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DOCTORAL THESIS

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Geometric and algebraic properties of discrete structures

Department of Applied Mathematics

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Finally, I would like to thank my family for their support and my friends.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Geometrické a algebraické vlastnosti diskrétních struktur

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Abstrakt: V práci se zabýváme dvou-dimenzionálními simpliciálními komplexy a lineárními kódy. Řekneme, že lineární kód $\mathcal{C}$ nad tělesem $\mathbb{F}$ je trojúhelníkově reprezentovatelný, pokud existuje dvou-dimenzionální simpliciální komplex Δ takový, že kód $\mathcal{C}$ je propíchnutým kódem jádra $\ker \Delta$ incidenční matice simpliciálního komplexu Δ nad $\mathbb{F}$ a $\dim \mathcal{C} = \dim \ker \Delta$. Tento simpliciální komplex nazveme geometrickou reprezentací kódu $\mathcal{C}$.

Dokážeme, že každý lineární kód nad prvotělesem je trojúhelníkově reprezentovatelný. Pro konečná prvotělesa sestrojíme geometrickou reprezentaci takovou, že váhový polynom kódu $\mathcal{C}$ je dán jednoduchou formulí váhového polynomu prostoru cyklů simpliciálního komplexu Δ. Tedy geometrická reprezentace kódu $\mathcal{C}$ určuje jeho váhový polynom.

Naše motivace pochází z teorie pfaffiánovských orientací grafů, která poskytuje polynomiální algoritmus pro výpočet váhového polynomu prostoru řezů grafu s omezeným rodem. Tento algoritmus využívá geometrických vlastností nakreslení grafu na orientovatelnou riemannovskou plochu. Prostor řezů je lineární kód a odpovídající graf je jeho užitečnou geometrickou reprezentací.

Dále studujeme vnořitelnost geometrických reprezentací do euklidovských prostorů. Ukážeme, že každý binární lineární kód má geometrickou reprezentaci v $\mathbb{R}^4$. Charakterizujeme binární lineární kódy, které mají geometrickou reprezentaci v $\mathbb{R}^3$.

Ukážeme, že váhový polynom každého binárního lineárního kódu je polynomiálně převeditelný na permanent troj-rozměrné nezáporné matice. Dále studujeme Pfaffiánovské troj-rozměrné matice a ukážeme aplikaci našich výsledků ve statistické fyzice.

Klíčová slova: simpliciální komplex, lineární kód, váhový polynom, geometrické reprezentace

Title: Geometric and algebraic properties of discrete structures

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Abstract: In the thesis we study two dimensional simplicial complexes and linear codes. We say that a linear code $\mathcal{C}$ over a field $\mathbb{F}$ is triangular representable if there exists a two dimensional simplicial complex Δ such that $\mathcal{C}$ is a punctured code of the kernel $\ker \Delta$ of the incidence matrix of Δ over $\mathbb{F}$ and $\dim \mathcal{C} = \dim \ker \Delta$. We call this simplicial complex a geometric representation of $\mathcal{C}$.

We show that every linear code $\mathcal{C}$ over a primefield is triangular representable. In the case of finite primefields we construct a geometric representation such that the weight enumerator of $\mathcal{C}$ is obtained by a simple formula from the weight enumerator of the cycle space of Δ. Thus the geometric representation of $\mathcal{C}$ carries its weight enumerator.

Our motivation comes from the theory of Pfaffian orientations of graphs which provides a polynomial algorithm for weight enumerator of the cut space of a graph of bounded genus. This algorithm uses geometric properties of an embedding of the graph into an orientable Riemann surface. Viewing the cut space of a graph as a linear code, the graph is thus a useful geometric representation of this linear code.

We study embeddability of the geometric representations into Euclidean spaces. We show that every binary linear code has a geometric representation that can be embedded into $\mathbb{R}^4$. We characterize binary linear codes that have a geometric representation embeddable into $\mathbb{R}^3$.

We further show that the weight enumerator of any binary linear code is polynomial reducible to the permanent of a non-negative three dimensional matrix. We give some applications of our results to statistical physics, by studying the Pfaffian three dimensional matrices.

Keywords: simplicial complex, linear code, weight enumerator, geometric representations

Contents

Introduction

In this thesis we study relations between geometric and algebraic properties of discrete structures. We mainly study two discrete structures: linear codes and simplicial complexes.

A simplex X is the convex hull of an affine independent set V in $\mathbb{R}^d$. The convex hull of any non-empty subset of V that defines a simplex is called a face of the simplex. A simplicial complex [10] Δ is a set of simplices fulfilling the following conditions: Every face of a simplex from Δ belongs to Δ and the intersection of every two simplices of Δ is a face of both.

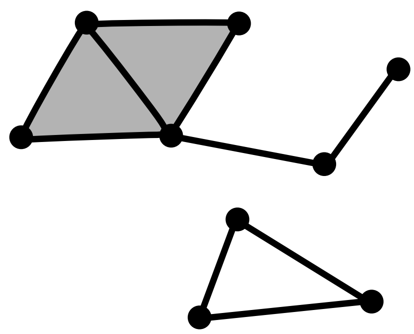

Figure 1: A simplicial complex

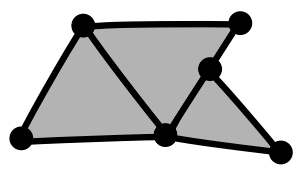

Figure 2: This set of simplices is not simplicial complex

A linear code [33] $\mathcal{C}$ of length n over a field $\mathbb{F}$ is a linear subspace of the vector space $\mathbb{F}^n$. Let $\mathcal{C} \subseteq \mathbb{F}^n$ be a linear code over a field $\mathbb{F}$ and let S be a subset of $\{1, \ldots, n\}$. Puncturing a code $\mathcal{C}$ along S means deleting the entries indexed by the elements of S from each codeword of $\mathcal{C}$. The resulting code is denoted by $\mathcal{C}/S$.

An important function associated with a linear code $\mathcal{C}$ over a finite field is its weight enumerator. It is defined by the formula

$$W_{\mathcal{C}}(x) := \sum_{c \in \mathcal{C}} x^{w(c)},$$

where $w(c)$ denotes the weight (the number of non-zero coordinates) of c. This polynomial encodes many interesting properties of the linear code.

A graph [4] is an ordered pair $G = (V, E)$ consisting of a set of vertices V and a set of edges E, which are two elements subsets of V. A matching M in G is a set of pairwise non-adjacent edges. A perfect matching in G is a matching in G that covers all vertices of G. The problem of finding a perfect matching in G is polynomially solvable by Edmonds's algorithm [5, 27]. On the other hand, there is no known polynomial algorithm for the number of perfect matchings even in bipartite graphs. The number of perfect matchings in a bipartite graph is equal to the permanent of the biadjacency matrix of the graph. The permanent of a matrix $B = (b_{ij})$ is defined according to the formula

$$\mathrm{per}(B) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} b_{i\sigma(i)}.$$

Valiant [31] showed that the computation of the permanent is #P-complete. By a result of Jerrum et al. [11], the permanent of a matrix B with all entries non-negative can be computed approximately in probabilistic polynomial time.

Our motivation for studying linear codes and simplicial complexes comes from the theory of Pfaffian orientations of graphs.

This theory provides a polynomial algorithm for the number of perfect matchings in planar graphs. This algorithm is due to Fisher, Kasteleyn and Temperley [12, 13, 14, 6]. It was extended by Gallucio and Loebl [7] to graphs of bounded genus. This algorithm uses geometric properties of the embedding of the input graph into an orientable Riemann surface. Viewing the cut space of a graph as a linear code, graphs are useful geometric representations of the cut space. For detailed survey of theory of Pfaffian orientations see Thomas [30].

There is an interesting question posed by M. Loebl whether the theory of Pfaffian orientations can be extended to general linear codes. We study this question in this thesis.

We say that a linear code $\mathcal{C}$ over a field $\mathbb{F}$ is triangular representable if there exists a two dimensional simplicial complex (geometric representation) Δ such that $\mathcal{C}$ is a punctured code of the kernel $\ker \Delta$ of the incidence matrix of Δ over $\mathbb{F}$ and $\dim \mathcal{C} = \dim \ker \Delta$.

We start with study of geometric representations of binary linear codes. We show that every linear binary code is triangular representable. For every linear binary code $\mathcal{C}$, we construct a two dimensional simplicial complex Δ so that the weight enumerator of $\mathcal{C}$ is obtained by a simple formula from the weight enumerator of the cycle space of Δ. The triangular configuration Δ thus provides a geometric representation of $\mathcal{C}$ which carries its weight enumerator. This is the first step in the research program suggested by M. Loebl to extend the Pfaffian theory. Then we carry out also the second step by constructing, for every triangular configuration Δ, a triangular configuration Δ' and a bijection between the cycle space of Δ and the set of the perfect matchings of Δ'. These results are presented in Chapter 1, they extend a result from the author's master's thesis [23] and they were published in Rytíř [24].

Next, we show that the weight enumerator of any binary linear code is polynomial reducible to the permanent of a 3-dimensional matrix (3-matrix).

This accomplishes a generalization of the first cornerstone of the Kasteleyn method: rewriting the Ising partition function as the dimer partition function, that is, the generating function of the perfect matchings. The second cornerstone is expressing the dimer partition function of planar graphs as a determinant.

In analogy to the standard (2-dimensional) matrices we say that a 3-matrix A is *Kasteleyn* if signs of its entries may be changed so that, denoting by A' the resulting 3-matrix, we have $\mathrm{per}(A) = \det(A')$. It was proved in a seminal paper Robertson et al. [22] that 2-dimensional Kasteleyn matrices are essentially biadjacency matrices of planar bipartite graphs and thus they form a restrictive class. We show that in contrast with the 2-dimensional case the class of Kasteleyn 3-matrices is rich; namely, for each matrix M there is a Kasteleyn 3-matrix A so that $\mathrm{per}(M) = \mathrm{per}(A) = \det(A')$.

In particular, the dimer partition function of a finite 3-dimensional cubic lattice may be written as the determinant of the vertex-adjacency 3-matrix of a 2-dimensional simplicial complex which preserves the natural embedding of the cubic lattice. This result accomplishes the second step of the Pfaffian method for the binary linear codes. These results are presented in Chapter 2 and they were submitted for publication [17]. Results in Chapter 2 are based on a common work with my advisor Martin Loebl.

Next, we study geometric properties of the geometric representations. An interesting geometric property of simplicial complexes is its embedding into a Euclidean space. We study embeddings of simplicial complexes in Chapter 3. See Matoušek et al. [19] for a survey of embeddings of simplicial complexes. We show that every binary linear code has a geometric representation that can be embedded into $\mathbb{R}^4$. Moreover, we show that a binary linear code $\mathcal{C}$ has a geometric representation in $\mathbb{R}^3$ if and only if there exists a graph G such that $\mathcal{C}$ equals the cut space of G. This is a polynomially testable property and hence we can conclude that there is a polynomial algorithm that decides the minimal dimension of a geometric representation of a binary linear code. These results are presented in Chapter 3 and they were submitted for publication [26].

Finally, we generalize our results from binary linear codes to linear codes over primefields. We show that the linear codes over rationals and over $GF(p)$, where p is a prime, are triangular representable. In case of linear codes over $GF(p)$, we show that this representation determines the weight enumerator of $\mathcal{C}$. We present one application of this result to the partition function of the Potts model.

On the other hand, we show that there exist linear codes over any field different from rationals and $GF(p)$, p prime, that are not triangular representable. We show that every construction of a triangular representation for these codes fails on a very weak condition that a linear code and its triangular representation have to have the same dimension. These results are presented in Chapter 4 and they were submitted for publication [25].

1. Geometric representations of binary codes

1.1 Introduction

A seminal result of Galluccio and Loebl [7] asserts that the weight enumerator of the cut space $\mathcal{C}$ of a graph G may be written as a linear combination of $4^{g(G)}$ Pfaffians, where $g(G)$ is the minimal genus of a surface in which G can be embedded. Recently, a topological interpretation of this result was given by Cimasoni and Reshetikhin [2]. Viewing the cut space $\mathcal{C}$ as a binary linear code, a graph G may be considered as a useful geometric representation of C which provides an important structure for the weight enumerator of $\mathcal{C}$.

This motivated Martin Loebl to ask, about 15 years ago, the following question: Which binary codes are cycle spaces of simplicial complexes? In general, for the binary codes with a geometric representation, one may hope to obtain a formula analogous to that of Galluccio and Loebl [7]. This question remains open. We construct geometric representations which carry over only the weight enumerator. We note that this construction is still sufficient for the extension of the theory of Pfaffian orientations.

We present a construction which shows that a useful geometric representation exists for all binary codes. The first main result is as follows:

Theorem 1.1.1. *For each binary linear code $\mathcal{C}$ of length n, one can construct a triangular configuration Δ and a positive integer e linear in n, so that if the weight enumerator of the cycle space of Δ equals $\sum_{i=0}^{m} a_i x^i$ then the weight enumerator of $\mathcal{C}$ satisfies*

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{m} a_i x^{(i \bmod e)/2}.$$

The second main result of this chapter is to construct, for every triangular configuration Δ, a triangular configuration Δ' and a bijection between the cycle space of Δ and the set of the perfect matchings of Δ'. This carries over the second step in the Loebl's suggestion to extend the theory of Pfaffian orientations to the general binary linear codes.

1.2 Preliminaries

We begin with definitions of the basic concepts. Let n be a positive integer. A *binary linear code $\mathcal{C}$ of length n* is a subspace of $GF(2)^n$, and each vector in $\mathcal{C}$ is called a *codeword*. The *weight* of a codeword c is the number of non-zero coordinates, denoted by $w(c)$. A binary linear code $\mathcal{C}$ is *even* if all codewords have an even weight. We define a partial order on $\mathcal{C}$ as follows: Let $c = (c^1, \dots, c^n), d = (d^1, \dots, d^n)$ be codewords of $\mathcal{C}$. Then $c \preceq d$ if $c^i = 1$ implies $d^i = 1$ for all $i = 1, \dots, n$. A codeword d is *minimal* if $c \preceq d$ implies $c = d$ for all c. The *weight enumerator* of the code $\mathcal{C}$ is defined according to the formula

$$W_{\mathcal{C}}(x) := \sum_{c \in \mathcal{C}} x^{w(c)}.$$

An *abstract simplicial complex* on a finite set V is a family Δ of subsets of V closed under taking subsets. Let X be an element of Δ. The *dimension* of X is $|X|-1$, denoted by $\dim X$. The dimension of Δ is $\max\{\dim X | X \in \Delta\}$, denoted by $\dim \Delta$.

A *simplex* in $\mathbb{R}^n$ is the convex hull of an affine independent set V in $\mathbb{R}^d$. The *dimension* of the simplex is $|V|-1$. The convex hull of any non-empty subset of V that defines a simplex is called a *face* of the simplex. A *simplicial complex* Δ is a set of simplices fulfilling the following conditions:

- Every face of a simplex from Δ belongs to Δ.
- The intersection of every two simplices of Δ is a face of both.

We denote the subset of d-dimensional simplices of Δ by Δ^d. Every simplicial complex defines an abstract simplicial complex on the set of vertices V, namely the family of sets of vertices of simplexes of Δ. We denote this abstract simplicial complex by $\mathcal{A}(\Delta)$.

The *geometric realization* of an abstract simplicial complex Δ is a simplicial complex Δ' such that $\Delta = \mathcal{A}(\Delta')$. It is well known that every finite d-dimensional abstract simplicial complex can be realized as a simplicial complex in $\mathbb{R}^{2d+1}$. We choose a geometric realization of an abstract simplicial complex Δ and denote it by $\mathcal{G}(\Delta)$. This chapter studies 2-dimensional simplicial complexes where each maximal simplex is a triangle. We call them *triangular configurations.* The number of triangles in an (abstract) simplicial complex Δ is denoted by $|\Delta|$. A *subconfiguration* of a triangular configuration Δ is a triangular configuration Δ' such that $\Delta' \subseteq \Delta$. A *cycle* of a triangular configuration is a subconfiguration such that every edge is incident with an even number of triangles. A *circuit* is a minimal non-empty cycle under inclusion.

Let Δ_1, Δ_2 be subconfigurations of a triangular configuration Δ. The *difference* of Δ_1 and Δ_2, denoted by $\Delta_1 - \Delta_2$, is defined to be the triangular configuration obtained from $\Delta_1^0 \cup \Delta_1^1 \cup \Delta_1^2 \setminus \Delta_2^2$ by removing the edges and vertices that are not contained in any triangle in $\Delta_1^2 \setminus \Delta_2^2$. The *symmetric difference* of Δ_1 and Δ_2, denoted by $\Delta_1 \triangle \Delta_2$, is defined to be $\Delta_1 \triangle \Delta_2 := (\Delta_1 \cup \Delta_2) - (\Delta_1 \cap \Delta_2)$. Let Δ_1, Δ_2 be triangular configurations. The union of Δ_1, Δ_2 is defined to be $\Delta_1 \cup \Delta_2 := \mathcal{G}(\mathcal{A}(\Delta_1) \cup \mathcal{A}(\Delta_1))$.

Let Δ be a d-dimensional simplicial complex. We define the *incidence matrix* $A = (A_{ij})$ as follows: the rows are indexed by $(d-1)$-dimensional simplices and the columns by d-dimensional simplices. We set

$$a_{ij} := \begin{cases} 1 & \text{if } (d-1)\text{-simplex } i \text{ belongs to } d\text{-simplex } j, \\ 0 & \text{otherwise.} \end{cases}$$

The *cycle space* $\mathcal{C}$ of Δ is the kernel $\ker \Delta$ of the incidence matrix of Δ over $GF(2)$, and $\mathcal{C} = \ker \Delta$ is said to be *represented* by Δ. For a subconfiguration C of Δ, we let $\chi(C) = (\chi(C)^{t_1}, \dots, \chi(C)^{t_{|\Delta|}}) \in \{0,1\}^{|\Delta|}$ denote its *incidence vector*, where $\chi(C)^t = 1$ if C contains the triangle t, and $\chi(C)^t = 0$ otherwise. It is well known that the kernel of Δ is the set of incidence vectors of cycles of Δ. Let $\mathcal{C} \subseteq \{0,1\}^n$ be a binary linear code and let S be a subset of $\{1, \dots, n\}$. *Puncturing* a code $\mathcal{C}$ along S means deleting the entries indexed by the elements of S from each codeword of $\mathcal{C}$. The resulting code is denoted by $\mathcal{C}/S$.

1.3 Triangular representation of binary codes

First, we define three basic triangular configurations.

1.3.1 Triangular configuration B^n

The triangular configuration B^n consists of n disjoint triangles as is depicted in Figure 1.1. We denote the triangles of B^n by $B_1^n, \ldots, B_n^n$.

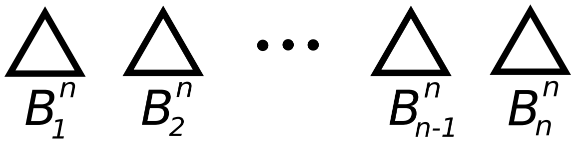

Figure 1.1: Triangular configuration B^n.

1.3.2 Triangular sphere $\mathcal{S}^m$

The triangular sphere $\mathcal{S}^m$, depicted in Figure 1.2, is a triangulation of a 2-dimensional sphere by m triangles. This triangulation exists for every even $m \geq 4$. We denote the triangles of $\mathcal{S}^m$ by $\mathcal{S}_1^m, \ldots, \mathcal{S}_m^m$

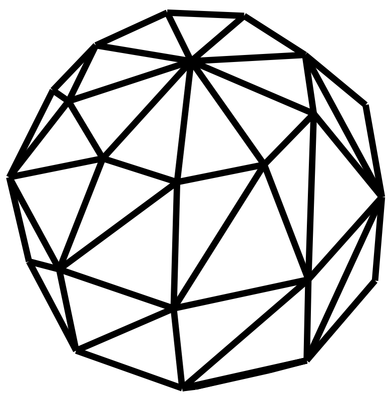

Figure 1.2: Triangular sphere $\mathcal{S}^m$.

1.3.3 Triangular tunnel T

The triangular tunnel T is depicted in Figure 1.3. In particular, triangles $\{1, 2, 3\}$ and $\{a, b, c\}$ are not elements of T.

1.3.4 Joining triangles by tunnels

Let Δ be a triangular configuration. Let t_1 and $t_2 \in \Delta$ be two disjoint triangles of Δ. The join of t_1 and t_2 in Δ is the triangular configuration Δ' defined as follows. Let T be a triangular tunnel as in Figure 1.3. Let t_1^1, t_1^2, t_1^3 and t_2^1, t_2^2, t_2^3 be edges of t_1 and t_2, respectively. We relabel edges of T such that $\{a, b, c\} = \{t_1^1, t_1^2, t_1^3\}$ and $\{1, 2, 3\} = \{t_2^1, t_2^2, t_2^3\}$. Then Δ' is defined to be $\Delta \cup T$.

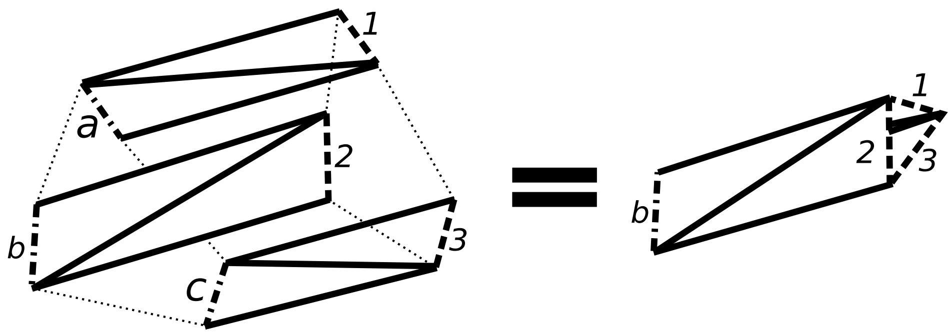

Figure 1.3: Triangular tunnel T.

