighbours. Choose a nowhere-zero function $\psi : \{vw_1, vw_2\} \to \mathbb{Z}_3$ so that $\partial\psi(v) = \mu(v)$. Then define $\mu' : V(G - v) \to \mathbb{Z}_3$ by the rule $\mu'(w_i) = \mu(w_i) - \partial\psi(w_i)$ for $i = 1, 2$ and otherwise $\mu'(w) = \mu(w)$. It follows from (i) and the zero-sum rule for ψ that $\sum_{w \in V \setminus \{v\}} \mu'(v) = 0$. So, we may apply induction to $G - v$ together with μ', ϕ_2^u, and ϕ_3^u to obtain ϕ_2' and ϕ_3'. Extend ϕ_3' to a function $\phi_3 : E(G) \to \mathbb{Z}_3$ by defining $\phi_3(vw_i) = \psi(vw_i)$ for $i = 1, 2$. Extend ϕ_2' to a function $\phi_2 : E(G) \to \mathbb{Z}_2$ by defining $\phi_2(vw_i) = \partial\phi_2'(w_i)$. Now ϕ_2, ϕ_3 give a solution.

In the only remaining case $\mu = 0$ and we choose an edge vw with $v, w \neq u$. Define $\mu' : V(G) \to \mathbb{Z}_3$ by $\mu'(v) = 1$, $\mu'(w) = -1$ and $\mu'(x) = 0$ for all $x \in V(G) \setminus \{v, w\}$. Now we may apply induction to $G - vw$ together with μ, ϕ_2^u, and ϕ_3^u and (by the zero-sum rule) extend the resulting functions to the desired flows in G. □

Proof of Theorem 1. If G is an (oriented) 2-edge-connected graph with a vertex v of degree at least four, then we may choose distinct edges vw, vw' which are contained in a common cycle (not nec. directed) and uncontract an edge at v (i.e. the reverse of contracting an edge to form v) so that each newly formed vertex has degree at least three and is incident to one of vw, vw'. Orient this new edge arbitrarily, and note that by our choice the underlying graph will still be 2-edge-connected. Repeat this process to obtain a sub-cubic oriented 2-edge-connected graph G^*. It follows from the previous lemma that G^* has a nowhere-zero $\mathbb{Z}_6$-flow, and by contracting edges, the graph G inherits such a flow. □