1.3.5 Construction

Let $\mathcal{C}$ be a binary code of length n and dimension d. Let $B = \{b_1, \ldots, b_d\}$ be a basis of $\mathcal{C}$. We construct its triangular representation $\Delta_B^{\mathcal{C}}$ as follows. For every basis vector b_i we construct a triangular configuration $\Delta_{b_i}^{\mathcal{C}}$. The triangular configuration $\Delta_{b_i}^{\mathcal{C}}$ is obtained from $B^n \cup \mathcal{S}^m$, where m is even and $m \geq n$, $m \geq 4$. Let J^i be the set of indices of non-zero entries of b_i. For each $j \in J^i$ we join the triangle $\mathcal{S}_j^m$ of $\mathcal{S}^m$ with the triangle B_j^n. Then we remove the triangle $\mathcal{S}_j^m$ from $\mathcal{S}^m$. Finally, we remove the triangles of B^n that are not joined with the sphere. An example of $\Delta_{b_i}^{\mathcal{C}}$ for $b_i = (1, 0, \ldots, 1, 0)$ is depicted in Figure 1.4. Thus, the

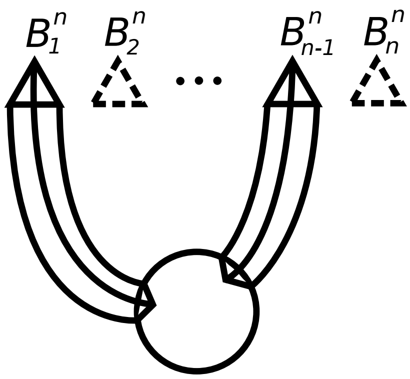

Figure 1.4: $\Delta_{b_i}^{\mathcal{C}}$ represents a basis vector $(1, 0, \ldots, 1, 0)$ of $\mathcal{C}$.

triangular configuration $\Delta_{b_i}^{\mathcal{C}}$ contains B_j^n if and only if $j \in J^i$. We note that

Proposition 1.3.1. *The number* $|\Delta_{b_i}^{\mathcal{C}}|$ *is always even.* □

Triangular configurations $\Delta_{b_i}^{\mathcal{C}}$, $i = 1, \ldots, d$, share triangles of B^n and do not share spheres $\mathcal{S}^m$. Hence, $\mathcal{A}(\Delta_{b_i}^{\mathcal{C}}) \cap \mathcal{A}(\Delta_{b_j}^{\mathcal{C}}) \subseteq \mathcal{A}(B_n)$ holds for $i < j$, $i, j \in \{1, \ldots, d\}$.

Finally, the triangular representation $\Delta_B^{\mathcal{C}}$ of $\mathcal{C}$ is the union of $\Delta_{b_i}^{\mathcal{C}}$, $i = 1, \ldots, d$. An example of a triangular representation $\Delta_B^{\mathcal{C}}$ of $\mathcal{C}$ is depicted in Figure 1.5. A triangular representation $\Delta_B^{\mathcal{C}}$ of $\mathcal{C}$ is *balanced* if there is an integer e such that $\left|\Delta_{b_i}^{\mathcal{C}}\right| - w(b_i) = e$ for all $i = 1, \ldots, d$. This e is denoted by $e(\Delta_B^{\mathcal{C}})$. We denote the addition modulo 2 by $+^2$ or $\sum^2$. Let c be a codeword of $\mathcal{C}$ and let $c = \sum^2_{i \in I} b_i$

be the unique expression of c, where $b_i \in B$. The *degree* of c with respect to a basis B is defined to be the cardinality $|I|$ of the index set. The degree is denoted by $d(c)$.

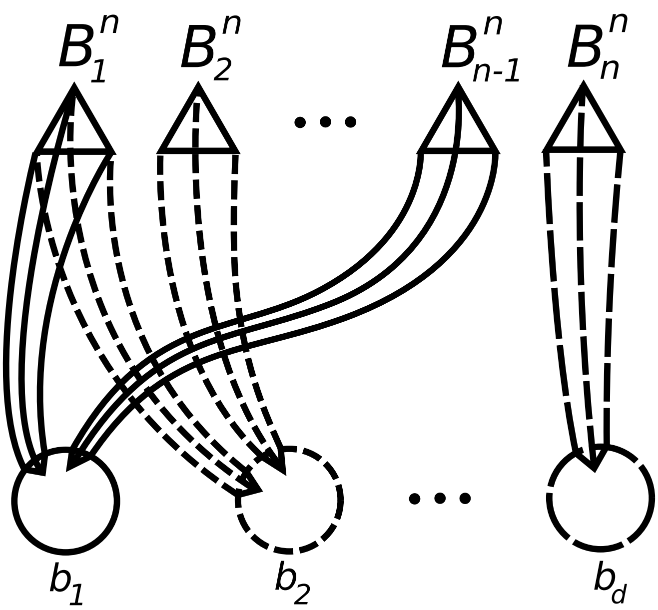

Figure 1.5: An example of triangular representation $\Delta_B^{\mathcal{C}}$ of $\mathcal{C}$.

We denote by $\ker \Delta_B^{\mathcal{C}}$ the cycle space of the triangular configuration $\Delta_B^{\mathcal{C}}$. We define a linear mapping $f\colon \mathcal{C} \mapsto \ker \Delta_B^{\mathcal{C}}$ in the following way: Let c be a codeword of $\mathcal{C}$ and let $c = \sum^2_{i\in I} b_i$ be the unique expression of c, where $b_i \in B$. We define $f(c) := \chi(\triangle_{i\in I}\Delta_{b_i}^{\mathcal{C}})$. The entries of $f(c)$ are indexed by the triangles of $\Delta_B^{\mathcal{C}}$. We have $f(c)^{B_j^n} = 1$ if and only if $\triangle_{i\in I}\Delta_{b_i}^{\mathcal{C}}$ contains the triangle B_j^n.

Proposition 1.3.2. *Denote* $\left|\Delta_B^{\mathcal{C}}\right|$ *by* m*. Let* $c = (c^1, \ldots, c^n)$ *and*

$$f(c) = \left(f(c)^{B_1^n}, \ldots, f(c)^{B_n^n}, f(c)^{n+1}, \ldots, f(c)^m\right).$$

Then $f(c)^{B_j^n} = c^j$ *for all* $j = 1, \ldots, n$ *and all* $c \in \mathcal{C}$*.*

Proof. We show the proposition by induction on the degree $d(c)$ of c. The codeword c is equal to $\sum^2_{i\in I} b_i$. If $d(c) = 0$, then $c = 0$ and $f(c) = 0$. Thus, $f(c)$ is the incidence vector of the empty triangular configuration. Hence, the proposition holds for vectors of degree 0. If $d(c)$ is greater than 0, then $|I| \geq 1$. We choose some k from I. The codeword $c +^2 b_k$ has a degree less than c. By the induction assumption, the proposition holds for $c +^2 b_k$. Let $b_k = (b_k^1, \ldots, b_k^n)$. From the definition of $\Delta_{b_k}^{\mathcal{C}}$, the equality $b_k^j = \chi(\Delta_{b_k}^{\mathcal{C}})^{B_j^n}$ holds for all $j = 1, \ldots, n$. Therefore,

$$c^j = (c^j +^2 b_k^j) +^2 b_k^j = \chi(\triangle_{i\in I\setminus\{k\}}\Delta_{b_i}^{\mathcal{C}})^{B_j^n} +^2 \chi(\Delta_{b_k}^{\mathcal{C}})^{B_j^n} = f(c)^{B_j^n}$$

for all $j = 1, \ldots, n$. □

Corollary 1.3.1. *The mapping* f *is injective.* □

Lemma 1.3.1. *Every non-empty cycle of $\Delta_B^{\mathcal{C}}$ contains $\Delta_{b_i}^{\mathcal{C}} - B^n$ as a subconfiguration for some $i \in \{1, \ldots, d\}$.*

Proof. Every cycle of $\Delta_B^{\mathcal{C}}$ contains either all triangles or no triangle of $\Delta_{b_i}^{\mathcal{C}} - B^n$, since $\Delta_{b_i}^{\mathcal{C}} \cap \Delta_{b_j}^{\mathcal{C}} \subseteq B^n$ for all distinct $i, j \in \{1, \ldots, d\}$. The configuration B^n does not contain non-empty cycles, since the triangles of B^n are disjoint. Therefore, every non-empty cycle contains a triangle of $\Delta_{b_i}^{\mathcal{C}} - B^n$ for some $i \in \{1, \ldots, d\}$. Hence, every non-empty cycle contains $\Delta_{b_i}^{\mathcal{C}} - B^n$ for some $i \in \{1, \ldots, d\}$. ☐

Theorem 1.3.1 (Rytíř [23])**.** *Let $\mathcal{C}$ be a binary code and let $\Delta_B^{\mathcal{C}}$ be its triangular representation with respect to a basis B. The mapping f defined above is a bijection of the binary linear codes $\mathcal{C}$ and $\ker \Delta_B^{\mathcal{C}}$ which maps minimal codewords to minimal codewords.*

Proof. By Corollary 1.3.1, the mapping f is injective. It remains to be proven that $\dim \mathcal{C} = \dim \ker \Delta_B^{\mathcal{C}}$. Suppose on the contrary that some codeword of $\ker \Delta_B^{\mathcal{C}}$ is not in the span of $\{f(b_1), \ldots, f(b_d)\}$. Let c be such a codeword with the minimal possible weight $w(c)$. Let K be a cycle of $\Delta_B^{\mathcal{C}}$ such that $\chi(K) = c$. By Lemma 1.3.1, the cycle K contains $\Delta_{b_i}^{\mathcal{C}} - B^n$ for some $i \in \{1, \ldots, d\}$. Since $\left|\Delta_{b_i}^{\mathcal{C}} - B^n\right| > |B^n|$, the inequality $\left|K \bigtriangleup \Delta_{b_i}^{\mathcal{C}}\right| < |K|$ holds. Therefore, $w(c) > w(\chi(K \bigtriangleup \Delta_{b_i}^{\mathcal{C}}))$. This is a contradiction.

Finally, we show that f maps minimal codewords to minimal codewords. Let d be a minimal codeword. Suppose on the contrary that $f(d)$ is not a minimal codeword of $\ker \Delta_B^{\mathcal{C}}$. Then $f(c) \prec f(d)$ for some codeword c. However, $c^i = f(c)^i = 1$ implies that $d^i = f(d)^i = 1$. Therefore, $c \prec d$. This contradicts the minimality of d. ☐

Let t be a triangle of a triangular configuration Δ. The *subdivision* of the triangle t is the triangular configuration obtained from Δ by exchanging the triangle t by triangles t_1, t_2, t_3 in the way depicted in Figure 1.6.

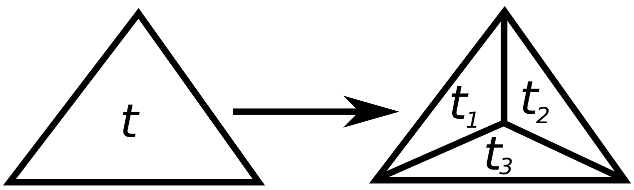

Figure 1.6: Triangle subdivision

Proposition 1.3.3. *Every even binary code $\mathcal{C}$ of length n and dimension d has a balanced triangular representation $\Delta_B^{\mathcal{C}}$ such that $e(\Delta_B^{\mathcal{C}}) > n$, where B is an arbitrary basis of $\mathcal{C}$.*

Proof. Let $\Delta_B^{\mathcal{C}}$ be an arbitrary triangular representation of $\mathcal{C}$ with respect to a basis $B = \{b_1, \ldots, b_d\}$. We denote by k_i the number $\left|\Delta_{b_i}^{\mathcal{C}}\right| - w(b_i)$. By Proposition 1.3.1, $\left|\Delta_{b_i}^{\mathcal{C}}\right|$ is even. Since $\mathcal{C}$ is an even binary code, every k_i is even. Let n' be the smallest even number greater than n and let k denote $\max\{n', k_i | i = 1, \ldots, d\}$. For each $i \in \{1, \ldots, d\}$ such that $k_i \neq k$, the following step is applied. We choose a triangle t from $\Delta_{b_i}^{\mathcal{C}} - B^n$ and subdivide it. The number k_i is increased by 2. If k_i still does not equal to k, then we repeat this step. After this procedure, the configuration $\Delta_B^{\mathcal{C}}$ is balanced and $e(\Delta_B^{\mathcal{C}}) > n$. ☐

Proposition 1.3.4. *Let $\mathcal{C}$ be an even binary linear code and let $\Delta_B^{\mathcal{C}}$ be its balanced triangular representation with respect to a basis B. Then $w(f(c)) = w(c) + d(c)e(\Delta_B^{\mathcal{C}})$ for every codeword $c \in \mathcal{C}$.*

Proof. Write c as $\sum^2_{i\in I} b_i$, where $b_i \in B$. Then $f(c) = \chi(\triangle_{i\in I}\Delta_{b_i}^{\mathcal{C}})$. Now, the configuration $\triangle_{i\in I}\Delta_{b_i}^{\mathcal{C}}$ contains all triangles of $\Delta_{b_i}^{\mathcal{C}} - B^n$ for all $i \in I$. The number of these triangles is $d(c)e(\Delta_B^{\mathcal{C}})$, since $\left|\Delta_{b_i}^{\mathcal{C}} - B^n\right| = e(\Delta_B^{\mathcal{C}})$ and $|I| = d(c)$. By Proposition 1.3.2, the configuration $\triangle_{i\in I}\Delta_{b_i}^{\mathcal{C}}$ contains the triangle B_k^n if and only if $c_k = 1$. The number of these triangles is $w(c)$. Therefore, $w(f(c)) = w(c) + d(c)e(\Delta_B^{\mathcal{C}})$. □

1.4 Weight enumerator

In this section, we state the connection between the weight enumerator of a code and the weight enumerator of its triangular representation. This provides a proof of Theorem 1.1.1. The *double code*, denoted by $\mathcal{C}^2$, of a binary linear code $\mathcal{C}$ of length n is the code $\{(c_1, \ldots, c_n, c_1, \ldots, c_n) : c \in C\}$.

Proposition 1.4.1. *Let $\mathcal{C}$ be a binary linear code and let $\mathcal{C}^2$ be the double code of $\mathcal{C}$. Then $\mathcal{C}^2$ is an even binary linear code and*

$$W_{\mathcal{C}}(x^2) = W_{\mathcal{C}^2}(x).$$

□

We define the *extended weight enumerator* (with respect to a fixed basis) by

$$W_{\mathcal{C}}^k(x) := \sum_{\substack{c\in\mathcal{C}\\ d(c)=k}} x^{w(c)}.$$

If a code $\mathcal{C}$ has dimension d, then

$$W_{\mathcal{C}}(x) = \sum_{k=0}^{d} W_{\mathcal{C}}^k(x).$$

Proposition 1.4.2 (Rytíř [23])**.** *Let $\mathcal{C}$ be an even binary code and let $\Delta_B^{\mathcal{C}}$ be its balanced triangular representation $\Delta_B^{\mathcal{C}}$ with respect to the fixed basis B. Then*

$$W^k_{\ker \Delta_B^{\mathcal{C}}}(x) = W_{\mathcal{C}}^k(x) x^{ke(\Delta_B^{\mathcal{C}})}.$$

Proof. Let f be the mapping defined in Section 1.3. For every codeword c of degree k of $\mathcal{C}$ there is codeword $f(c)$ of degree k of $\ker \Delta_B^{\mathcal{C}}$. By Proposition 1.3.4, $w(f(c)) = w(c) + ke(\Delta_B^{\mathcal{C}})$. Therefore,

$$W^k_{\ker \Delta_B^{\mathcal{C}}}(x) = \sum_{\substack{f(c)\in\ker \Delta_B^{\mathcal{C}}\\ d(f(c))=k}} x^{w(f(c))} = \sum_{\substack{c\in\mathcal{C}\\ d(c)=k}} x^{w(c)+ke(\Delta_B^{\mathcal{C}})} = W_{\mathcal{C}}^k(x) x^{ke(\Delta_B^{\mathcal{C}})}.$$

□

Proposition 1.4.3. *Let $\mathcal{C}$ be an even binary code of length n and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular representation of $\mathcal{C}$. The inequality $ke(\Delta_B^{\mathcal{C}}) \leq w(c) \leq ke(\Delta_B^{\mathcal{C}}) + n$ holds for every codeword c of degree k of $\ker \Delta_B^{\mathcal{C}}$.*

Proof. By Proposition 1.3.4, $w(c) = w(f^{-1}(c)) + ke(\Delta_B^{\mathcal{C}})$. Since $0 \leq w(f^{-1}(c)) \leq n$ for every $c \in \ker \Delta_B^{\mathcal{C}}$, the inequality $ke(\Delta_B^{\mathcal{C}}) \leq w(c) \leq ke(\Delta_B^{\mathcal{C}}) + n$ holds. □

Corollary 1.4.1. *Let $\mathcal{C}$ be an even binary code of dimension d and length n and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular representation of $\mathcal{C}$ such that $n < e(\Delta_B^{\mathcal{C}})$. Denote $e(\Delta_B^{\mathcal{C}})$ by e. Let $\sum_{i=0}^{de+n} a_i x^i$ be the weight enumerator of $\ker \Delta_B^{\mathcal{C}}$. Then*

$$W^k_{\ker \Delta_B^{\mathcal{C}}}(x) = \sum_{i=ke}^{ke+n} a_i x^i.$$

Proof. By Proposition 1.4.3, $w(c) \leq (k-1)e + n$ for all codewords $c \in \ker \Delta_B^{\mathcal{C}}$ of a degree less than k. Since $n < e$, the inequality $w(c) \leq ke - e + n < ke$ holds. By Proposition 1.4.3, $(j+1)e \leq w(c)$ for all codewords $c \in \ker \Delta_B^{\mathcal{C}}$ of a degree greater than k. Since $n < e$, the inequality $ke + e < ke + n \leq w(c)$ holds. Hence, the enumerator $W^k_{\ker \Delta_B^{\mathcal{C}}}(x)$ is the sum over all codewords of a weight between ke and $ke + n$. □

Theorem 1.4.1. *Let $\mathcal{C}$ be an even binary code of dimension d and length n and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular representation of $\mathcal{C}$ such that $n < e(\Delta_B^{\mathcal{C}})$. Denote $e(\Delta_B^{\mathcal{C}})$ by e. Let $\sum_{i=0}^{de+n} a_i x^i$ be the weight polynomial of $\ker \Delta_B^{\mathcal{C}}$. Then*

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{de+n} a_i x^{i \bmod e}.$$

Proof. The inequality $w(c) \leq n$ holds for every codeword $c \in \mathcal{C}$. Let f be the mapping defined in Section 1.3. By Proposition 1.3.4, $w(f(c)) = w(c) + d(c)e$ for every codeword c of $\mathcal{C}$. Since $n < e$, the following equality holds.

$$w(f(c)) \bmod e = (w(c) + d(c)e) \bmod e = w(c).$$

Hence,

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{de+n} a_i x^{i \bmod e}.$$

□

Now, we prove Theorem 1.1.1.

Proof. Proof of Theorem 1.1.1. Let $\mathcal{C}$ be a linear binary code of length n and dimension d. Let $\mathcal{C}^2$ be the double code of $\mathcal{C}$ of length $2n$ and dimension d. The double code $\mathcal{C}^2$ is even. By Proposition 1.3.3, we can construct a balanced triangular representation Δ of $\mathcal{C}^2$ such that $e(\Delta) > 2n$. Denote $e(\Delta)$ by e. Let $W_\Delta(x) = \sum_{i=0}^{de+2n} a_i x^i$ be the weight enumerator of Δ. By Theorem 1.4.1, the following equality holds.

$$W_{\mathcal{C}^2}(x) = \sum_{i=0}^{de+2n} a_i x^{i \bmod e}.$$

By Proposition 1.4.1, we have

$$W_{\mathcal{C}}(x) = W_{\mathcal{C}^2}(x^{1/2}) = \sum_{i=0}^{de+2n} a_i x^{(i \bmod e)/2}.$$

□

1.5 Matching

In this section we reduce the computation of the weight enumerator of the even subconfigurations to the computation of the weight enumerator of the perfect matchings.

Let Δ be a triangular configuration. A *matching* of Δ is a subconfiguration M of Δ such that $t_1 \cap t_2$ does not contain an edge for every distinct $t_1, t_2 \in T(M)$. Let Δ be a triangular configuration. Let M be a matching of Δ. Then the *defect* of M is the set $E(T) \setminus E(M)$. We denote the matching with this defect by $M_{E(T)\setminus E(M)}$. The *perfect matching* of Δ is a matching with empty defect. We denote the set of all perfect matchings of Δ by $\mathcal{P}(\Delta)$. The weight enumerator of perfect matchings in Δ is defined to be $P_\Delta(x) = \sum_{P\in\mathcal{P}(\Delta)} x^{w(P)}$, where $w(P) := \sum_{t\in P} w_t$.

Now, we define some basic triangular configurations.

1.5.1 Triangular configuration P

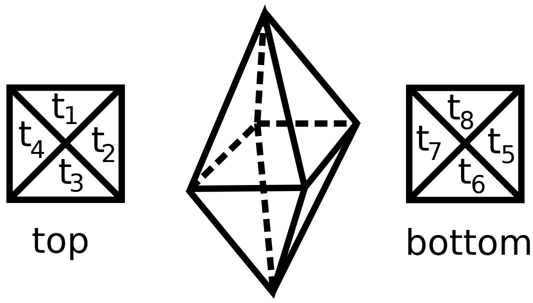

Figure 1.7: Triangular configuration P

The triangular configuration P is depicted in Figure 1.7.

Proposition 1.5.1. *The triangular configuration P has exactly two perfect matchings $\{t_1, t_3, t_5, t_7\}$, $\{t_2, t_4, t_6, t_8\}$.* □

1.5.2 Closed triangular tunnel T

The closed triangular tunnel T is depicted in Figure 1.8. We call triangles $\{a, b, c\} = t_2$ and $\{1, 2, 3\} = t_1$ ending triangles.

Proposition 1.5.2. *A closed triangular tunnel T has two perfect matchings $M^T_{t_1} = \{t_1, s_4, s_5, s_6\}$, $M^T_{t_2} = \{t_2, s_1, s_2, s_3\}$.* □

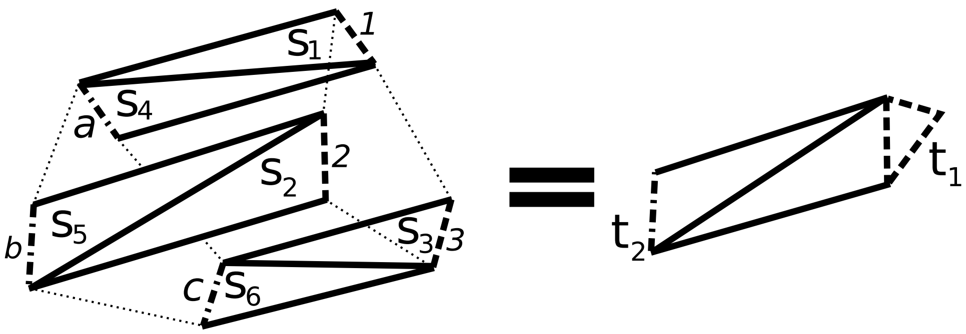

Figure 1.8: Closed triangular tunnel T.

1.5.3 Triangular configuration E_{pq}

The *matching triangular edge* is the triangular configuration which is obtained from the triangular configuration P and two closed triangular tunnels T in the following way: Let T_1 and T_2 be closed triangular tunnels. Let $t_1^{T_1}, p^{T_1}$ and $t_1^{T_2}, q^{T_2}$ be the ending triangles of T_1 and T_2, respectively. We identify $t_1^{T_1}$ with t_1^P and $t_1^{T_2}$ with t_3^P. The configuration E_{pq} is defined to be $T_1 \triangle P \triangle T_2$. The triangular configuration E_{pq} is depicted in Figure 1.9.

Proposition 1.5.3. *A matching triangular edge has two perfect matchings.*

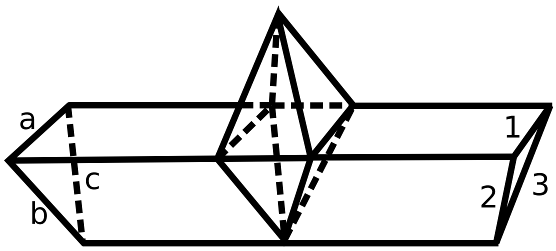

Figure 1.9: Matching triangular edge

Proof. There are two matchings. The first matching is $N^0_{pq} := M^{T_1}_{t_1} \cup M^{T_2}_{t_1} \cup \{t_5^P, t_7^P\}$. The second matching is $N^1_{pq} := M^{T_1}_p \cup M^{T_2}_q \cup \{t_2^P, t_4^P, t_6^P, t_8^P\}$.

Any perfect matching of E_{pq} contains $\{t_5^P, t_7^P\}$ or $\{t_2^P, t_4^P, t_6^P, t_8^P\}$. This determines remaining triangles in a perfect matching. Hence, there are just two perfect matchings. □

We denote the matching N^1_{pq} by M^1_{pq} and the matching $N^0_{pq} \setminus p, q$ by M^0_{pq}.

1.5.4 Triangular configuration T_{pqr}

The *matching triangular triangle* is the triangular configuration which is obtained from the triangular configuration P and three closed triangular tunnels T in the following way: Let T_1, T_2 and T_3 be closed triangular tunnels. Let $t_1^{T_1}, p^{T_1}$; $t_1^{T_2}, q^{T_2}$ and $t_1^{T_3}, r^{T_3}$ be the ending triangles of T_1, T_2 and T_3, respectively. We identify $t_1^{T_1}$ with t_1^P; $t_1^{T_2}$ with t_3^P and $t_1^{T_3}$ with t_5^P. The configuration T_{pqr} is defined to be $T_1 \triangle P \triangle T_2 \triangle T_3$. The triangular configuration T_{pqr} is depicted in Figure 1.10.

Proposition 1.5.4. *A matching triangular triangle has two perfect matchings.*

Proof. There are two matchings. The first matching is $N^0_{pqr} := M^{T_1}_{t_1} \cup M^{T_2}_{t_1} \cup M^{T_3}_{t_1} \cup \{t^P_7\}$. The second matching is $N^1_{pqr} := M^{T_1}_p \cup M^{T_2}_q \cup M^{T_3}_r \cup \{t^P_2, t^P_4, t^P_6, t^P_8\}$.

Any perfect matching of T_{pqr} contains $\{t^P_5, t^P_7\}$ or $\{t^P_2, t^P_4, t^P_6, t^P_8\}$. This determines remaining triangles in a perfect matching. Hence, there are just two perfect matchings. □

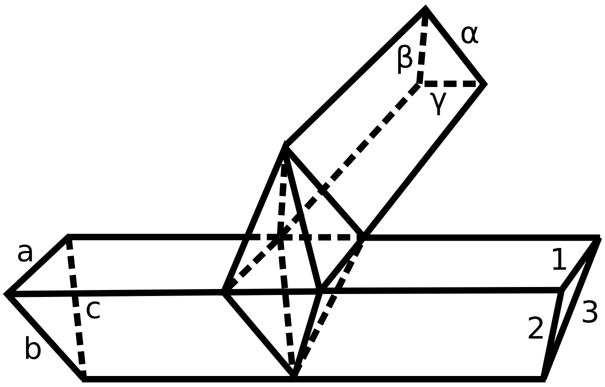

Figure 1.10: Matching triangle

We denote the matching N^1_{pqr} by M^1_{pqr} and the matching $N^0_{pqr} \setminus p, q, r$ by M^0_{pqr}.

1.5.5 Triangular configuration $C_{t_1 t_2 \ldots t_n}$

This part of the reduction is analogous to the reduction for graphs described in Galluccio et al. [9]. Let t_1, t'_1 be empty disjoint triangles. Let $t_2, \ldots, t_n, t'_2, \ldots, t'_n$ be disjoint triangles. Then $C_{t_1 t_2 \ldots t_n}$ is defined to be

$$(\triangle^n_{i=1} t_i) \triangle (\triangle^n_{i=1} t'_i) \triangle \left(\triangle^n_{i=1} E_{t_i t'_i}\right) \triangle \left(\triangle^n_{i=2} E_{t_i t'_{i-1}}\right) \triangle \left(\triangle^{n-1}_{i=1} E_{t'_i t'_{i+1}}\right).$$

The configuration is depicted in Figure 1.11.

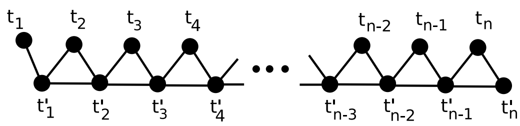

Figure 1.11: Triangular configuration $C_{t_1 t_2 \ldots t_n}$

Proposition 1.5.5. *Let M^I_C denote the perfect matching containing triangles $t_i, i \in I$. Then there exists exactly one perfect matching M^I_C of $C_{t_1 t_2 \ldots t_n}$ if and only if $|I|$ is even.*

Proof. We construct the perfect matching M by the following algorithm. The first step is defined as follows. If $t_1 \in I$ then we set M_1 to $M^0_{t_1 t'_1} \cup \{t_1\}$ otherwise we set M_1 to $M^1_{t_1 t'_1}$.

Let $i \geq 2$. In the i-th step, we extend the matching M_{i-1} in the following way.

(a) If t'_{i-1} is covered by M_{i-1} and $t_i \in I$ then $M_i := M_{i-1} \cup M^0_{t'_{i-1}t_i} \cup \{t_i\} \cup M^0_{t_i t'_i} \cup M^0_{t'_{i-1}t'_i}$.

(b) If t'_{i-1} is not covered by M_{i-1} and $t_i \in I$ then $M_i := M_{i-1} \cup M^0_{t'_{i-1}t_i} \cup \{t_i\} \cup M^0_{t_i t'_i} \cup M^1_{t'_{i-1}t'_i}$.

(c) If t'_{i-1} is covered by M_{i-1} and $t_i \notin I$ then $M_i := M_{i-1} \cup M^0_{t'_{i-1}t_i} \cup M^1_{t_i t'_i} \cup M^0_{t'_{i-1}t'_i}$.

(d) If t'_{i-1} is not covered by M_{i-1} and $t_i \notin I$ then $M_i := M_{i-1} \cup M^1_{t'_{i-1}t_i} \cup M^0_{t_i t'_i} \cup M^0_{t'_{i-1}t'_i}$.

Let $i \geq 1$. We say that the i-th step is even if t'_i is covered by M_i otherwise it is odd. Every step is determined by the previous steps and the set I. Therefore, the perfect matching exists if and only if the algorithm succeeds. The algorithm succeeds if and only if the last step is even. The parity of the i-th step is different from the previous step if $t_i \in I$. Hence, the algorithm succeeds if and only if the cardinality $|I|$ is even. The desired matching M is M_n. □

1.5.6 Reduction

Let Δ be a triangular configuration. We construct the triangular configuration Δ' such that every even subconfiguration of Δ uniquely corresponds to one perfect matching of Δ' and a natural weight-preserving bijection between the set of the even subconfiguration of Δ and the set of the perfect matchings of Δ'. We put into Δ' empty disjoint triangles t_e for every tuple (t, e), where $e \in E(\Delta)$ and $t \in T(\Delta)$. We add to Δ' matching triangles $T_{t_a t_b t_c}$ for every triangle $t \in T(\Delta)$, where a, b, c are edges of t. We assign weight 1 to one arbitrary triangle in the matching M^1_t and weight 0 to all remaining triangles of $T_{t_a t_b t_c}$. We add to Δ' triangular configurations $C_{t^1_e \ldots t^n_e}$ for every edge $e \in E(\Delta)$, where $t^1_e, \ldots, t^n_e$ are triangles incident with e in Δ. We assign weight 0 to all triangles of $C_{t^1_e \ldots t^n_e}$.

Theorem 1.5.1. *Let Δ be a triangular configuration and let Δ' be a matching reduction of Δ and let C be an even subconfiguration of Δ. Then there exists exactly one perfect matching M_C in Δ', and Δ' does not contain any others perfect matchings.*

Proof. Let C be an even subconfiguration of Δ. We construct a perfect matching M_C in Δ'. We denote matchings $M^1_{t_a t_b t_c}$ and $M^0_{t_a t_b t_c}$ of $T_{t_a t_b t_c}$ by M^1_t and M^0_t, respectively. We denote the set $\{i | e \in T(t_i), t_i \in C\}$ by I_e and define

$$M_C := \{M^1_t | t \in C\} \cup \{M^0_t | t \notin C, t \in T(\Delta)\} \cup \{M^{I_e}_e | e \in E(\Delta)\}.$$

The matching M_C is perfect.

We show that there is no other perfect matching. Every matching triangle T_t is covered by M^1_t or M^0_t. Thus C_e is covered by M^I_e for some even I. Therefore, every perfect matching in Δ' defines an even subset in Δ. □

Proposition 1.5.6. *Let Δ be a triangular configuration and let Δ' be its matching representation and let C be an even subconfiguration and let M_C be the corresponding perfect matching. Then $|C| = w(M_C)$.*

Proof.

$$\begin{aligned} w(M_C) &= \sum_{t\in C} w(M_t^1) + \sum_{t\notin C, t\in T(\Delta)} w(M_t^0) + \sum_{e\in E(\Delta)} w(M_e^{\{i|e\in T(t_i), t_i\in C\}}) \\ &= \sum_{t\in C} 1 + \sum_{t\notin C, t\in T(\Delta)} 0 + \sum_{e\in E(\Delta)} 0 \\ &= |C| \end{aligned}$$

□

The following theorem is a consequence of Proposition 1.5.6.

Theorem 1.5.2. *Let Δ be a triangular configuration and let Δ' be its matching representation. Then $W_\Delta(x) = P_{\Delta'}(x)$.* □

2. Kasteleyn 3-matrices

2.1 Introduction

The *Kasteleyn method* is a way how to calculate the Ising partition function on a finite graph G. It goes as follows. We first realize that the Ising partition function is equivalent to a multivariable weight enumerator of the cut space of G. We modify G to graph G' so that this weight enumerator is equal to the generating function of the perfect matchings of G', perhaps better known as the dimer partition function on G'. Such generating functions are hard to calculate. In particular, if G' is bipartite then the generating function of the perfect matchings of G' is equal to the permanent of the biadjacency matrix of G'. If however this permanent may be turned into the determinant of a modified matrix then the calculation can be successfully carried over since the determinants may be calculated efficiently. Already in 1913 Polya [21] asked for which non-negative matrix M we can change signs of its entries so that, denoting by M' the resulting matrix, we have $\mathrm{per}(M) = \det(M')$. We call these matrices *Kasteleyn* after the physicist Kasteleyn who invented the Kasteleyn method [12, 13, 14, 6]. Kasteleyn [14] proved in 1960's that all biadjacency matrices of the planar bipartite graphs are Kasteleyn. We say that a bipartite graph is Pfaffian if its biadjacency matrix is Kasteleyn. The problem to characterize the Kasteleyn matrices (or equivalently Pfaffian bipartite graphs) was open until 1993, when Robertson, Seymour and Thomas [22] found a polynomial recognition method and a structural description of the Kasteleyn matrices. They showed that the class of the Kasteleyn matrices is rather restricted and extends only moderately beyond the biadjacency matrices of the planar bipartite graphs.

In this chapter we carry out the Kasteleyn method for general binary linear codes. We show that the weight enumerator of any binary linear code is polynomial reducible to the permanent of the triadjacency 3-matrix of a 2-dimensional simplicial complex. In analogy to the standard (2-dimensional) matrices we say that a 3-dimensional non-negative matrix A is *Kasteleyn* if signs of its entries may be changed so that, denoting by A' the resulting 3-dimensional matrix, we have $\mathrm{per}(A) = \det(A')$. We show that in contrast with the 2-dimensional case the class of Kasteleyn 3-dimensional matrices is rich; namely, for each 2-dimensional non-negative matrix M there is a 3-dimensional non-negative Kasteleyn matrix A so that $\mathrm{per}(M) = \mathrm{per}(A)$. Finally we conclude with some remarks directed towards possible application for the 3-dimensional Ising and dimer problems.

2.1.1 Basic definitions

We start with basic definitions. A linear code $\mathcal{C}$ of length n and dimension d over a field $\mathbb{F}$ is a linear subspace with dimension d of the vector space $\mathbb{F}^n$. Each vector in $\mathcal{C}$ is called a codeword. The weight $w(c)$ of a codeword c is the number of non-zero entries of c. The weight enumerator of a finite code $\mathcal{C}$ is defined according to the formula

$$W_{\mathcal{C}}(x) := \sum_{c \in \mathcal{C}} x^{w(c)}.$$

A simplex X is the convex hull of an affine independent set V in $\mathbb{R}^d$. The dimension of X is $|V|-1$, denoted by $\dim X$. The convex hull of any non-empty subset of V that defines a simplex is called a face of the simplex. A simplicial complex Δ is a set of simplices fulfilling the following conditions: Every face of a simplex from Δ belongs to Δ and the intersection of every two simplices of Δ is a face of both. The dimension of Δ is $\max\{\dim X | X \in \Delta\}$. Let Δ be a d-dimensional simplicial complex. We define the incidence matrix $A = (A_{ij})$ as follows: The rows are indexed by $(d-1)$-dimensional simplices and the columns by d-dimensional simplices. We set

$$A_{ij} := \begin{cases} 1 & \text{if } (d-1)\text{-simplex } i \text{ belongs to } d\text{-simplex } j, \\ 0 & \text{otherwise.} \end{cases}$$

This chapter studies 2-dimensional simplicial complexes where each maximal simplex is a triangle or an edge. We call them triangular configurations. We denote the set of vertices of Δ by $V(\Delta)$, the set of edges by $E(\Delta)$ and the set of triangles by $T(\Delta)$. The cycle space of Δ over a field $\mathbb{F}$, denoted $\ker \Delta$, is the kernel of the incidence matrix A of Δ over $\mathbb{F}$, that is $\{x | Ax = 0\}$.

Let Δ be a triangular configuration. A matching of Δ is a subconfiguration M of Δ such that $t_1 \cap t_2$ does not contain an edge for every distinct $t_1, t_2 \in T(M)$. Let Δ be a triangular configuration. Let M be a matching of Δ. Then the defect of M is the set $E(T) \setminus E(M)$. The perfect matching of Δ is a matching with empty defect. We denote the set of all perfect matchings of Δ by $\mathcal{P}(\Delta)$. Let $w : T(\Delta) \mapsto \mathbb{R}$ be weights of the triangles of Δ. The generating function of perfect matchings in Δ is defined to be $P_\Delta(x) = \sum_{P \in \mathcal{P}(\Delta)} x^{w(P)}$, where $w(P) := \sum_{t \in P} w(t)$.

A triangular configuration Δ is tripartite if the edges of Δ can be divided into three disjoint sets E_1, E_2, E_3 such that every triangle of Δ contains edges from all sets E_1, E_2, E_3. We call the sets E_1, E_2, E_3 tripartition of Δ.

The triadjacency 3-matrix $A(x) = (a_{ijk})$ of a tripartite triangular configuration Δ with tripartition E_1, E_2, E_3 is the $|E_1| \times |E_2| \times |E_3|$ three dimensional array of numbers, defined as follows: We set

$$a_{ijk} := \begin{cases} x^{w(t)} & \text{if } e_i \in E_1, e_j \in E_2, e_k \in E_3 \text{ form a triangle } t \text{ with weight } w(t), \\ 0 & \text{otherwise.} \end{cases}$$

The permanent of a $n \times n \times n$ 3-matrix A is defined to be

$$\operatorname{per}(A) = \sum_{\sigma_1, \sigma_2 \in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

The determinant of a $n \times n \times n$ 3-matrix A is defined to be

$$\det(A) = \sum_{\sigma_1, \sigma_2 \in S_n} \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2) \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

We recall that biadjacency matrix $A(x) = (a_{ij})$ of a bipartite graph $G = (V, W, E)$ is the $|V| \times |W|$ matrix, defined as follows: We set

$$a_{ij} := \begin{cases} x^{w(e)} & \text{if } v_i \in V, v_j \in W \text{ form an edge } e \text{ with weight } w(e), \\ 0 & \text{otherwise.} \end{cases}$$

2.1.2 Main results

Theorem 2.1.1. *Let $\mathcal{C}$ be a linear binary code. Then there exists a tripartite triangular configuration Δ such that: If $\mathrm{per}(A_\Delta(x)) = \sum_{i=0}^m a_i x^i$, where $A_\Delta(x)$ is triadjacency matrix of Δ, then*

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{m} a_i x^{(i \bmod e)/2},$$

where e is an integer linear in length of $\mathcal{C}$.

Proof. Follows from Theorems 2.1.2, 2.1.3 and 2.1.4 below. ☐

Theorem 2.1.2 (Chapter 4)**.** *Let $\mathcal{C}$ be a linear code over $GF(p)$, where p is a prime. Then there exists a triangular configuration Δ such that: if $\sum_{i=0}^m a_i x^i$ is the weight enumerator of $\ker \Delta$ then*

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{m} a_i x^{(i \bmod e)/2},$$

where e is an integer linear in length of $\mathcal{C}$.

Theorem 2.1.3 (Chapter 1)**.** *Let Δ be a triangular configuration. Then there exists a triangular configuration Δ' and weights $w' : T(\Delta') \mapsto \mathbb{R}$ such that $W_{\ker \Delta}(x) = P_{\Delta'}(x)$.*

Theorem 2.1.4. *Let Δ be a triangular configuration with weights $w : T(\Delta) \mapsto \mathbb{R}$. Then there exists a tripartite triangular configuration Δ' and weights $w' : T(\Delta') \mapsto \mathbb{R}$ such that $P_\Delta(x) = \mathrm{per}(A_{\Delta'}(x))$ where $A_{\Delta'}(x)$ is the triadjacency matrix of Δ'.*

Proof. Follows directly from Proposition 2.2.9 and Proposition 2.2.10 of Section 2.2. ☐

Definition 2.1.5. We say that an $n \times n \times n$ 3-matrix A is *Kasteleyn* if there is 3-matrix A' obtained from A by changing signs of some entries so that $\mathrm{per}(A) = \det(A')$.

Theorem 2.1.6. *Let M be $n \times n$ matrix. Then one can construct $m \times m \times m$ Kasteleyn 3-matrix A with $m \leq n^2 + 2n$ and $\mathrm{per}(M) = \mathrm{per}(A)$. Moreover, Kasteleyn signing is trivial, i.e., $\mathrm{per}(A) = \det(A)$, and if M is non-negative then A is non-negative.*

Theorem 2.1.6 is proved in Section 2.3.

2.2 Triangular configurations and permanents

In this section we prove Theorem 2.1.4. We use basic building blocks as in Chapter 1. However, the use is novel and we need to stress the tripartitness of basic blocks. Hence we briefly describe them again.

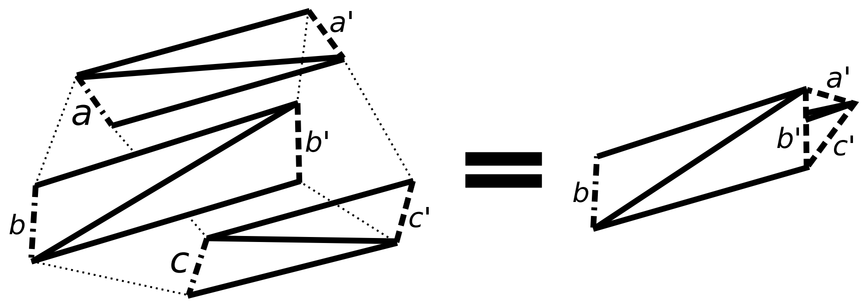

Figure 2.1: Triangular tunnel

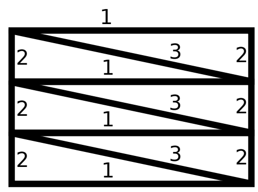

Figure 2.2: Tunnel tripartition

2.2.1 Triangular tunnel

Triangular tunnel is depicted in Figure 2.1. An *empty triangle* is a set of three edges forming a boundary of a triangle. We call the empty triangles $\{a, b, c\}$ and $\{a', b', c'\}$ ending.

Proposition 2.2.1. *The triangular tunnel has exactly one matching M^L with defect $\{a, b, c\}$ and exactly one matching M^R with defect $\{a', b', c'\}$.* □

Proposition 2.2.2. *The triangular tunnel is tripartite.*

Proof. Follows from Figure 2.2. □

2.2.2 Triangular configuration S^5

Triangular configuration S^5 is depicted in Figure 2.3. Letter "X" denotes empty triangles. We call these empty triangles ending.

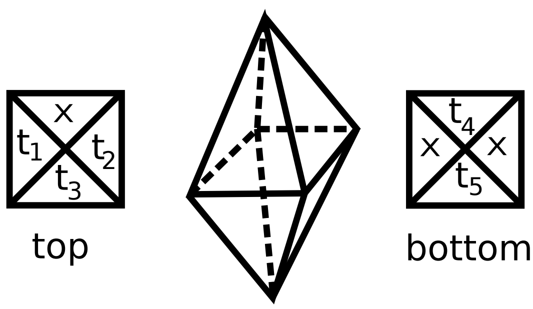

Figure 2.3: Triangular configuration S^5

Proposition 2.2.3. *Triangular configuration S^5 has one exactly perfect matching and exactly one matching with defect on edges of all empty triangles.*

Proof. The unique perfect matching is $\{t_1, t_2, t_4, t_5\}$. We denote it by $M^1(S^5)$. The unique matching with defect on edges of all empty triangles is $\{t_3\}$. We denote it by $M^0(S^5)$. ☐

Proposition 2.2.4. *Triangular configuration S^5 is tripartite.*

Proof. Follows from Figure 2.4.

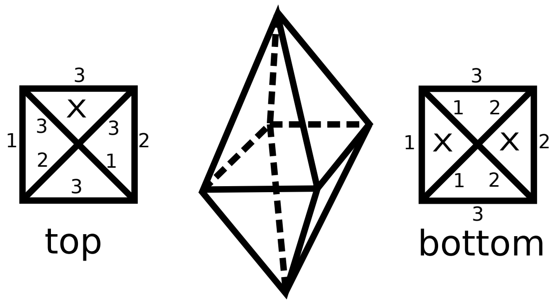

Figure 2.4: Triangular configuration S^5 with partitioning

☐

2.2.3 Matching triangular triangle

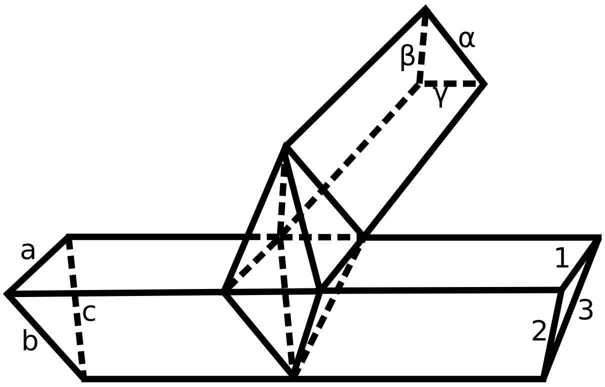

Figure 2.5: Matching triangular triangle

The *matching triangular triangle* is obtained from the triangular configuration S^5 and three triangular tunnels in the following way: Let T_1, T_2 and T_3 be triangular tunnels. Let $t_1^{T_1}, p^{T_1}$; $t_1^{T_2}, q^{T_2}$ and $t_1^{T_3}, r^{T_3}$ be the ending empty triangles of T_1, T_2 and T_3, respectively. Let $t_1^{S^5}, t_2^{S^5}, t_3^{S^5}$ be ending empty triangles of S^5. We identify $t_1^{T_1}$ with $t_1^{S^5}$; $t_1^{T_2}$ with $t_2^{S^5}$ and $t_1^{T_3}$ with $t_3^{S^5}$. The matching triangular triangle is defined to be $T_1 \cup S^5 \cup T_2 \cup T_3$. The matching triangular triangle is depicted in Figure 2.5.

Proposition 2.2.5. *The matching triangular triangle has exactly one perfect matching M^1 and exactly one matching M^0 with defect $\{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$. It has no matching with defect E, where $\emptyset \neq E \subsetneq \{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$.*

Proof. The perfect matching is $M^1 := M^1(S^5) \cup M^L(T_1) \cup M^L(T_2) \cup M^L(T_3)$. The matching M^0 is $M^0(S^5) \cup M^R(T_1) \cup M^R(T_2) \cup M^R(T_3)$.

Any matching with defect $E \subset \{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$ of the matching triangular triangle contains $M^1(S^5)$ or $M^0(S^5)$. This determines remaining triangles in a matching with defect $E \subseteq \{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$. Hence, there are just two matchings M^1 and M^0 with defect $E \subseteq \{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$. □

Proposition 2.2.6. *Matching triangular triangle T is tripartite and there is a tripartition of T such that $a, b, c \in E_1$; $1, 2, 3 \in E_2$; $\alpha, \beta, \gamma \in E_3$.*

Proof. Follows from Figure 2.4 and Figure 2.2. □

2.2.4 Linking three triangles by matching triangular triangle

Let Δ be a triangular configuration. Let t_1, t_2 and t_3 be three edge disjoint triangles of Δ.

The *link by matching triangular triangle* between t_1, t_2 and t_3 in Δ is the triangular configuration Δ' defined as follows. Let T be a matching triangular triangle defined in Section 2.2.3. Let $\{a, b, c\}, \{1, 2, 3\}, \{\alpha, \beta, \gamma\}$ be ending empty triangles of T. Let t_1^1, t_1^2, t_1^3 and t_2^1, t_2^2, t_2^3 and t_3^1, t_3^2, t_3^3 be edges of t_1 and t_2 and t_3, respectively. We relabel edges of T such that $\{a, b, c\} = \{t_1^1, t_1^2, t_1^3\}$ and $\{1, 2, 3\} = \{t_2^1, t_2^2, t_2^3\}$ and $\{\alpha, \beta, \gamma\} = \{t_3^1, t_3^2, t_3^3\}$. We let $\Delta' := \Delta \cup T$.

2.2.5 Construction

Let Δ be a triangular configuration and let $w : T(\Delta) \mapsto \mathbb{R}$ be weights of triangles. We construct a tripartite triangular configuration Δ' and weights $w' : T(\Delta') \mapsto \mathbb{R}$ in two steps. First step: We start with triangular configuration

$$\Delta_1' := \Delta_1 \cup \Delta_2 \cup \Delta_3$$

where $\Delta_1, \Delta_2, \Delta_3$ are disjoint copies of Δ. Let t be a triangle of Δ. We denote the corresponding copies of t in $\Delta_1, \Delta_2, \Delta_3$ by t_1, t_2, t_3, respectively.

Second step: For every triangle t of Δ, we link t_1, t_2, t_3 in Δ_1' by triangular matching triangle T. We denote this triangular matching triangle by T_t. Then we remove triangles t_1, t_2, t_3 from Δ_1'. We choose a triangle t' from $M^1(T_t)$ and set $w'(t') := w(t)$. We set $w'(t') := 0$ for $t' \in T(T_t) \setminus \{t\}$. The resulting configuration is desired configuration Δ'.

Proposition 2.2.7. *Triangular configuration Δ' is tripartite.*

Proof. The triangular configuration Δ' is constructed from three disjoint triangular configurations $\Delta_1, \Delta_2, \Delta_3$. From these configurations all triangles are removed. Hence, we can put edges $E(\Delta_i)$ to set E_i for $i = 1, 2, 3$. The remainder of Δ' is formed by matching triangular triangles. Every matching triangular triangle connects edges of $\Delta_1, \Delta_2, \Delta_3$. By Proposition 2.2.6 the matching triangular triangle is tripartite and its ends belong to different partities.

□

We recall 2^X denotes the set of all subsets X. We define a mapping $f : 2^{T(\Delta)} \mapsto 2^{T(\Delta')}$ as: Let S be a subset of $T(\Delta)$ then

$$f(S) := \{M^1(T_t) | t \in S\} \cup \{M^0(T_t) | t \in T(\Delta) \setminus S\}.$$

Proposition 2.2.8. *The mapping f is a bijection between the set of perfect matchings of Δ and the set of perfect matchings of Δ' and $w(M) = w'(f(M))$ for every $M \subseteq T(\Delta)$.*

Proof. By definition, the mapping f is an injection. By Proposition 2.2.5, every inner edge of T_t, $t \in T(\Delta)$, is covered by $f(S)$ for any subset S of $T(\Delta)$. Let M be a perfect matching of Δ. We show that $f(M)$ is perfect matching of Δ'.

$$f(M) = \{M^1(T_t) | t \in M\} \cup \{M^0(T_t) | t \in T(\Delta) \setminus M\}.$$

Let e be an edge of Δ and let e_1, e_2, e_3 be corresponding copies in $\Delta_1, \Delta_2, \Delta_3$. Let $t_1, t_2, \ldots, t_l$ be triangles incident with edge e in Δ. Let t_k be the triangle from perfect matching M incident with e. By definition of Δ', the edges e_1, e_2, e_3 are incident only with triangles of T_{t_i}, $i = 1, \ldots, l$. The edges e_1, e_2, e_3 are covered by $M^1(t_k)$. The edges of T_{t_i}, $i = 1, \ldots, l$, $i \neq k$ are covered by $M^0(T_{t_i})$. Hence $f(M)$ is a perfect matching of Δ'.

Let M' be a perfect matching of Δ'. By Proposition 2.2.5, $M' = \{M^1(T_t) | t \in S\} \cup \{M^0(T_t) | t \in T(\Delta) \setminus S\}$ for some set S. The set S is a perfect matching of Δ. Thus, the mapping f is a bijection.

□

Corollary 2.2.1. $P_\Delta(x) = P_{\Delta'}(x)$. □

Proposition 2.2.9. *Let Δ be a triangular configuration with weights $w : T(\Delta) \mapsto \mathbb{R}$. Then there exist a tripartite triangular configuration Δ' and weights $w' : T(\Delta') \mapsto \mathbb{R}$ such that there is a bijection f between the set of perfect matchings $\mathcal{P}(\Delta)$ and the set of perfect matchings of $\mathcal{P}(\Delta')$. Moreover, $w(M) = w'(f(M))$ for every $M \in \mathcal{P}(\Delta)$, and $P_\Delta(x) = P_{\Delta'}(x)$.*

Proof. Follows directly from Propositions 2.2.7 and 2.2.8 and Corollary 2.2.1. □

Proposition 2.2.10. *Let Δ be a tripartite triangular configuration with tripartition E_1, E_2, E_3 such that $|E_1| = |E_2| = |E_3|$ and let $A_\Delta(x)$ be its triadjacency matrix. Then $P_\Delta(x) = \mathrm{per}(A_\Delta(x))$.*

Proof. We have

$$\mathrm{per}(A_\Delta(x)) = \sum_{\sigma_1, \sigma_2 \in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

Every perfect triangular matching between partities E_1, E_2, E_3 can be encoded by two permutations σ_1, σ_2 and vice versa. If matching M is a subset of $T(\Delta)$, then

$$\prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)} = \prod_{i=1}^{n} x^{w([i\sigma_1(i)\sigma_2(i)])} = x^{w(M)},$$

where $[ijk]$ denotes a triangle of Δ with edges i, j, k. If M is not a subset of $T(\Delta)$, then there is i such that $a_{i\sigma_1(i)\sigma_2(i)} = 0$. Hence $\prod_{i=1}^{n} x^{w([i\sigma_1(i)\sigma_2(i)])} = 0$. Therefore

$$\sum_{\sigma_1, \sigma_2 \in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)} = P_\Delta(x).$$

□

2.3 Kasteleyn 3-matrices

We first introduce a sufficient condition for a 3-matrix to be Kasteleyn. Let A be a $|V_0| \times |V_1| \times |V_2|$ non-negative 3-matrix, where $|V_i| = m, i = 1, 2, 3$. We first define two bipartite graphs G_1, G_2 as follows. We let, for $i = 1, 2$, $G_i^A = G_i = (V_0, V_i, E_i)$ where

$$E_1 = \{\{a, b\} | a \in V_0, b \in V_1 \text{ and } A_{abc} \neq 0 \text{ for some } c\},$$

and

$$E_2 = \{\{a, c\} | a \in V_0, c \in V_2 \text{ and } A_{abc} \neq 0 \text{ for some } b\}.$$

Theorem 2.3.1. *If A is such that both G_1^A, G_2^A are Pfaffian bipartite graphs then A is Kasteleyn.*

Proof. Let M_i be the biadjacency matrix of G_i and let $\text{sign}_i : E(G_i^A) \mapsto \{-1, 1\}$ be the signing of the entries of M_i which defines matrix M_i' such that $\text{per}(M_i) = \det(M_i')$. We define 3-matrix A' by

$$A'_{abc} = \text{sign}_1(\{a, b\})\text{sign}_2(\{a, c\})A_{abc}.$$

We have $\det(A')$ equals

$$\sum_{\sigma_1} \text{sign}(\sigma_1) \times \sum_{\sigma_2} \text{sign}(\sigma_2) \prod_j \text{sign}_2(\{j, \sigma_2(j)\})[\text{sign}_1(\{j, \sigma_1(j)\})A_{j\sigma_1(j)\sigma_2(j)}].$$

By the construction of sign_2 we have that for each σ_2 and each σ_1, if

$$\prod_j A_{j\sigma_1(j)\sigma_2(j)} \neq 0$$

then

$$\text{sign}(\sigma_2) \prod_j \text{sign}_2(\{j, \sigma_2(j)\}) = 1.$$

Hence

$$\begin{aligned}\det(A') &= \sum_{\sigma_1} \text{sign}(\sigma_1) \times \sum_{\sigma_2} \prod_j [\text{sign}_1(\{j, \sigma_1(j)\})A_{j\sigma_1(j)\sigma_2(j)}] \\ &= \sum_{\sigma_2} \times \sum_{\sigma_1} \text{sign}(\sigma_1) \prod_j \text{sign}_1(\{j, \sigma_1(j)\})A_{j\sigma_1(j)\sigma_2(j)}.\end{aligned}$$

Analogously by the construction of sign_1 we have that for each σ_1 and each σ_2, if

$$\prod_j A_{j\sigma_1(j)\sigma_2(j)} \neq 0$$

then

$$\text{sign}(\sigma_1) \prod_j \text{sign}_1(\{j, \sigma_1(j)\}) = 1.$$

Hence

$$\det(A') = \sum_{\sigma_1, \sigma_2} \prod_j A_{j\sigma_1(j)\sigma_2(j)} = \text{per}(A).$$

□

In the introduction we defined the triadjacency 3-matrix of a triangular configuration as the adjacency matrix of the *edges* of the triangles. We also defined a *matching* of a triangular configuration as a set of edge-disjoint triangles. In this section it is advantageous to consider any 3-matrix with $0,1$ entries as the adjacency matrix of *vertices* of a triangular configuration. Hence we need the following notions.

A triangular configuration Δ is *vertex-tripartite* if vertices of Δ can be divided into three disjoint sets V_1, V_2, V_3 such that every triangle of Δ contains one vertex from each set V_1, V_2, V_3. We call the sets V_1, V_2, V_3 vertex-tripartition of Δ.

The *vertex-adjacency* 3-matrix $A(x) = (a_{ijk})$ of a tripartite triangular configuration Δ with vertex-tripartition V_1, V_2, V_3 is the $|V_1| \times |V_2| \times |V_3|$ three dimensional array of numbers, defined as follows: We set

$$a_{ijk} := \begin{cases} x^{w(t)} & \text{if } i \in V_1, j \in V_2, k \in V_3 \text{ forms a triangle } t \text{ with weight } w(t), \\ 0 & \text{otherwise.} \end{cases}$$

We will need the following modification of the notion of a matching. A set of triangles of a triangular configuration is called *strong matching* if its triangles are mutually vertex-disjoint.

Proof of Theorem 2.1.6. Let M be a $n \times n$ matrix and let $G = (V_1, V_2, E)$ be the adjacency bipartite graph of its non-zero entries. We have $|V_1| = |V_2| = n$. We order vertices of each V_i, $i = 1, 2$ arbitrarily and let $V_i = \{v(i,1), \dots, v(i,n)\}$. Let $V_i' = \{v'(i,1), \dots, v'(i,n)\}$ be disjoint copy of V_i, $i = 1, 2$.

We next define three sets of vertices W_1, W_2, W_0 and system of triangles $\Delta(G) = \Delta$ so that each triangle intersects each W_i in exactly one vertex.

$$\begin{aligned} W_1 &= V_1 \cup V_1' \cup \{w(1,e) | e \in E\}, \\ W_2 &= V_2 \cup V_2' \cup \{w(2,e) | e \in E\}, \\ W_0 &= \{w(0,e) | e \in E\} \cup \{w(0,i,j) | i = 1,2; j = 1, \dots, n\}, \end{aligned}$$

$$\begin{aligned} \Delta = \cup_{e=ab \in E} & \{(a, b, w(0,e)), (w(0,e), w(1,e), w(2,e))\} \cup \\ & \cup_{j=1}^{n} \{(w(0,1,j), v'(2,j), w(1,e)) | v(1,j) \in e\} \cup \\ & \cup_{j=1}^{n} \{(w(0,2,j), v'(1,j), w(2,e)) | v(2,j) \in e\}. \end{aligned}$$

We let A be the vertex-adjacency 3-matrix of the triangular configuration $\mathcal{T}(G) = \mathcal{T} = (W_0, W_1, W_2, \Delta)$. We first observe that both bipartite graphs G_1, G_2 of this triangular configuration are planar; let us consider only G_1, the reasoning for G_2 is the same. First, vertices $v'(1,j)$ and $w(0,2,j)$ are connected only among themselves in G_1. Further, the component of G_1 containing vertex $v(1,j)$ contains also vertex $w(0,1,j)$ and consists of $deg_G(v(1,j))$ disjoint paths of length 3 between these two vertices. Here $deg_G(v(1,j))$ denotes the degree of $v(1,j)$ in graph G, i.e., the number of edges of G incident with $v(1,j)$. Thus, by Theorem 2.3.1, A is Kasteleyn.

We next observe that Kasteleyn signing is trivial. Let D_1 be the orientation of G_1 in which each edge is directed from W_0 to W_1. In each planar drawing of G_1, each inner face has an odd number of edges directed in D_1 clockwise. This means that D_1 is a *Pfaffian orientation* of G_1, and $\mathrm{per}(A) = \det(A)$ (see e.g. Loebl [15] for basic facts on Pfaffian orientations and Pfaffian signings).

Finally there is a bijection between the perfect matchings of G and the perfect strong matchings of $\mathcal{T}$: if $P \subset E$ is a perfect matching of G then let

$$P(\mathcal{T}) = \{(a, b, w(0, e)) | e = ab \in P\}.$$

We observe that $P(\mathcal{T})$ can be uniquely extended to a perfect strong matching of $\mathcal{T}$, namely by the set of triples $S_1 \cup S_2 \cup S_3$ where

$$\begin{aligned}
S_1 &= \cup_{e \in E \setminus P}\{(w(0, e), w(1, e), w(2, e))\},\\
S_2 &= \cup_{j=1}^{n}\{(w(0, 1, j), v'(2, j), w(1, e)) | v(1, j) \in e \in P\},\\
S_3 &= \cup_{j=1}^{n}\{(w(0, 2, j), v'(1, j), w(2, e)) | v(2, j) \in e \in P\}.
\end{aligned}$$

Set S_1 is inevitable in any perfect strong matching containing $P(\mathcal{T})$ since the vertices $w(0, e); e \notin P$ must be covered. This immediately implies that sets S_2, S_3 are inevitable as well.

On the other hand, if Q is a perfect strong matching of $\mathcal{T}$ then Q contains $P(\mathcal{T})$ for some perfect matching P of G.

□

2.4 Application to 3D dimer problem

Let Q be cubic $n \times n \times n$ lattice. The dimer partition function of Q, which is equal to the generating function of the perfect matchings of Q, can be by Theorem 2.1.6 identified with the permanent of the Kasteleyn vertex-adjacency matrix of triangular configuration $\mathcal{T}(Q)$. Natural question arises whether this observation can be used to study the 3D dimer problem.

We first observe that the natural embedding of Q in 3-space can be simply modified to yield an embedding of $\mathcal{T}(Q)$ in 3-space. This can perhaps best be understood by figures, see Figure 2.6; this figure depicts configuration $\mathcal{T}(Q)$ around vertex v of Q with neighbors $u_1, \ldots, u_6$.

Triangular configuration $\mathcal{T}(Q)$ is obtained by identification of vertices $v_i, i = 1, \ldots, 6$ in the left and right parts of Figure 2.6. Now assume that the embedding of left part of Figure 2.6 is such that for each vertex v of Q, the vertices $v_1, \ldots, v_6$ belong to the same plane and the convex closure of $v_1, \ldots, v_6$ intersects the rest of the configuration only in $v_1, \ldots, v_6$. Then we add the embedding of the right part, for each vertex v of Q, so that x_1 belongs to the plane of the v_i's and x_2 is very near to x_1 but outside of this plane.

Summarizing, the dimer partition function of a finite 3-dimensional cubic lattice Q may be written as the determinant of the vertex-adjacency 3-matrix of triangular configuration $\mathcal{T}(Q)$ which preserves the natural embedding of the cubic lattice. Calculating the determinant of a 3-matrix is hard, but perhaps formulas for the determinant of the particular vertex-adjacency 3-matrix of $\mathcal{T}(Q)$, illuminating the 3-dimensional dimer problem, may be found. An example of a formula valid for the determinant of a 3-matrix is shown in the next subsection. It is new as far as we know but its proof is basically identical to the proof of Lemma 3.3 of Barvinok [1].

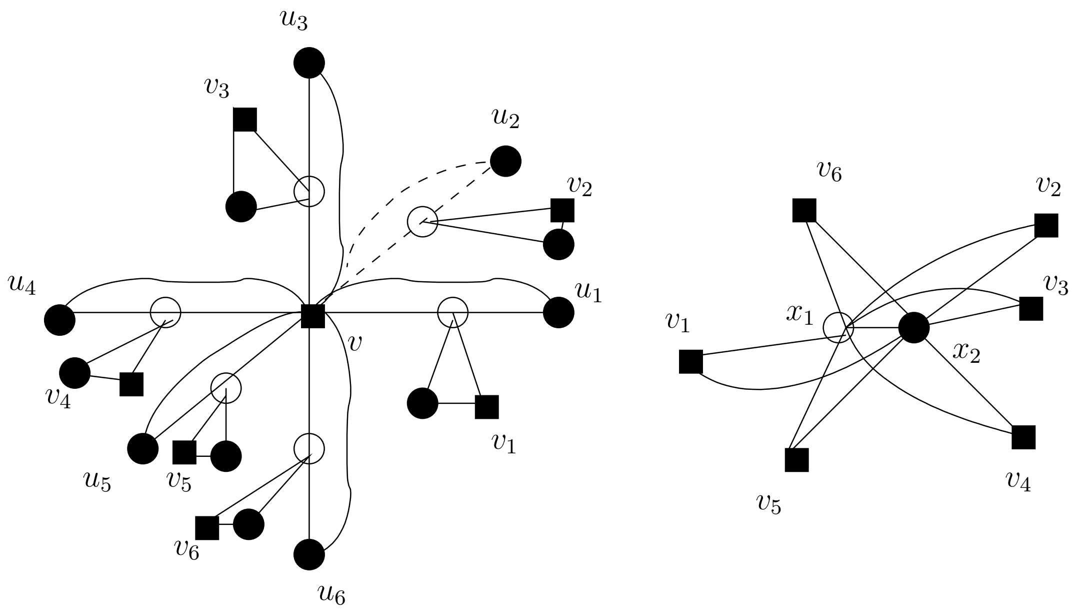

Figure 2.6: Configuration $\mathcal{T}(Q)$ around vertex v of Q with neighbors $u_1, \dots, u_6$ so that u_1, u_3, u_4, u_6 belong to the same plane in the 3-space, u_2 is 'behind' this plane and u_5 is 'in front of' this plane. Empty vertices belong to W_0, square vertices belong to W_1 and full vertices belong to W_2.

2.4.1 Binet-Cauchy formula for determinants of 3-matrices

We recall from the introduction that the permanent of a $n \times n \times n$ 3-matrix A is defined to be

$$\mathrm{per}(A) = \sum_{\sigma_1, \sigma_2 \in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

The determinant of a $n \times n \times n$ 3-matrix A is defined to be

$$\det(A) = \sum_{\sigma_1, \sigma_2 \in S_n} \mathrm{sign}(\sigma_1)\mathrm{sign}(\sigma_2) \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

The next formula is a generalization of Binet-Cauchy formula (see the proof of Lemma 3.3 in Barvinok [1]).

Lemma 2.4.1. *Let A^1, A^2, A^3 be real $r \times n$ matrices, $r \le n$. For a subset $I \subset \{1, \dots, n\}$ of cardinality r we denote by A^s_I the $r \times r$ submatrix of the matrix A^s consisting of the columns of A^s indexed by the elements of the set I. Let C be the 3-matrix defined, for all i_1, i_2, i_3 by*

$$C_{i_1,i_2,i_3} = \sum_{j=1}^{n} A^1_{i_1,j} A^2_{i_2,j} A^3_{i_3,j}.$$

Then

$$\det(C) = \sum_{I} \mathrm{per}(A^1_I) \det(A^2_I) \det(A^3_I),$$

where the sum is over all subsets $I \subset \{1, \dots, n\}$ of cardinality r

Proof.

$$
\begin{aligned}
\det(C) &= \sum_{\sigma_1,\sigma_2\in S_r} \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2) \prod_{i=1}^{r}\sum_{j=1}^{n} A^1_{i,j}A^2_{\sigma_1(i),j}A^3_{\sigma_2(i),j} \\
&= \sum_{\sigma_1,\sigma_2\in S_r} \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2) \times \sum_{1\le j_1,\dots,j_r\le n}\prod_{i=1}^{r} A^1_{i,j_i}A^2_{\sigma_1(i),j_i}A^3_{\sigma_2(i),j_i} \\
&= \sum_{1\le j_1,\dots,j_r\le n}\sum_{\sigma_1,\sigma_2\in S_r} \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2) \times \prod_{i=1}^{r} A^1_{i,j_i}A^2_{\sigma_1(i),j_i}A^3_{\sigma_2(i),j_i}.
\end{aligned}
$$

Now, for all $J = (j_1, \dots, j_r)$ we have

$$
\begin{aligned}
&\sum_{\sigma_1,\sigma_2\in S_r} \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2) \times \prod_{i=1}^{r} A^1_{i,j_i}A^2_{\sigma_1(i),j_i}A^3_{\sigma_2(i),j_i} \\
=&(\prod_{i=1}^{r} A^1_{i,j_i})(\sum_{\sigma_1}\operatorname{sign}(\sigma_1)\prod_{i=1}^{r} A^2_{\sigma_1(i),j_i})(\sum_{\sigma_2}\operatorname{sign}(\sigma_2)\prod_{i=1}^{r} A^3_{\sigma_2(i),j_i}) \\
=&(\prod_{i=1}^{r} A^1_{i,j_i})\det(\tilde{A}^2_J)\det(\tilde{A}^3_J),
\end{aligned}
$$

where $\tilde{A}^s_J$ denotes the $r \times r$ matrix whose ith column is the j_ith column of matrix A^s.

If sequence J contains a pair of equal numbers then the corresponding summand is zero, since $\det(\tilde{A}^2_J)$ is zero. Moreover, if J is a permutation, and J' is obtained from J by a transposition, then

$$\det(\tilde{A}^2_J)\det(\tilde{A}^3_J) = \det(\tilde{A}^2_{J'})\det(\tilde{A}^3_{J'}).$$

Therefore Lemma 2.4.1 follows.

$\square$

3. Geometric representations in three dimensions

3.1 Introduction

This chapter extends results from Chapter 1 where it was proven that every binary linear code has a geometric representation. Here we show that each binary linear code has a geometric representation that can be embedded into $\mathbb{R}^4$. Moreover we characterize those $\mathcal{C}$ which admit a geometric representation in $\mathbb{R}^3$.

A *linear code* $\mathcal{C}$ *of length* n *and dimension* d *over a field* $\mathbb{F}$ is a linear subspace with dimension d of the vector space $\mathbb{F}^n$. Each vector in $\mathcal{C}$ is called a *codeword*. Let B be a basis of a binary code $\mathcal{C}$. A basis B is k-basis if every entry is non-zero in at most k vectors of B.

Let $\mathcal{C} \subseteq \mathbb{F}^n$ be a linear code over a field $\mathbb{F}$ and let S be a subset of $\{1, \ldots, n\}$. *Puncturing* a code $\mathcal{C}$ along S means deleting the entries indexed by the elements of S from each codeword of $\mathcal{C}$. The resulting code is denoted by $\mathcal{C}/S$.

A *simplex* X is the convex hull of an affine independent set V in $\mathbb{R}^d$. The *dimension* of X is $|V|-1$, denoted by $\dim X$. The convex hull of any non-empty subset of V that defines a simplex is called a *face* of the simplex. A *simplicial complex* Δ is a set of simplices fulfilling the following conditions: Every face of a simplex from Δ belongs to Δ and the intersection of every two simplices of Δ is a face of both.

The dimension of Δ is $\max\{\dim X | X \in \Delta\}$. Let Δ be a d-dimensional simplicial complex. We define the *incidence matrix* $A = (A_{ij})$ as follows: The rows are indexed by $(d-1)$-dimensional simplices and the columns by d-dimensional simplices. We set

$$A_{ij} := \begin{cases} 1 & \text{if } (d-1)\text{-simplex } i \text{ belongs to } d\text{-simplex } j, \\ 0 & \text{otherwise.} \end{cases}$$

This chapter studies two dimensional simplicial complexes where each maximal simplex is a triangle or a segment. We call them *triangular configurations*. Let Δ be a triangular configuration. A *subconfiguration* of Δ is a subset of Δ that is a triangular configuration. We denote the set of triangles of Δ by $T(\Delta)$. The *cycle space* of Δ over a field $\mathbb{F}$, denoted $\ker \Delta$, is the kernel of the incidence matrix A of Δ over $\mathbb{F}$, that is $\{x | Ax = 0\}$. Let T be a subset of the set of triangles of Δ. We denote by $\mathcal{K}(T)$ the triangular configuration that is defined by the set of triangles T. The *even subset* or *cycle* of Δ is a subset E of the set of triangles of Δ such that all edges of the triangular configuration $\mathcal{K}(E)$ have an even degree.

Let $\{t_1, \ldots, t_m\}$ be the set of triangles of Δ. For a subconfiguration Δ' of Δ, we let $\chi(\Delta') = (\chi(\Delta')_1, \ldots, \chi(\Delta')_m) \in \{0,1\}^m$ denote its *characteristics vector*, where $\chi(\Delta')_i = 1$ if Δ' contains triangle t_i, and $\chi(\Delta')_i = 0$ otherwise. Note that, the characteristics vectors of even subsets of Δ forms the cycle space of Δ.

Let E_1 and E_2 be sets. Then the *symmetric difference* of E_1 and E_2, denoted by $E_1 \triangle E_2$, is defined to be $E_1 \triangle E_2 := (E_1 \cup E_2) \setminus (E_1 \cap E_2)$. Note that, the symmetric difference of two even subsets E_1 and E_2 of Δ is also even subset of Δ and it holds $\chi(\mathcal{K}(E_1)) + \chi(\mathcal{K}(E_2)) = \chi(\mathcal{K}(E_1 \triangle E_2))$ over $GF(2)$.

A linear code $\mathcal{C}$ has a *geometric representation* if there exists a triangular configuration Δ such that $\mathcal{C} = \ker \Delta / S$ for some set S and $\dim \mathcal{C} = \dim \ker \Delta$. For such S we write $S = S(\ker \Delta, \mathcal{C})$.

Theorem 3.1.1 (Chapter 4)**.** *Let $\mathcal{C}$ be a linear code over rationals or over $GF(p)$, where p is a prime. Then $\mathcal{C}$ has a geometric representation.*

3.1.1 Main Results

A basis B of a binary linear code $\mathcal{C} \subseteq GF(2)^n$ is 2-basis if every entry $i \leq n$ is non-zero in at most two vectors of B.

Theorem 3.1.2. *Let Δ be a triangular configuration embeddable into R^3 then* $\ker \Delta$ *has a 2-basis.*

Proof. The proof follows from Theorem 3.2.2 in Section 3.2.1. □

By Whitney's theorem, the cycle space of a 3-connected graph G determines G. It is therefore natural to ask whether our result can help to answer the question: Given a 2 dimensional simplicial complex, is it embeddable into $\mathbb{R}^3$? Theorem 3.1.2 gives only a necessary condition. For example no triangulation of the Klein bottle can be embedded into $\mathbb{R}^3$ and its cycle space has a 2-basis. The topic of embedding of simplicial complexes is treated in Matoušek et al. [19].

The main result of this chapter is that existence of a 2-basis characterize geometric representations in $\mathbb{R}^3$.

Theorem 3.1.3. *A binary linear code $\mathcal{C}$ has a geometric representation embeddable into $\mathbb{R}^3$ if and only if $\mathcal{C}$ has a 2-basis.*

The above theorem is an analogy of Mac Lane's planarity criterion [18] for graphs.

Theorem 3.1.4. *A binary linear code $\mathcal{C}$ has a geometric representation embeddable into $\mathbb{R}^3$ if and only if there exists a graph G such that $\mathcal{C}$ equals the cut space of G.*

It is well known that every two dimensional simplicial complex can be embedded into $\mathbb{R}^5$. Hence, every binary linear code has a geometric representation embeddable into $\mathbb{R}^5$. We further show:

Theorem 3.1.5. *Every binary linear code $\mathcal{C}$ has a geometric representation embeddable into $\mathbb{R}^4$.*

Theorem 3.1.5 extends a main result of Chapter 1 where it is shown that every binary linear code has a geometric representation.

Corollary 3.1.1. *There is a polynomial algorithm that decides the minimal dimension of a geometric representation of a binary code $\mathcal{C}$.*

This positive result complements the results of Matoušek et al. [19] on embeddings of simplicial complexes.

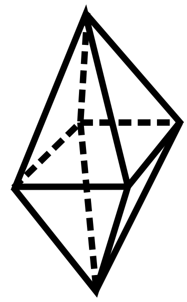

Figure 3.1: An example of strong boundary in $\mathbb{R}^3$

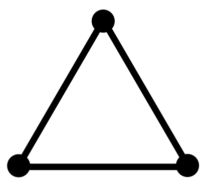

Figure 3.2: A polygon, counterpart of strong boundary in $\mathbb{R}^2$

3.2 Proof of main results

Proof of Theorem 3.1.3. The necessary condition of the theorem follows from Theorem 3.1.2. The sufficient condition is proven in Section 3.2.2. ☐

3.2.1 Bases of triangular configurations embedded into $\mathbb{R}^3$

In this section we suppose that all triangular configurations are embedded into $\mathbb{R}^3$ with the standard Euclidean metric $\rho(x, y) := \sqrt{\sum_{i=1}^{3}(x_i - y_i)^2}$. Let x be an element of $\mathbb{R}^3$ and let $\epsilon \in \mathbb{R}$ and $\epsilon > 0$. The ϵ-neighborhood of x is the set $N_\epsilon(x) := \{y \in \mathbb{R}^3 \mid \rho(x, y) < \epsilon\}$. If no confusion can arise we let $N_\epsilon(x) = N(x)$. Let $(x_1, \ldots, x_m)$ be a sequence points in a space. A *polygonal path* along the sequence $(x_1, \ldots, x_m)$ is a sequence of line segments connecting the consecutive points. Let Δ be a triangular configuration embedded into $\mathbb{R}^3$. A *cell* X of Δ is a non-empty maximal subset of $\mathbb{R}^3 \setminus \Delta$ with respect to inclusion such that between any two points of X there is a polygonal path that does not intersect Δ. A *bounded cell* is a cell that is contained in some sphere of a finite diameter. A *strong boundary* is a triangular configuration C such that C has at least two cells and every subconfiguration C' has fewer cells than C. An example is depicted in Figure 3.1. A one dimensional counterpart of strong boundary is a polygon, for example see Figure 3.2. Let X be a subset of $\mathbb{R}^3$. The closure of X, denoted by $cl(X)$, is the set $cl(X) := \{y \in \mathbb{R}^3 \mid \forall \epsilon > 0; N_\epsilon(y) \cap X \neq \emptyset\}$. We say that a triangle t is *incident* with a cell S if $t \subseteq cl(S)$.

Proposition 3.2.1. *Let Δ be a triangular configuration embedded into $\mathbb{R}^3$. Then every triangle t of Δ is incident with at least one cell of Δ and at most two cells of Δ.*

Proof. Let t be a triangle of Δ. For a contradiction, suppose that t is incident with three cells X_1, X_2, X_3 of Δ. Let p a point of t that does not belong to any edge of t. It holds that $p \in cl(X_1)$, $p \in cl(X_2)$ and $p \in cl(X_3)$. Let $N(p)$

be a neighborhood of p such that $N(p)$ does not intersect an edge of Δ. The neighborhood $N(p)$ intersects the cells X_1, X_2, X_3. Let x_1, x_2, x_3 be points of $cl(X_1) \cap N(p)$, $cl(X_2) \cap N(p)$, $cl(X_3) \cap N(p)$, respectively. Then, the segments x_1x_2, x_2x_3, x_1x_3 intersect triangle t. Let H be a hyperplane of $\mathbb{R}^3$ that contains triangle t. Then two points of x_1, x_2, x_3 belong to the same half-space defined by H. The segment connecting these two points do not intersect t. This is the contraction. Hence, t is incident with at most two cells of Δ.

Now, we show that t is incident with at least one cell. Let v_1, v_2, v_3 be vertices of t and let p be a point of t that belongs to no edge of Δ. Let v be a vector orthogonal to triangle t and let $\epsilon > 0$. Let P_ϵ^+ be a convex hull of set $\{v_1, v_2, v_3, p + \epsilon v\}$ and let P_ϵ^- be a convex hull of set $\{v_1, v_2, v_3, p - \epsilon v\}$. We choose $\epsilon > 0$ sufficiently small such that $\Delta \cap P_\epsilon^+ = t$ and $\Delta \cap P_\epsilon^- = t$. The sets $P_\epsilon^+ \setminus t$ and $P_\epsilon^- \setminus t$ are convex and disjoint with Δ. Thus, $P_\epsilon^+ \setminus t$ is a part of one cell of Δ. Let X^+ be the cell of Δ that contains $P_\epsilon^+ \setminus t$. Clearly $t \subseteq cl(X^+)$. Thus, triangle t is incident with at least cell X^+.

□

Corollary 3.2.1. *Let C be a strong boundary embedded into $\mathbb{R}^3$. Then every triangle t of C is incident with two cells of C.*

Proof. By Proposition 3.2.1, triangle t is incident with one or two cells of C. If t is incident with one cell, we can remove it from C and the number of cells of C does not change. Thus, $C \setminus \{t\}$ is also a strong boundary. This contradict with the minimality of C. Hence, t is incident with exactly two cells. □

Lemma 3.2.1. *Let Δ be a triangular configuration embedded into $\mathbb{R}^3$. Let t be a triangle of Δ incident with two cells of Δ. Then the number of cells of $\Delta \setminus \{t\}$ is equal to the number of cells of Δ minus one.*

Proof. Let X_1 and X_2 be cells incident with t. Let x be a point of t. Then there are points x_1 and x_2 of X_1 and X_2, respectively, such that $N(x_1) \cap x \neq \emptyset$ and $N(x_2) \cap x \neq \emptyset$. Hence, there is a polygonal path between x_1 and x_2 disjoint from $\Delta \setminus \{t\}$. The set $X_1 \cup t \cup X_2$ is a cell of $\Delta \setminus \{t\}$ and the proposition follows. □

Proposition 3.2.2. *Let C be a strong boundary embedded into $\mathbb{R}^3$. Then C has exactly two cells.*

Proof. For a contradiction suppose that C has more than two cells. Let t be a triangle of C. By Corollary 3.2.1, triangle t is incident with exactly two cells. By Lemma 3.2.1, by removing t from C, we join two cells into one. If C has more than two cells, the subconfiguration $C \setminus \{t\}$ has at least two cells. Let C' be the minimal subconfiguration (with respect to inclusion) of $C \setminus \{t\}$ that has at least two cells. Then C' is a smaller strong boundary than C. This is a contradiction with the minimality of C. □

Proposition 3.2.3. *Let C be a strong boundary embedded into $\mathbb{R}^3$. Then one of the cells of C is bounded and the second one is unbounded.*

Proof. By proposition 3.2.2, C has two cells. By definition, every triangular configuration Δ is finite. Thus, every strong boundary C is finite. Hence, C is contained in a sufficiently large sphere S. The complement of the ball of S is

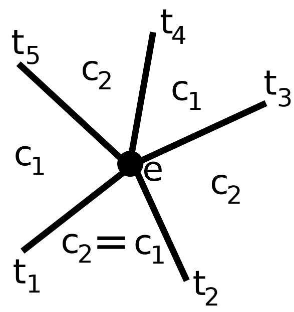

Figure 3.3: A contradicting example of an edge e of a strong boundary. The boundary has two cells c_1, c_2. The edge e is incident with triangles $t_1, \ldots, t_5$ and it has odd Degree 5. Then the strong boundary has only one cell. This contradict the definition of strong boundary.

contained in one cell of C, thus this cell is unbounded. The other cell of C is inside this ball and thus it is bounded.

□

Let C be a strong boundary. We call the bounded cell of C *inner cell* of C and denote it by $\mathrm{int}(C)$. The unbounded cell of C we denote by $\mathrm{ext}(C)$. We denote $C \cup \mathrm{int}(C)$ and $C \cup \mathrm{ext}(C)$ by $\overline{\mathrm{int}(C)}$ and $\overline{\mathrm{ext}(C)}$, respectively. So far we considered strong boundary as a triangular configuration in $\mathbb{R}^3$. Now we consider strong boundaries in a triangular configuration Δ. We say that a strong boundary C is a *strong boundary of* Δ if C is a subconfiguration of Δ. We say that a triangular configuration Δ is connected if every two triangles of Δ belong to a common strong boundary of Δ. The *connected component* of Δ is a maximal connected subconfiguration (under inclusion) of Δ.

Proposition 3.2.4. *Let C be a strong boundary. Then $cl(\mathrm{int}(C)) = \overline{\mathrm{int}(C)}$ and $cl(\mathrm{ext}(C)) = \overline{\mathrm{ext}(C)}$.*

Proof. By Corollary 3.2.1, every triangle of C is incident exactly with two cells $\mathrm{int}(C)$ and $\mathrm{ext}(C)$. By definition of incidence, it holds $cl(\mathrm{int}(C)) = \overline{\mathrm{int}(C)}$ and $cl(\mathrm{ext}(C)) = \overline{\mathrm{ext}(C)}$. □

Proposition 3.2.5. *Let C be a strong boundary embedded into $\mathbb{R}^3$. Then the set of triangles of C is an even subset.*

Proof. For a contradiction suppose that a strong boundary C contains an edge e with an odd degree. By Proposition 3.2.2, C has two cells. By Corollary 3.2.1, every triangle of C is incident with two cells. Let T be the set of triangles incident with e. Since edge e has an odd degree and every triangle of T is incident with two cells, we set a contradiction. (see Figure 3.3). □

Elementary strong boundaries

A strong boundary C of Δ is *elementary* if there is no strong boundary C' of Δ such that $\mathrm{int}(C) \cap \mathrm{int}(C') \neq \emptyset$ and $\mathrm{int}(C) \cap \mathrm{ext}(C') \neq \emptyset$. First, we illustrate this definition on one dimensional simplicial complexes embedded into $\mathbb{R}^2$. One dimensional simplicial complexes embedded into $\mathbb{R}^2$ correspond to planar embeddings

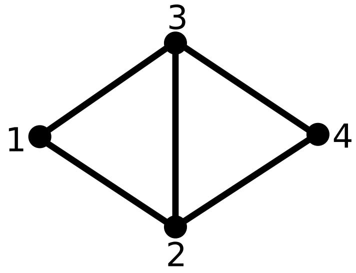

Figure 3.4: One dimensional complex embedded into $\mathbb{R}^2$ (a plane graph).

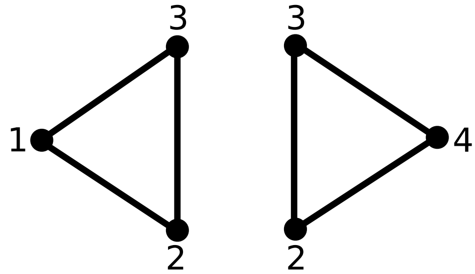

Figure 3.5: Two elementary strong boundaries of the complex in Figure 3.4.

of planar graphs. The graphs counterpart of our definition of elementary strong boundary is a boundary of a face of a 2-connected plane graph. The 2-connected plane graph depicted in Figure 3.4 has two boundaries of faces (elementary strong boundaries) depicted in Figure 3.5 and one circuit (strong boundary) that is not a boundary of a face (elementary strong boundary) depicted in Figure 3.6.

Now, we give example of triangular configuration embedded into $\mathbb{R}^3$ with two elementary strong boundaries. The triangular configuration in Figure 3.7 has two elementary strong boundaries (Figure 3.8) and one strong boundary that is not elementary (Figure 3.9).

Lemma 3.2.2. *Let Δ be a connected triangular configuration embedded into $\mathbb{R}^3$. Let X be a bounded cell of Δ. Let E be the set of the triangles of Δ incident with X. Then $\mathcal{K}(E)$ (triangular configuration defined by the set of triangles E) is an elementary strong boundary of Δ.*

Proof. Since cell X is bounded, set E is nonempty. Triangular configuration $\mathcal{K}(E)$ has at least two cells: If $\mathcal{K}(E)$ has only one cell, there is a triangle of E that belongs to no strong boundary of Δ. This contradict the connectivity of Δ.

Now we show that triangular configuration $\mathcal{K}(E)$ has at most two cells: For a

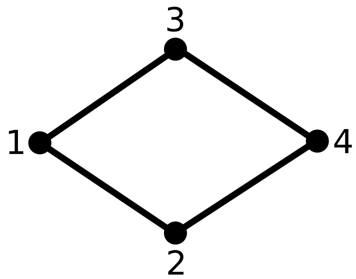

Figure 3.6: This strong boundary of the complex in Figure 3.4 is not elementary.

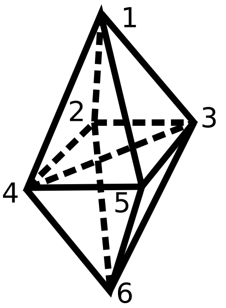

Figure 3.7: Triangular configuration embedded into $\mathbb{R}^3$.

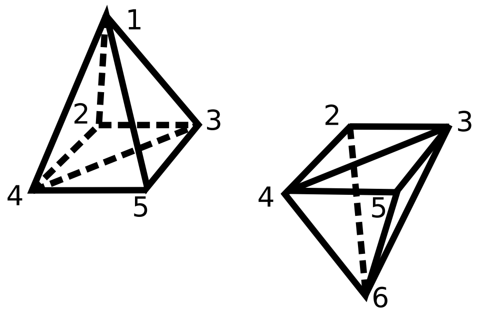

Figure 3.8: Two elementary strong boundaries of the triangular configuration in Figure 3.7.

contradiction suppose that $\mathcal{K}(E)$ has at least three cells X_1, X_2, X_3. Let t_1 be a triangle of E incident with X_1 and X_2 and let t_2 be a triangle of E incident with X_2, X_3. Since Δ is connected, there is a strong boundary D that contains t_1 and t_2. The cell X is a subset of X_2 and the cells X, X_2 are subsets of $\mathrm{int}(D)$. Then $E \subseteq \overline{\mathrm{int}(D)}$ and $X_1, X_3 \subseteq \mathrm{ext}(D)$. Hence, the triangular configuration $\mathcal{K}(E)$ does not have cells X_1, X_3, the contradiction.

Let t be a triangle of E. We show that triangle t is incident with two cells X_1, X_2 of $\mathcal{K}(E)$. For a contradiction suppose that t is incident only with cell X_1. It holds $X \subseteq X_1$. Then t is incident only with cell X of Δ. By connectivity, triangle t belongs to a strong boundary D of Δ. By Proposition 3.2.2, triangle t is incident with two cells of D. Since D is a subconfiguration of Δ, triangle t is incident with two cells X^1_Δ, X^2_Δ of Δ. This is the contradiction.

By Lemma 3.2.1, $\mathcal{K}(E) \setminus \{t\}$ has only one cell. Hence $\mathcal{K}(E)$ is a strong

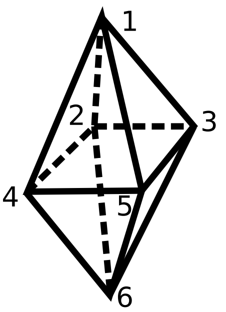

Figure 3.9: This strong boundary of the triangular configuration in Figure 3.7 is not elementary.

boundary.

For a contradiction suppose that $\mathcal{K}(E)$ is not an elementary strong boundary of Δ. Then there is a strong boundary C of Δ such that $\mathrm{int}(\mathcal{K}(E)) \cap \mathrm{int}(C) \neq \emptyset$ and $\mathrm{int}(\mathcal{K}(E)) \cap \mathrm{ext}(C) \neq \emptyset$. Then there is a triangle t' of C that belongs to $\mathrm{int}(\mathcal{K}(E))$. If there is not such a triangle t', then $\mathrm{int}(\mathcal{K}(E)) \subseteq \mathrm{int}(C)$ and $\mathrm{int}(\mathcal{K}(E)) \subseteq \mathrm{ext}(C)$. This contradict that $\mathrm{ext}(C)$ and $\mathrm{int}(C)$ are disjoint. Since the cell X is a subset of $\mathrm{int}(\mathcal{K}(E))$, triangle t' also belongs to E. Since $t' \in \mathrm{int}(\mathcal{K}(E))$, triangle t' is incident only with cell $\mathrm{int}(\mathcal{K}(E))$ of $\mathcal{K}(E)$. This is the contradiction. Thus, $\mathcal{K}(E)$ is an elementary strong boundary of Δ. ☐

Lemma 3.2.3. *Let Δ be a connected triangular configuration embedded into $\mathbb{R}^3$. Let t be a triangle of Δ that belongs to a strong boundary C of Δ. Then t belongs to exactly one elementary strong boundary C' of Δ such that $C' \subseteq \overline{\mathrm{int}(C)}$.*

Proof. Let X be a cell of Δ such that X is incident with t and $X \subseteq \mathrm{int}(C)$. Cell X is bounded. Let C' be the set of triangles of Δ incident with X. By lemma 3.2.2, the triangular configuration $\mathcal{K}(C')$ is an elementary strong boundary of Δ. By Proposition 3.2.4 and from $X \subseteq \mathrm{int}(C)$, we have $C' \subseteq \overline{\mathrm{int}(C)}$.

If there is an elementary strong boundary C'' of Δ different from C' that contains t such that $C'' \subseteq \overline{\mathrm{int}(C)}$. We have $\mathrm{int}(C') \cap \mathrm{int}(C'') \neq \emptyset$. Since $C' \neq C''$, we have $\mathrm{ext}(C') \cap \mathrm{int}(C'') \neq \emptyset$ or $\mathrm{ext}(C'') \cap \mathrm{int}(C') \neq \emptyset$. This a contradiction with the definition of elementary strong boundary. Thus, t is contained only in one elementary strong boundary that is contained in $\overline{\mathrm{int}(C)}$. ☐

Lemma 3.2.4. *Let Δ be a connected triangular configuration embedded into $\mathbb{R}^3$. Let t be a triangle of Δ. Then t is incident with two cells of Δ.*

Proof. Since Δ is connected, there is a strong boundary C of Δ that contains t. By Corollary 3.2.1, t is incident with two cells of C. Since C is a subconfiguration of Δ, t is also incident with two cells of Δ. ☐

Lemma 3.2.5. *Let Δ be a connected triangular configuration embedded into $\mathbb{R}^3$. Let t be a triangle of Δ such that t is contained in* $\mathrm{int}(C)$ *where C is a strong boundary of Δ. Then t belongs to exactly two elementary strong boundaries C_1, C_2 of Δ and $C_1 \subseteq \overline{\mathrm{int}(C)}$ and $C_2 \subseteq \overline{\mathrm{int}(C)}$.*

Proof. By lemma 3.2.4, triangle t is incident with two cells X_1 and X_2 of Δ. Let C_1 and C_2 be the sets of triangles incident with X_1 and X_2, respectively. By lemma 3.2.2, the sets C_1 and C_2 are elementary strong boundaries of Δ.

Since $t \in \mathrm{int}(C)$, cells X_1 and X_2 are subsets of $\mathrm{int}(C)$. Thus, $C_1 \subseteq \overline{\mathrm{int}(C)}$ and $C_2 \subseteq \overline{\mathrm{int}(C)}$.

For a contradiction suppose that there is a third elementary strong boundary C_3 of Δ that contains t. Since t is incident with two cells, we have $\mathrm{int}(C_3) \cap \mathrm{int}(C_1) \neq \emptyset$ or $\mathrm{int}(C_3) \cap \mathrm{int}(C_2) \neq \emptyset$. Without loose of generality we can suppose $\mathrm{int}(C_3) \cap \mathrm{int}(C_1) \neq \emptyset$. Since $C_3 \neq C_1$, we have $\mathrm{ext}(C_1) \cap \mathrm{int}(C_3) \neq \emptyset$ or $\mathrm{ext}(C_3) \cap \mathrm{int}(C_1) \neq \emptyset$. This a contradiction with the definition of elementary strong boundary. Thus, t is contained in exactly two elementary strong boundaries of Δ. ☐

Proposition 3.2.6. *Let Δ be a connected triangular configuration embedded into $\mathbb{R}^3$ and let C be a strong boundary of Δ and let $ESB(C)$ be the set of elementary* *strong boundaries of Δ contained in $\overline{\mathrm{int}(C)}$. Then $\chi(C)$ equals the sum of the characteristics vectors of the elements of $ESB(C)$ over $GF(2)$. Thus, $\chi(C) = \sum_{S \in ESB(C)} \chi(S)$.*

Proof. Each element of $ESB(C)$ is contained in $\overline{\mathrm{int}(C)}$. Therefore,

$$\triangle_{S \in ESB(C)} T(S) \subseteq \overline{\mathrm{int}(C)},$$

where $T(S)$ denotes the set of triangles of S. Let t be a triangle of Δ such that $t \subseteq \mathrm{int}(C)$. By Lemma 3.2.5, t is incident with two elementary boundaries C_1 and C_2 such that $C_1, C_2 \in ESB(C)$. Therefore,

$$\triangle_{S \in ESB(C)} T(S) \subseteq T(C).$$

Let t be a triangle of C. By Lemma 3.2.3, t belongs to exactly one elementary strong boundary from $ESB(C)$. Therefore,

$$\triangle_{S \in ESB(C)} T(S) \supseteq T(C).$$

Hence,

$$\triangle_{S \in ESB(C)} T(S) = T(C)$$

and

$$\sum_{S \in ESB(C)} \chi(S) = \chi(C)$$

over $GF(2)$. □

Non-empty even subsets divide $\mathbb{R}^3$

Proposition 3.2.7. *Let Δ be a non-empty triangular configuration embedded into $\mathbb{R}^3$ with all edges of an even degree. Then Δ has at least two cells.*

Proof. This proof is a variation of a proof of Jordan curve theorem for polygonal paths that can be found in Courant et al. [3]. First, we introduce some notation. Let t be a triangle. We denote by $\mathring{t}$ the interior of t, i.e., $\mathring{t} := t \setminus (e_1 \cup e_2 \cup e_3)$ where e_1, e_2, e_3 are the edges of t. Let e be an edge. We denote by $\mathring{e}$ the interior of e, i.e., $\mathring{e} := e \setminus (v_1 \cup v_2)$ where v_1, v_2 are the vertices of e.

Let r be a vector in $\mathbb{R}^3$ that is neither parallel with a triangle nor an edge of Δ. Let x be a point of $\mathbb{R}^3 \setminus \Delta$. Let $R(x)$ be the ray from x in direction r. Suppose that $R(x)$ does not intersect any vertex of Δ. We define the following quantities: Let $I_T(R(x), \Delta)$ denote the number of intersection of $R(x)$ with interiors of triangles of Δ. Let e be an edge of Δ that is intersected by $R(x)$. Let H be the plane defined by the edge e and the ray $R(x)$. Let n be the number of triangles incident with e on one side of H and m number of triangles on the other side of H. Then we define $I_e(R(x), \Delta)$ as the minimum of n and m. Let $I_E(R(x), \Delta)$ be the sum of $I_e(R(x), \Delta)$ over all edges of Δ that are intersected by $R(x)$ on interiors.

We define the sum $I(R(x), \Delta) := I_T(R(x), \Delta) + I_E(R(x), \Delta)$ and the parity of x as $P(R(x), \Delta) := I(R(x), \Delta) \bmod 2$.

Let P be a polygonal path in $\mathbb{R}^3 \setminus \Delta$. We show that all points of P have the same parity. First, we prove the following lemma.

Lemma 3.2.6. *Let x and x' be points in $\mathbb{R}^3$ such that*

1. *the segment xx' does not intersect Δ,*
2. *$R(x')$ intersect at least one edge,*
3. *$R(x')$ does not intersect a vertex,*
4. *$R(y)$ does not intersect an edge for $y \in xx' \setminus x'$.*

Then $P(R(x), \Delta) = P(R(y), \Delta)$ for all $y \in xx'$.

Proof. All points of xx' except x' have the same parity, since the parity can only change when the ray hits or leave an edge. A nontrivial case is to show that x and x' have the same parity. Let $E(R(x'))$ be the set of edges of Δ that are intersected by $R(x')$. Let $H(R, e)$ be the hyperplane defined by R and e. Let n_e be the number of triangles incident with e on the same side of H_e as x and let m_e denote the number of triangles incident with e on the other side of H_e.

By definition,

$$I(R(x), \Delta) = I_T(R(x), \Delta) + I_E(R(x), \Delta),$$

and

$$I(R(x'), \Delta) = I_T(R(x), \Delta) - \sum_{e \in E(R(x'))} n_e + I_E(R(x), \Delta) + \sum_{e \in E(R(x'))} \min\{n_e, m_e\} \tag{3.1}$$

Since every edge e of Δ has an even degree, $n_e + m_e$ is even. Hence, $n_e \equiv m_e \mod 2$. Therefore $I(R(x), \Delta) \equiv I(R(x'), \Delta) \mod 2$ and $P(R(x), \Delta) = P(R(x'), \Delta)$. □

By repeatedly using the above lemma, we get the following corollary.

Corollary 3.2.2. *Let P be a polygonal path such that $P \cap \Delta = \emptyset$ and no ray from any point of P hits a vertex of Δ. Then all points of P have the same parity.* □

Corollary 3.2.3. *Let x be a point from $\mathbb{R}^3$ such that $\Delta \cap x = \emptyset$ and $R(x)$ hits a vertex of Δ. Then there is a neighborhood $U(x)$ of x such that all points from $U(x) \setminus x$ have the same parity.*

Proof. Let $U(x)$ be a neighborhood of x such that $R(y)$ does not hit a vertex of Δ for $y \in U(x) \setminus x$. We can connect any two points of $U(x) \setminus x$ by a polygonal path and use the previous corollary. □

Let x be a point of $\mathbb{R}^3$ such that $R(x)$ intersect a vertex of Δ. We define the parity of x to be the same as a parity of a sufficiently small neighborhood of x.

Corollary 3.2.4. *Let P be a polygonal path such that $P \cap \Delta = \emptyset$. Then all points of P have the same parity.* □

Finish of the proof of Proposition 3.2.7. Any two points of a connected region of triangular configuration can be connected by a polygonal path. Hence any two points of a connected region have the same parity.

Let a and b be two different points of $\mathbb{R}^3$ such that a and b lie close to a triangle t of Δ and the segment from a to b intersects Δ only on the interior of t. Then a and b have different parities. Hence, Δ has at least two cells. □

Proof of Theorem 3.1.2

Proposition 3.2.8. *Let Δ be a triangular configuration embedded into $\mathbb{R}^3$. Then the set $\mathcal{S}$ of characteristics vectors of elementary strong boundaries of Δ is linear independent.*

Proof. We prove the proposition by the induction along the size of the set $\mathcal{S}$. If $|\mathcal{S}| \leq 1$, the proposition is clear. Let $|\mathcal{S}| > 1$ and let $\chi(C)$ be an element of $\mathcal{S}$ and let C be the corresponding elementary strong boundary such that C is incident with the unbounded cell of Δ. Let t be a triangle of C that is incident with the unbounded cell. Now we show that the triangle t belongs only to one elementary strong boundary C. For a contradiction suppose that t belongs to an elementary strong boundary C' of Δ different from C. Let X_1 and X_2 be the cell of Δ incident with t. One of the cells is unbounded, suppose that X_2 is unbounded. Then $X_2 \subseteq \mathrm{int}(C)$ and $X_2 \subseteq \mathrm{int}(C')$. Thus, $\mathrm{int}(C) \cap \mathrm{int}(C') \neq \emptyset$. Since $C \neq C'$, $\mathrm{int}(C') \cap \mathrm{ext}(C) \neq \emptyset$. Hence, C' is not elementary strong boundary. The contradiction. Thus, t belongs to only one elementary strong boundary C.

Hence, $\chi(C)$ is not linear combination of the other elements $\mathcal{S} \setminus \{\chi(C)\}$. By the induction assumption, the set $\mathcal{S} \setminus \{\chi(C)\}$ is linear independent. Hence, the set $\mathcal{S}$ is linear independent. □

Theorem 3.2.1. *Let Δ be a connected triangular configuration embedded into $\mathbb{R}^3$. Let $\mathcal{S}$ be the set of characteristics vectors of elementary strong boundaries of Δ. Then the set $\mathcal{S}$ is a 2-basis of the cycle space* $\ker \Delta$ *of Δ.*

Proof. Let $\chi(C_0)$ be an element of $\ker \Delta$ and let E_0 be the subset of triangles of Δ such that $C_0 = \mathcal{K}(E_0)$. The set E_0 is an even subset of Δ.

By Proposition 3.2.7, the triangular configuration $\mathcal{K}(E_0)$ has at least two cells. Therefore, $\mathcal{K}(E_0)$ contains a strong boundary C_1. Let E_1 be the subset of triangles of Δ such that $C_1 = \mathcal{K}(E_1)$. By Proposition 3.2.5, the set E_1 is an even subset. The symmetric difference $E_0 \triangle E_1$ is also even subset. For $i = 2, \ldots, k$ we define the sets E_i in the following way: Until $E_0 \triangle \cdots \triangle E_{i-1} \neq \emptyset$ we set E_i to be the set of triangles of a strong boundary contained in $\mathcal{K}(E_0 \triangle \cdots \triangle E_{i-1})$. The triangular configuration $\mathcal{K}(E_0 \triangle \cdots \triangle E_{i-1})$ contains a strong boundary, because $E_0 \triangle \cdots \triangle E_{i-1}$ is even subset and by Proposition 3.2.7, triangular configuration $\mathcal{K}(E_0 \triangle \cdots \triangle E_{i-1})$ has at least two cells. Since Δ is finite, this sequence of even subsets is finite.

Thus, the set E_0 is the symmetric difference of the even subsets $E_1, \ldots, E_k$ and $\chi(C_0) = \chi(C_1) + \cdots + \chi(C_k)$ over $GF(2)$. By proposition 3.2.6, characteristics vector of each strong boundary $\chi(C_i)$, $i = 1, \ldots, k$; is a linear combination of characteristics vectors of elementary strong boundaries over $GF(2)$. Therefore, $\chi(C)$ is a linear combination of characteristics vectors of elementary strong boundaries $\mathcal{S}$. By Proposition 3.2.8, the set $\mathcal{S}$ is linear independent. Thus, the set $\mathcal{S}$ is a basis.

Every strong boundary has exactly two cells. By definition of elementary strong boundary, the inner cell of any elementary strong boundary contains no triangle of other strong boundary. Hence, every triangle of Δ is contained in at most two elementary strong boundaries and at most two characteristics vectors of elementary strong boundaries are non-zero on the same coordinate. Thus, the set $\mathcal{S}$ is a 2-basis. □

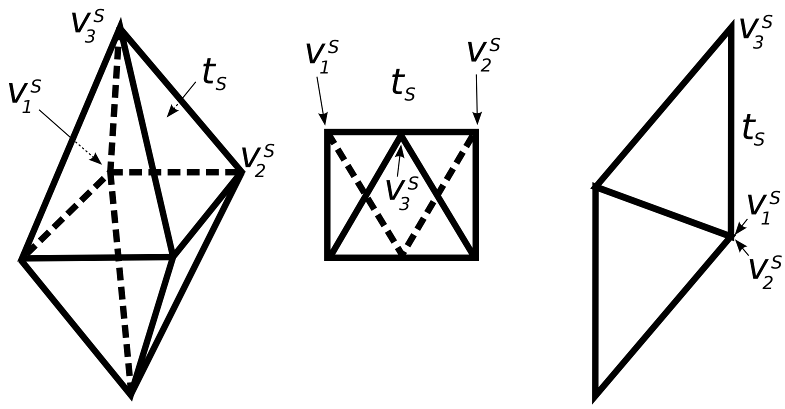

Figure 3.10: Eight triangles forming triangular sphere S. The picture on the left is a perspective view, the middle picture is a view from top, the picture on the right is a view from the right side.

Theorem 3.2.2. *Let Δ be a triangular configuration embedded into $\mathbb{R}^3$. Then the cycle space of Δ has a 2-basis.*

Proof. Let $\Delta_1, \ldots, \Delta_m$ be connected components of the triangular configuration Δ. Let $B_1, \ldots, B_m$ be bases of $\Delta_1, \ldots, \Delta_m$, respectively, provided by Theorem 3.2.1. Since characteristics vectors that corresponds to strong boundaries from different connected components have no common non-zero coordinate, the set $B := B_1 \cup \cdots \cup B_m$ is a 2-basis of the cycle space of Δ. $\square$

3.2.2 Proof of Theorem 3.1.3 (Representations in $\mathbb{R}^3$)

It remains to prove sufficiency of the condition of Theorem 3.1.3 for geometric representations in $\mathbb{R}^3$. We show that the construction from Chapter 1 and 4 for binary linear codes with 2-basis can be embedded into $\mathbb{R}^3$.

Basic building blocks

We start with definition of basic building blocks.

Triangular configuration S^n

First, we define triangular configuration S as a triangulation of a two dimensional sphere by 8 triangles. It is depicted in Figure 3.10. The triangle t_S has vertices v_1^S, v_2^S, v_3^S. All triangles of S have the same size. Therefore, the size of S and position of S in a space is determined by the coordinates of the points v_1^S, v_2^S, v_3^S. We denote the triangular configuration S with prescribed vertices $v_1^S = x, v_2^S = y, v_3^S = z$ by $S(x, y, z)$.

Proposition 3.2.9. *Triangular configuration S can be embedded into $\mathbb{R}^3$.* $\square$

Let n be a positive integer. We subdivide the triangle t_S of S in the way depicted in figure 3.11. Note that, the resulting object is a triangular configuration. We denote the resulting triangular configuration by S^n. Clearly, S^n can be embedded into $\mathbb{R}^3$. We denote the triangle i of S^n by $S^n(i)$, for $i = 1, \ldots, n$.

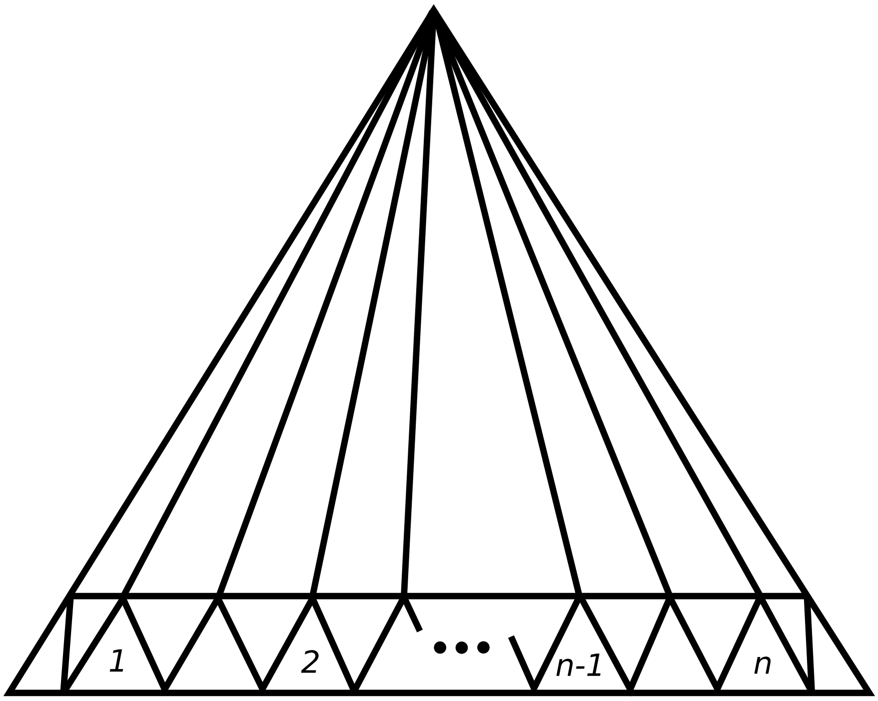

Figure 3.11: Subdivision of a triangle, triangles $1, \ldots, n$ are equilateral.

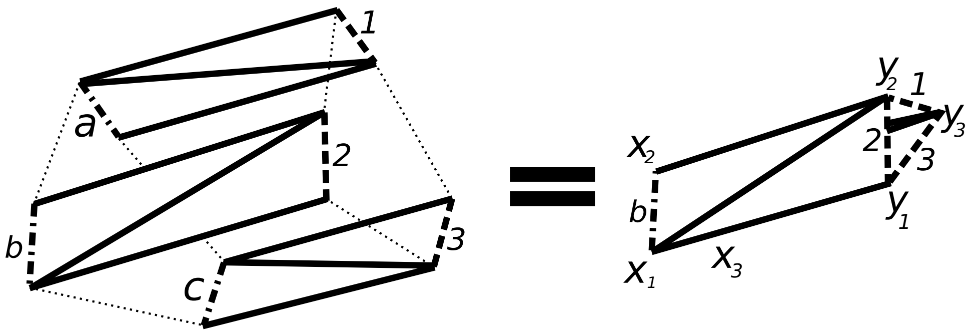

Figure 3.12: Triangular tunnel $T(t_1, t_2)$

Triangular tunnel

Let t_1 and t_2 be two empty triangles. Let x_1, x_2, x_3 be vertices of t_1 and y_1, y_2, y_3 be vertices of t_2. The triangular tunnel between t_1 and t_2 denoted by $T(t_1, t_2)$ is the six triangles that form a tunnel as is depicted in Figure 3.12. The vertices of the empty triangle abc lies on the points x_1, x_2, x_3 and the vertices of the empty triangle 123 lies on the points y_1, y_2, y_3.

Triangular tunnel bridge

Let t_1 and t_2 be empty triangles embedded into $\mathbb{R}^3$ such that t_1 and t_2 belong to the hyperplane given by equation $x_3 = 0$ and one edge of both t_1 and t_2 belongs to x_1 axis of $\mathbb{R}^3$ and t_2 is a shifted copy of t_1 in the direction of x_1 axis of $\mathbb{R}^3$, $t_2 = t_1 + a(1, 0, 0)$, $a \in \mathbb{R}$. Let l be the size of edge of t_1. We suppose that a is greater than l. See Figure 3.13.

Let $b > a$ and $c > a$. Let $alt(t_1)$ and $alt(t_2)$ denote the altitude of t_1 and t_2, respectively. Let t_1' and t_2' be copies of triangle t_1 and t_2 shifted by $(0, b, 0)$

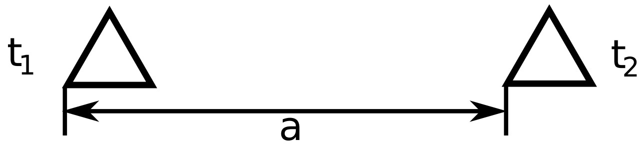

Figure 3.13: Empty triangles of bridge

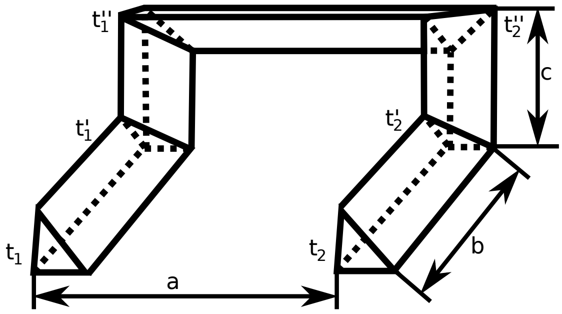

Figure 3.14: Triangular tunnel bridge

with top vertex shifted by $(0, -l/2, 0)$, respectively. Let t_1'' be a copy of t_1' shifted by $(0, 0, c)$ with the left vertex shifted by $(0, 0, alt(t_1))$ and let t_2'' be a copy of t_2' shifted by $(0, 0, c)$ with the right vertex shifted by $(0, 0, alt(t_2))$. Then the triangular tunnel bridge is

$$TB(t_1, t_2, b, c) := T(t_1, t_1') \cup T(t_1', t_1'') \cup T(t_1'', t_2'') \cup T(t_2'', t_2') \cup T(t_2', t_2).$$

The triangular tunnel bridge is depicted in Figure 3.14.

Proposition 3.2.10. *Let t_1, t_2, t_3, t_4 be disjoint triangles embedded into $\mathbb{R}^3$ such that the triangles belong to the hyperplane given by equation $x_3 = 0$ and one edge of each t_1, t_2, t_3, t_4 belongs to x_1 axis of $\mathbb{R}^3$ and t_2, t_3, t_4 are shifted copies of t_1 in the direction of the first coordinate of $\mathbb{R}^3$. Let l be the size of the longest edge of t_1. Let $a > l$ and $b > 2a$. Then the triangular tunnel bridges $TB(t_1, t_2, a, a)$ and $TB(t_3, t_4, b, a)$ are disjoint.*

Proof. The proposition follows from Figure 3.15. $\square$

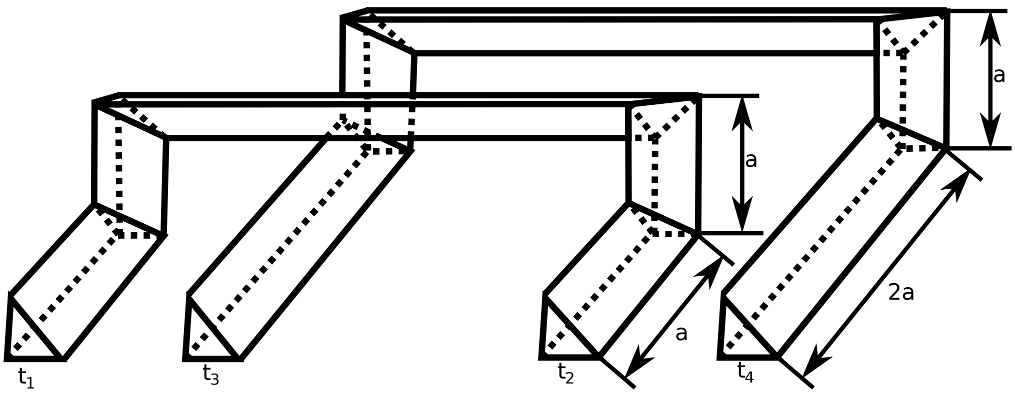

Figure 3.15: The proof of Proposition 3.2.10.

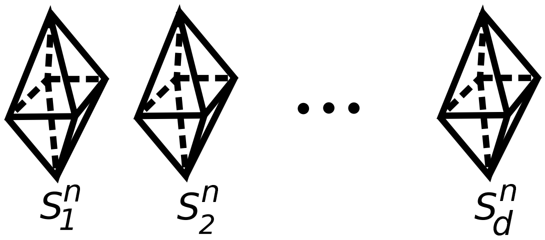

Figure 3.16: Spheres

Construction

Let $\mathcal{C}$ be a binary code with a 2-basis $B = \{b_1, \ldots, b_d\}$. We construct the following triangular configuration $\Delta_B^{\mathcal{C}}$ and embed it into $\mathbb{R}^3$.

In the first step, we put d identical copies of S^n, denoted by $S_1^n, \ldots, S_d^n$; into $\mathbb{R}^3$ as is depicted in Figure 3.16. Formally: Let v_1^1 equals $(0,0,0)$ and v_2^1 equals $(2,0,0)$ and v_3^1 equals $(1,0,1)$. The points v_1^1, v_2^1, v_3^1 are vertices of the first copy of S^n. Thus S_1^n equals $S^n(v_1^1, v_2^1, v_3^1)$. Therefore, the size of every edge of every triangle of S_1^n is less or equal 2. The triangular configuration S_i^n is shifted by offset $5i$ from the origin. Let v_1^i equals $(0+5i, 0, 0)$ and v_2^i equals $(2+5i, 0, 0)$ and v_3^i equals $(1+5i, 0, 1)$. Then S_i^n equals $S^n(v_1^i, v_2^i, v_3^i)$, for $i = 1, \ldots, d$. Then

$$\Delta_B^{\mathcal{C}} := S_1^n \cup \cdots \cup S_n^d.$$

In the second step, we add to $\Delta_B^{\mathcal{C}}$ the tunnels. We also construct set of triangles $\{B_1^n, \ldots, B_n^n\}$ and triangular configurations $\Delta_{b_i}, i = 1, \ldots, d$. Initially we set $\Delta_{b_i} := S_i^n$ for $i = 1, \ldots, d$. We interconnect the triangular configurations S_i^n, for $i = 1, \ldots, d$, by tunnels in the following way:

We proceed from the first coordinate 1 to the last coordinate n.

- If the coordinate i is zero in all basis vectors, we add an isolated triangle to $\Delta_B^{\mathcal{C}}$ and denote it by B_i^n.
- If the coordinate i is non-zero only in one basis vector b_k, we denote the triangle $S_k^n(i)$ by B_i^n and we do nothing otherwise.
- If the coordinate i is non-zero in two basis vectors b_k and b_l, $k < l$, we add triangular tunnel bridge $TB(S_k^n(i), S_l^n(i), 5i, 5)$ to $\Delta_B^{\mathcal{C}}$. We also add this tunnel bridge to the set Δ_{b_l}. We remove the triangle $S_l^n(i)$ from $\Delta_B^{\mathcal{C}}$ and Δ_{b_l} and we denote the triangle $S_k^n(i)$ by B_i^n.

We denote the set of triangles $\{B_1^n, \ldots, B_n^n\}$ by B^n.

Proposition 3.2.11. *The triangular tunnel bridges added in the last step are mutually disjoint.*

Proof. Let $TB(S_{k_1}^n(i_1), S_{l_1}^n(i_1), 5i_1, 5)$ and $TB(S_{k_2}^n(i_2), S_{l_2}^n(i_2), 5i_2, 5)$ be two triangular tunnel bridges from the last step. If there is none or only one, the proposition follows. Since $i_1 \neq i_2$ and $k_1 \neq l_1$ and $k_2 \neq l_2$, the triangles $S_{k_1}^n(i_1), S_{l_1}^n(i_1), S_{k_2}^n(i_2), S_{l_2}^n(i_2)$ are disjoint. The size of each edge of the triangles is at most 2. Since $i_1 \geq 1, i_2 \geq 1$, it holds $5i_1 > 2$ and $5i_2 > 2$. We can suppose that $i_1 < i_2$. Therefore $2(5i_1) \leq 5i_2$. Now, we can use Proposition 3.2.10 and the proposition follows. $\square$

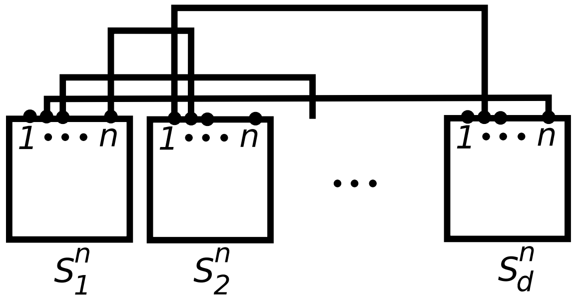

Figure 3.17: Top view on an example of the construction, the triangular tunnel bridges are depicted by lines connected to dots denoted by $1, \ldots, n$.

Corollary 3.2.5. *Triangular configuration $\Delta_B^{\mathcal{C}}$ can be embedded into $\mathbb{R}^3$.* □

An example of construction is depicted in Figure 3.17. To finish proof of Theorem 3.1.3, it remains to show that $\Delta_B^{\mathcal{C}}$ is geometric representation of $\mathcal{C}$. We prove that $\Delta_B^{\mathcal{C}}$ is indeed geometric representation of $\mathcal{C}$ in Subsection 3.2.2.

Proof of representability

We follow strategy described in Chapter 1 with the building blocks constructed in previous section. Before we state the proofs we introduce some definitions. In this section all operations are over the field $GF(2)$.

Let $\mathcal{C}$ be a binary linear code and let $B = \{b_1, \ldots, b_d\}$ be a basis of $\mathcal{C}$. Let $\Delta_B^{\mathcal{C}}$ be the geometric representation of $\mathcal{C}$ with respect to the basis B from Section 3.2.2 or Section 3.2.3. We suppose that $\Delta_B^{\mathcal{C}}$ exists. Let c be a codeword from $\mathcal{C}$. Then $c = \sum_{i \in I} b_i$. The *degree* of c with respect to the basis B is defined to be the cardinality $|I|$ of the index set. The degree is denoted by $d(c)$. Let $\Delta_{b_i}^{\mathcal{C}}$, $i = 1, \ldots, d$ be triangular configurations defined also in Section 3.2.2. We define a linear mapping $f\colon \mathcal{C} \mapsto \ker \Delta_B^{\mathcal{C}}$ in the following way: Let c be a codeword of $\mathcal{C}$ and let $c = \sum_{i \in I} b_i$ be the unique expression of c, where $b_i \in B$. We define $f(c) := \sum_{i \in I} \chi(\Delta_{b_i}^{\mathcal{C}})$. The entries of $f(c)$ are indexed by the triangles of $\Delta_B^{\mathcal{C}}$. We have $f(c)^{B_j^n} = 1$ if and only if $\triangle_{i \in I} T(\Delta_{b_i}^{\mathcal{C}})$ contains the triangle B_j^n.

Proposition 3.2.12. *Let m be the number of triangles of $\Delta_B^{\mathcal{C}}$. Let $c = (c^1, \ldots, c^n)$ and*

$$f(c) = \left(f(c)^{B_1^n}, \ldots, f(c)^{B_n^n}, f(c)^{n+1}, \ldots, f(c)^m\right).$$

Then $f(c)^{B_j^n} = c^j$ for all $j = 1, \ldots, n$ and all $c \in \mathcal{C}$.

Proof. We show the proposition by the induction on the degree $d(c)$ of c. The codeword c is equal to $\sum_{i \in I} b_i$. If $d(c) = 0$, then $c = 0$ and $f(c) = 0$. Thus, $f(c)$ is the characteristics vector of the empty triangular configuration. The proposition holds for vectors of degree 0. Suppose that $d(c)$ is greater than 0, then $|I| \geq 1$. We choose some k from I. The codeword $c + b_k$ has a degree less than c. By the induction assumption, the proposition holds for $c + b_k$. Let $b_k = (b_k^1, \ldots, b_k^n)$. From the definition of $\Delta_{b_k}^{\mathcal{C}}$, the equality $b_k^j = \chi(\Delta_{b_k}^{\mathcal{C}})^{B_j^n}$ holds for all $j = 1, \ldots, n$. Therefore,

$$c^j = (c^j + b_k^j) + b_k^j = \chi(\triangle_{i \in I \setminus \{k\}} \Delta_{b_i}^{\mathcal{C}})^{B_j^n} + \chi(\Delta_{b_k}^{\mathcal{C}})^{B_j^n} = f(c)^{B_j^n}$$

for all $j = 1, \ldots, n$. □

Corollary 3.2.6. *The mapping f is injective.* $\square$

Lemma 3.2.7. *Let E be a non-empty even subset of $\Delta_B^{\mathcal{C}}$. Then $\mathcal{K}(E)$ contains $\Delta_{b_i}^{\mathcal{C}} \setminus T(B^n)$ ($\Delta_{b_i}^{\mathcal{C}}$ with triangles of B^n removed) as a subconfiguration for some $i \in \{1, \ldots, d\}$.*

Proof. Triangular configuration $\mathcal{K}(E)$ contains either all triangles or no triangle of $\Delta_{b_i}^{\mathcal{C}} \setminus T(B^n)$, since all edges of $\Delta_{b_i}^{\mathcal{C}}$ incident with no triangle of B^n have degree equals 2, for $i = 1, \ldots, d$. The triangular configuration B^n have no non-empty cycle, since the triangles of B^n are disjoint. Hence, $\mathcal{K}(E)$ contains a triangle of $\Delta_{b_i}^{\mathcal{C}} \setminus T(B^n)$ for some $i \in \{1, \ldots, d\}$. Thus, $\mathcal{K}(E)$ contains $\Delta_{b_i}^{\mathcal{C}} \setminus T(B^n)$ for some $i \in \{1, \ldots, d\}$. $\square$

Theorem 3.2.3. *The mapping f defined above is a bijection between the binary linear code $\mathcal{C}$ and $\ker \Delta_B^{\mathcal{C}}$.*

Proof. By Corollary 3.2.6, the mapping f is injective. It remains to be proven that $\dim \mathcal{C} = \dim \ker \Delta_B^{\mathcal{C}}$. Suppose on the contrary that some codeword of $\ker \Delta_B^{\mathcal{C}}$ is not in the span of $\{f(b_1), \ldots, f(b_d)\}$. Let c be such a codeword with the minimal possible weight $w(c)$. The weight $w(c)$ means the number of non-zero coordinates of c. Let E be an even subset of $\Delta_B^{\mathcal{C}}$ such that $\chi(\mathcal{K}(E)) = c$. By Lemma 3.2.7, $\mathcal{K}(E)$ contains $\Delta_{b_i}^{\mathcal{C}} \setminus T(B^n)$ for some $i \in \{1, \ldots, d\}$. By definition of $\Delta_{b_i}^{\mathcal{C}}$, it holds $\left|T(\Delta_{b_i}^{\mathcal{C}} \setminus T(B^n))\right| > |T(B^n)|$. Therefore, the inequality $\left|E \triangle T(\Delta_{b_i}^{\mathcal{C}})\right| < |E|$ holds. Thus, $w(c) > w(\chi(\mathcal{K}(E \triangle T(\Delta_{b_i}^{\mathcal{C}}))))$. This is a contradiction.

$\square$

The entries of the vectors of $\ker \Delta_B^{\mathcal{C}}$ are indexed by triangles and the entries of vectors of $\mathcal{C}$ are indexed by integers, we make a convention that a coordinate of $\ker \Delta_B^{\mathcal{C}}$ indexed by triangle B_i^n corresponds to coordinate of $\mathcal{C}$ indexed by i. Now, we can state the following corollary.

Corollary 3.2.7. $\mathcal{C} = \ker \Delta_B^{\mathcal{C}} / (T(\Delta_B^{\mathcal{C}} \setminus T(B^n)))$ *and* $\dim \ker \Delta_B^{\mathcal{C}} = \dim \mathcal{C}$. $\square$

Thus, the triangular configuration $\Delta_B^{\mathcal{C}}$ is a geometric representation of $\mathcal{C}$.

3.2.3 Proof of Theorem 3.1.5 (Representation in $\mathbb{R}^4$)

In this section for every binary linear code we construct its geometric representation that can be embedded into $\mathbb{R}^4$.

Let $\mathcal{C}$ be a binary linear code of length n and let $B = \{b_1, \ldots, b_d\}$ be a basis of $\mathcal{C}$. For every basis vector b_i we construct triangular configuration Δ_{b_i} in this way: Let Q be a three dimensional cube of size $1 \times 1 \times 1$. We put in the middle of this cube the triangular configuration S^n defined in Section 3.2.2. We make an appropriate scaling of S^n such that S^n fits into the cube Q and put S^n into Q in the way depicted in Figure 3.18. The triangles $S^n(k)$, $k = 1, \ldots, n$ of S^n are in front. Let F be the front facet of Q (front in the Figure 3.18). We put triangles $\{B_1^n, \ldots, B_n^n\}$ to F as is depicted in Figure 3.18. Let b_i equals $(b_i^1, \ldots, b_i^n)$. We initially set $\Delta_{b_i} := S^n$. For every non-zero coordinate b_i^k we add tunnel (See Section 3.2.2) $T(S^n(k), B_k^n)$ between triangles $S^n(k)$ and B_k^n to Δ_{b_i}. Then we remove triangle $S^n(k)$ from Δ_{b_i}. An example of Δ_{b_i} for $b_i = (1, 1, \ldots, 0, 1)$ is depicted in Figure 3.18. The cube Q is not a part of Δ_{b_i}, it is important that Δ_{b_i} is embedded into Q. We denote this cube by $Q(\Delta b_i)$ and the facet F by $F(\Delta_{b_i})$.

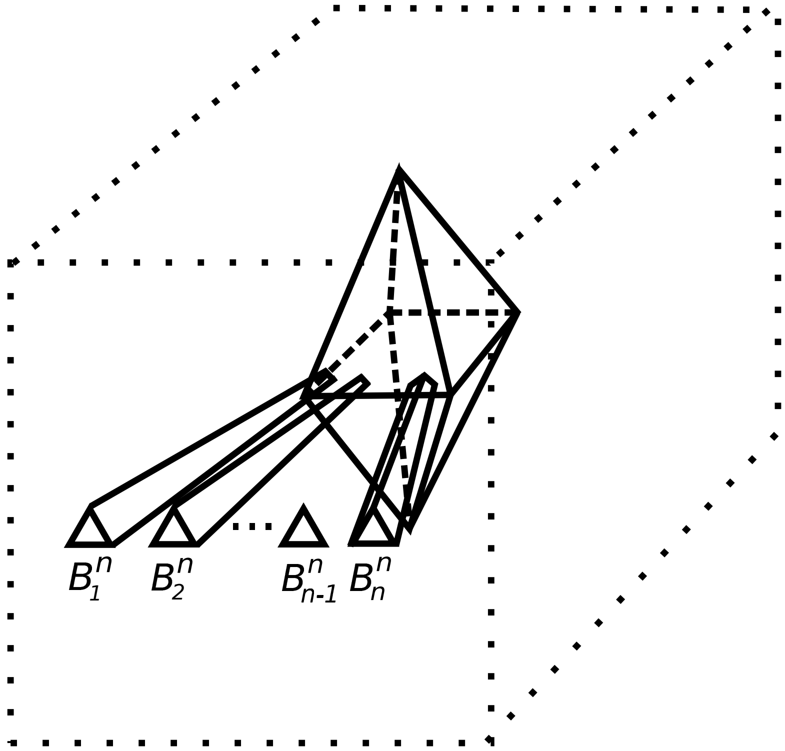

Figure 3.18: An example of Δ_{b_i} put into cube Q for $b_i = (1, 1, \ldots, 0, 1)$.

Proposition 3.2.13. *Let $Q_1, \ldots, Q_d$ be three dimensional cubes of the same size. Then the cubes can be embedded into $\mathbb{R}^4$ such that all cubes intersect at one facet and otherwise are disjoint.*

Proof. Fix a size l of the edges of the cubes. Let F be a square of size $l \times l$ embedded into $\mathbb{R}^4$. Let $v_1, \ldots, v_d$ be vectors of $\mathbb{R}^4$ of length l orthogonal to the square F such that every two vectors of $v_1, \ldots, v_d$ are linear independent. Such vectors exist in $\mathbb{R}^4$. Let Q_i be the cube defined as $\{f + \alpha v_i | f \in F, \alpha \in [0, 1]\}$. The cubes intersect at facet F: for a contradiction suppose that there are two cubes Q_i and Q_j such that $Q_i \cap Q_j \nsubseteq F$. Let x be a point of $(Q_i \cap Q_j) \setminus F$. Then $x = f_i + \alpha_i v_i = f_j + \alpha_j v_j$, where $f_i, f_j \in F$ and $\alpha_i, \alpha_j \in (0, 1]$. Since v_i is not a linear combination of v_j, the points f_i, f_j are different. The point $f_i + \alpha_i v_i - \alpha_j v_j$ belongs to F. Thus, the vector $\alpha_i v_i - \alpha_j v_j$ is parallel to F. Since f_i, f_j are different, we have $\alpha_i v_i - \alpha_j v_j \neq 0$. Since the vector $\alpha_i v_i - \alpha_j v_j$ is a linear combination of two vectors v_i, v_j orthogonal to F, the vector $\alpha_i v_i - \alpha_j v_j$ is also orthogonal to F. Thus, the vector $\alpha_i v_i - \alpha_j v_j$ is non-zero and orthogonal to itself. This is impossible in $\mathbb{R}^4$, a contradiction. The proposition follows. $\square$

By Proposition 3.2.13, we can embed the cubes $Q(\Delta b_1), \ldots, Q(\Delta b_d)$ with $\Delta_{b_1}, \ldots, \Delta_{b_d}$ into $\mathbb{R}^4$ such that all cubes $Q(\Delta b_1), \ldots, Q(\Delta b_d)$ intersect on the facets $F(\Delta_{b_1}), \ldots, F(\Delta_{b_d})$ and otherwise are disjoint and the triangular configurations $\Delta_{b_1}, \ldots, \Delta_{b_d}$ intersects on the triangles B^n and otherwise are disjoint. The resulting triangular configuration is the geometric representation $\Delta_B^{\mathcal{C}}$ of $\mathcal{C}$ embedded into $\mathbb{R}^4$. An example of the representation of a binary linear code that is generated by two basis vectors b_1, b_2 is depicted in Figure 3.19.

The proof that $\Delta_B^{\mathcal{C}}$ is indeed geometric representation of $\mathcal{C}$ is the same as the proof in Subsection 3.2.2. This completes the proof of Theorem 3.1.5.

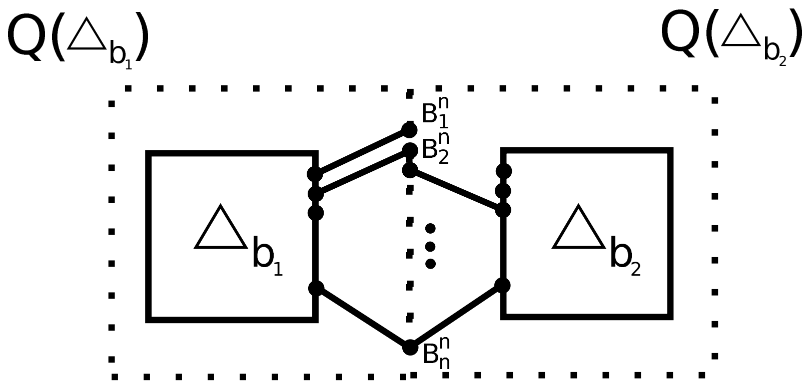

Figure 3.19: Top view on an example of the representation of a two dimensional code in $\mathbb{R}^4$

3.2.4 Proof of Theorem 3.1.4

Proposition 3.2.14. *Let $\mathcal{C}$ be a binary linear code. Then $\mathcal{C}$ has a 2-basis if and only if there is a graph G such that $\mathcal{C}$ is equal to the cut space of G.*

Proof. First, we prove that every binary linear code with a 2-basis is a cut space of a graph possibly with loops and parallel edges. Let $\mathcal{C}$ be a binary linear code of length n with a 2-basis $B = \{b_1, \ldots, b_d\}$. We define a graph $G = (V, E)$ possibly with parallel edges and loops as follows: We define the set of vertices V as:

$$V := B \cup \{u\}.$$

For $i = 1, \ldots, n$; we define edge e_i as follows: If all basis codewords of B have the entry indexed by coordinate i equals to zero, we set e_i to be a loop (u, u). If there is exactly one basis codeword $b_l \in B$ that has non-zero entry indexed by i, we set e_i to be (b_l, v). If there are exactly two basis codewords $b_l, b_k \in B$ that have non-zero entry indexed by i, we set e_i to be (b_l, b_k). Then the set of edges E of G is $\{e_i | i = 1, \ldots, n\}$.

Let $E(v)$ be the set of edges incident with a vertex v. Let E' be a subset of E. We define the incidence vector of E' is $\chi(E') := (\chi(E')_1, \ldots, \chi(E')_n)$, where $\chi(E')_i = 1$ if $e_i \in E'$ and $\chi(E')_i = 0$ otherwise. By definition, the set $B' := \{\chi(E(v)) | v \in (V \setminus \{u\})\}$ equals B. It is known fact that the set B' generates the cut space of G, for a proof see for example Diestel [4].

Now, we prove the reverse implication. Let G be a graph and let u be a vertex of G. Then the set $B' := \{\chi(E(v)) | v \in (V \setminus \{u\})\}$ is a basis of the cut space of G. Since every edge of G is incident at most with two vertices, the set B' is a 2-basis of the cut space of G. ☐

Now, we finish the proof of Theorem 3.1.4.

Proof of Theorem 3.1.4. By proposition 3.2.14, a binary linear code $\mathcal{C}$ has 2-basis if and only if $\mathcal{C}$ is a cut space of a graph. By Theorem 3.1.3, code $\mathcal{C}$ has representation in $\mathbb{R}^3$ if and only if it has a 2-basis. If code $\mathcal{C}$ has no 2-basis. Then from Theorem 3.1.5 follows that $\mathcal{C}$ has representation in $\mathbb{R}^4$. ☐

Proof of Corollary 3.1.1. It is a known fact that there is polynomial algorithm that decide if a given binary linear code is a cut space of a graph (See Seymour [28]). ☐

Proposition 3.2.15. *Let G be a non-planar graph. Then the cycle space of G has no 2-basis.*

Proof. Follows from the Mac Lane's planarity criterion. See Mac Lane [18] or O'Neil [20]. $\square$

4. Geometric representations of linear codes

4.1 Introduction

The aim of this chapter is to introduce a theory of geometric representations of general linear codes. A seminal result of Galluccio and Loebl [7] asserts that the Ising partition function on graph G may be written as a linear combination of $4^{g(G)}$ Pfaffians, where $g(G)$ is the minimal genus of the closed Riemann surface in which G can be embedded. Recently, a topological interpretation of this result was given by Cimasoni and Reshetikhin [2]. We explain in Section 4.1.1 that the Ising partition function on graph G may be described as the weight enumerator of the cycle space $\mathcal{C}$ of G. Viewing the cycle space $\mathcal{C}$ as a linear code over $GF(2)$, a graph G may be considered as a useful geometric representation of $\mathcal{C}$ which provides an important structure for the weight enumerator of $\mathcal{C}$, see Theorem 4.1.1. This motivated Martin Loebl to ask, about more than 15 years ago, the following question: Which binary linear codes are cycle spaces of simplicial complexes? In general, for the linear codes with a geometric representation, one may hope to obtain a formula analogous to that of Theorem 4.1.1. This question remains open. However, to extend the theory of Pfaffian orientations, it suffices to construct a geometric representation which carries over the weight enumerator only. This was achieved in Chapter 1 for binary linear codes. In this chapter we present results for general linear codes. By another result of Galluccio and Loebl [8], the q-Potts partition function of graph G is determined by the row space of the oriented incidence matrix O_G of graph G over $GF(q)$. The row space of O_G is a linear code, and so one surprising application of our results is a new formula for the q-Potts partition function, where q is a prime.

We start with basic definitions. A *linear code* $\mathcal{C}$ *of length* n *and dimension* d *over a field* $\mathbb{F}$ is a linear subspace with dimension d of the vector space $\mathbb{F}^n$. Each vector in $\mathcal{C}$ is called a *codeword*. We define a partial order on $\mathcal{C}$ as follows: Let $c = (c^1, \ldots, c^n), d = (d^1, \ldots, d^n)$ be codewords of $\mathcal{C}$. Then $c \preceq d$ if $c^i \neq 0$ implies $d^i \neq 0$ for all $i = 1, \ldots, n$. A codeword d is *minimal* if $c \preceq d$ implies $c = d$ for all c. The *weight* $w(c)$ of a codeword c is the number of non-zero entries of c. The *weight enumerator* of a finite code $\mathcal{C}$ is defined according to the formula

$$W_{\mathcal{C}}(x) := \sum_{c \in \mathcal{C}} x^{w(c)}.$$

Let $\mathcal{C} \subseteq \mathbb{F}^n$ be a linear code over a field $\mathbb{F}$ and let S be a subset of $\{1, \ldots, n\}$. *Puncturing* a code $\mathcal{C}$ along S means deleting the entries indexed by the elements of S from each codeword of $\mathcal{C}$. The resulting code is denoted by $\mathcal{C}/S$.

A *simplex* X is the convex hull of an affine independent set V in $\mathbb{R}^d$. The *dimension* of X is $|V| - 1$, denoted by $\dim X$. The convex hull of any non-empty subset of V that defines a simplex is called a *face* of the simplex. A *simplicial complex* Δ is a set of simplices fulfilling the following conditions: Every face of a simplex from Δ belongs to Δ and the intersection of every two simplices of Δ is a face of both. The dimension of Δ is $\max\{\dim X | X \in \Delta\}$. Let Δ be a

d-dimensional simplicial complex. We define the *incidence matrix* $A = (A_{ij})$ as follows: The rows are indexed by $(d-1)$-dimensional simplices and the columns by d-dimensional simplices. We set

$$A_{ij} := \begin{cases} 1 & \text{if } (d-1)\text{-simplex } i \text{ belongs to } d\text{-simplex } j, \\ 0 & \text{otherwise.} \end{cases}$$

This chapter studies 2-dimensional simplicial complexes where each maximal simplex is a triangle or an edge. We call them *triangular configurations.* The *cycle space* of Δ over a field $\mathbb{F}$, denoted $\ker \Delta$, is the kernel of the incidence matrix A of Δ over $\mathbb{F}$, that is $\{x|Ax = 0\}$. A linear code $\mathcal{C}$ is *triangular representable* if there exists a triangular configuration Δ such that $\mathcal{C} = \ker \Delta / S$ for some set S and there is a linear mapping between $\mathcal{C}$ and $\ker \Delta$ which is a bijection and maps minimal codewords to minimal codewords. For such S we write $S = S(\ker \Delta, \mathcal{C})$.

4.1.1 Motivation

Our motivation to study geometric representations of linear codes lies in the theory of Pfaffian orientation in the study of the Ising problem on graphs. In this section we use the notation from Loebl and Masbaum [16]. Let $G = (V(G), E(G))$ be a finite unoriented graph. We say that $E' \subseteq E(G)$ is *even* if the graph $(V(G), E')$ has even degree at each vertex. Let $\mathcal{E}(G)$ denote the set of all even sets of edges of G. The set of all even sets $\mathcal{E}(G)$ is called cycle space of graph G. Note that, graph G is 1-dimensional simplicial complex. Let A_G be its incidence matrix, then the set of characteristic vectors of the even sets forms the kernel of A_G over $GF(2)$.

We assume that a variable x_e is associated with each edge e, and define the generating polynomial for even sets, $\mathcal{E}_G$, in $\mathbb{Z}[(x_e)_{e \in E(G)}]$, as follows:

$$\mathcal{E}_G(x) = \sum_{E' \in \mathcal{E}(G)} \prod_{e \in E'} x_e \ .$$

Let $\mathcal{C}$ be the kernel of the incidence matrix A_G of graph G. Then there is the following relation between the weight enumerator of $\mathcal{C}$ and the generating polynomial of even sets of G

$$W_{\mathcal{C}}(x) = \mathcal{E}_G(z)\Big|_{z_e \,:=\, x \ \forall e \in E(G)}.$$

Next, we describe the equivalence of Ising partition function and the generating function of even subgraphs. The Ising partition function on graph G is defined by

$$Z_G^{\text{Ising}}(\beta) = Z_G^{\text{Ising}}(x)\Big|_{x_e \,:=\, e^{\beta J_e} \ \forall e \in E(G)}$$

where the J_e $(e \in E(G))$ are weights (coupling constants) associated with the edges of the graph G, the parameter β is the inverse temperature, and

$$Z_G^{\text{Ising}}(x) = \sum_{\sigma : V(G) \to \{1,-1\}} \ \prod_{e = \{u,v\} \in E(G)} x_e^{\sigma(u)\sigma(v)}.$$

The theorem of van der Warden [32] (see [15, Section 6.3] for a proof) states that $Z_G^{\text{Ising}}(x)$ is the same as $\mathcal{E}_G(x)$ up to change of variables and multiplication by a constant factor:

$$Z_G^{\text{Ising}}(x) = 2^{|V(G)|}\left(\prod_{e\in E(G)} \frac{x_e + x_e^{-1}}{2}\right)\mathcal{E}_G(z)\Big|_{z_e := \frac{x_e - x_e^{-1}}{x_e + x_e^{-1}}}.$$

The significance of geometric properties of graph G in studying the Ising partition function $Z_G^{\text{Ising}}(x)$ and the generating function of the even subsets of edges of G, $\mathcal{E}_G(x)$, is expressed by the following theorem.

Theorem 4.1.1 (Galluccio and Loebl [7])**.** *If G embeds into an orientable surface of genus g, then the even subgraph polynomial $\mathcal{E}_G(x)$ can be expressed as a linear combination of 4^g Pfaffians of matrices constructed from the embedding of G.*

4.1.2 Main results

Theorem 4.1.2. *Let $\mathcal{C}$ be a linear code over a field $\mathbb{F}$, $\mathcal{C} \subseteq \mathbb{F}^n$, and let B be a basis of $\mathcal{C}$ such that for each $b \in B$, each entry of b belongs to a cyclic subgroup $G(b)$ of the additive group of the field $\mathbb{F}$. Then $\mathcal{C}$ is triangular representable. Moreover, if $\mathcal{C}$ is finite, then there exists a triangular representation Δ such that: if $\sum_{i=0}^m a_i x^i$ is the weight enumerator of* $\ker\Delta$ *then*

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{m} a_i x^{(i \bmod e)/2},$$

where $e = (|S(\ker\Delta, \mathcal{C})| - n)/\dim\mathcal{C}$.

Corollary 4.1.1. *The conclusion of Theorem 4.1.2 holds for the linear codes over rationals and over $GF(p)$, where p is a prime.*

Corollary 4.1.1 was proved for $GF(2)$ in Chapter 1. On the other hand, we have the following negative results for the remaining fields.

Theorem 4.1.3. *Let $\mathcal{C}$ be a linear code over a field $\mathbb{F}$ such that every basis B of $\mathcal{C}$ contains a vector b so that its entries f, p do not belong to the same cyclic subgroup of the additive group of $\mathbb{F}$. Then code $\mathcal{C}$ is not triangular representable.*

Corollary 4.1.2. *Let $\mathbb{F}$ be a field different from rationals and $GF(p)$, where p is a prime. Then there exists a linear code over $\mathbb{F}$ that is not triangular representable.*

In Chapter 1, for every triangular configuration Δ, we constructed a triangular configuration Δ' such that there exists a bijection between the cycle space over $GF(2)$ of Δ and the set of the perfect matchings of Δ'. We conjecture that this result can be extended to other finite fields. Finally, see Section 4.3.2 for the multivariate versions of the above theorems.

4.1.3 Potts model

We give an application (suggested by Martin Loebl) of our results to the Potts partition function. The *q-Potts partition function* of a graph G is defined according to the formula

$$P^q(G,x) = \sum_{s:V\mapsto\{0,\dots,q-1\}} x^{H(s)}.$$

The Hamiltonian $H(s)$ is defined as $\sum_{uv\in E} w(uv)\delta(s(u),s(v))$ where δ is Kronecker delta and $w(uv)$ is the weight of edge uv. The Potts partition function is one of the extensively studied functions in statistical physics; see the book by Loebl [15]. We note that 2-Potts partition function is equivalent to the Ising partition function and that $P^q(G,x)$ is also called the multivariate Tutte polynomial of G; see the survey by Sokal [29].

Theorem 4.1.4. *Let G be a connected graph such that all edges of G have the same weight w. Let q be a prime. Then there exists triangular configuration Δ such that if $\sum_{i=0}^m a_i x^i$ is the weight enumerator of* $\ker\Delta$ *then*

$$P^q(G,x) = \frac{q}{\prod_{uv\in E} x^{-w}} \sum_{i=0}^{m} a_i x^{-w((i \bmod e)/2)},$$

where e is a positive integer linear in the number of edges of G.

The case of different weights is treated in Section 4.3.2.

4.2 Triangular representations

In this section all computations are done over a fixed field $\mathbb{F}$. Let Δ_1, Δ_2 be subconfigurations of a triangular configuration Δ. We denote the set of vertices of Δ by $V(\Delta)$, the set of edges by $E(\Delta)$ and the set of triangles by $T(\Delta)$. The *difference* of Δ_1 and Δ_2, denoted by $\Delta_1 - \Delta_2$, is defined to be the triangular configuration obtained from $V(\Delta_1) \cup E(\Delta_1) \cup T(\Delta_1) \setminus T(\Delta_2)$ by removing the edges and vertices that are not contained in any triangle in $T(\Delta_1) \setminus T(\Delta_2)$. An *abstract simplicial complex* on a finite set V is a family Δ of subsets of V closed under taking subsets. Let X be an element of Δ. The *dimension* of X is $|X|-1$, denoted by $\dim X$. The dimension of Δ is $\max\{\dim X | X \in \Delta\}$. Every simplicial complex defines an abstract simplicial complex on the set of vertices V, namely the family of sets of vertices of simplexes of Δ. We denote this abstract simplicial complex by $\mathcal{A}(\Delta)$. The *geometric realization* of an abstract simplicial complex Δ is a simplicial complex Δ' such that $\Delta = \mathcal{A}(\Delta')$. It is well known that every finite d-dimensional abstract simplicial complex can be realized as a simplicial complex in $\mathbb{R}^{2d+1}$. We choose a geometric realization of an abstract simplicial complex Δ and denote it by $\mathcal{G}(\Delta)$. Let Δ_1, Δ_2 be triangular configurations. The union of Δ_1, Δ_2 is defined to be $\Delta_1 \cup \Delta_2 := \mathcal{G}(\mathcal{A}(\Delta_1) \cup \mathcal{A}(\Delta_2))$. We start with construction of basic building blocks.

4.2.1 Triangular configuration B^n

The triangular configuration B^n consists of n disjoint triangles as is depicted in Figure 4.1. We denote the triangles of B^n by $B^n_1, \dots, B^n_n$.

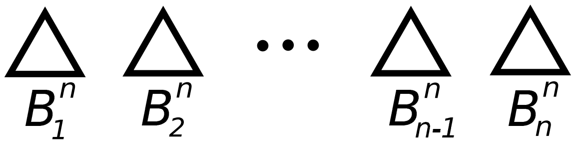

Figure 4.1: Triangular configuration B^n.

4.2.2 Orientation

An *oriented triangular configuration* $\vec{\Delta}$ is a triangular configuration Δ with an assignment $o : T(\Delta) \mapsto \{+,-\}$ of signs to triangles. We denote the subset of triangles $\{t \in T(\Delta) | o(t) = +\}$ by $\left[\vec{\Delta}\right]_+$ and $\{t \in T(\Delta) | o(t) = -\}$ by $\left[\vec{\Delta}\right]_-$.

4.2.3 Oriented tunnel $\vec{T}$

An *empty triangle* is a set of three edges forming a boundary of a triangle.

The *oriented tunnel* $\vec{T}$ is defined by Figure 4.2. The oriented tunnel has

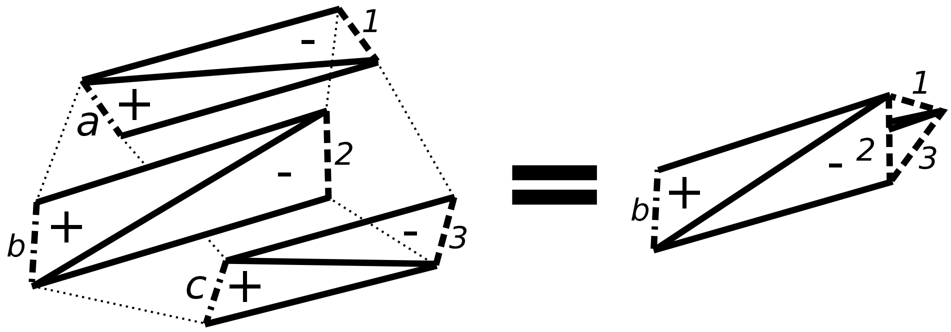

Figure 4.2: Oriented triangular tunnel $\vec{T}$.

two empty triangles $\{a,b,c\}$ and $\{1,2,3\}$. The empty triangle $\{a,b,c\}$ is called *positive end* of the tunnel, and the empty triangle $\{1,2,3\}$ is called *negative end*.

A *tunnel* is obtained from oriented tunnel by deleting the signs.

Proposition 4.2.1. *Let T be a tunnel and let $v = (v^t)_{t \in T(T)}$ be a vector whose entries are indexed by triangles of T satisfying: For each inner edge e, $\sum_{e \in t} v^t = 0$. Then $v^{t_a} = v^{t_b} = v^{t_c}$ and $v^{t_1} = v^{t_2} = v^{t_3}$, where for $f \in \{a,b,c,1,2,3\}$, t_f denotes the triangle of T containing the edge f. Moreover, $v^{t_a} = -v^{t_1}$.* $\square$

4.2.4 Linking triangles by oriented tunnel

Let $\vec{\Delta}$ be an oriented triangular configuration. Let t_1 and t_2 be two edge disjoint triangles of $\vec{\Delta}$.

The *link* from t_1 to t_2 in $\vec{\Delta}$ is the oriented triangular configuration $\vec{\Delta}'$ defined as follows. Let $\vec{T}$ be an oriented triangular tunnel as in Figure 4.2. Let t_1^1, t_1^2, t_1^3 and t_2^1, t_2^2, t_2^3 be edges of t_1 and t_2, respectively. We relabel edges of $\vec{T}$ such that $\{a,b,c\} = \{t_1^1, t_1^2, t_1^3\}$ and $\{1,2,3\} = \{t_2^1, t_2^2, t_2^3\}$. We let $\vec{\Delta}' := \vec{\Delta} \cup \vec{T}$.

Note that, the link has a direction, the triangle t_1 is incident with the positive end of $\vec{T}$ in $\vec{\Delta}'$, and t_2 is incident with the negative end of $\vec{T}$ in $\vec{\Delta}'$.

Analogously, we define linking from t_1 to t_2 of edge disjoint t_1, t_2 where at least one of t_1, t_2 is an empty triangle of $\vec{\Delta}$.

Further, if Δ is (non-oriented) triangular configuration and t_1, t_2 are edge disjoint triangles, then link from t_1 to t_2 is defined as above, but replacing oriented tunnel by tunnel.

4.2.5 Triangular configuration $\Delta_B^{\mathcal{C}}$

Let $\mathcal{C}$ be a linear code of length n and dimension d over a field $\mathbb{F}$. Let $B = \{b_1, \ldots, b_d\}$ be a basis of $\mathcal{C}$. We say that a basis B is *representable* if for each $b \in B$ each entry of b belongs to a cyclic subgroup $G(b)$ of the additive group of the field $\mathbb{F}$. In this section we suppose that the linear code $\mathcal{C}$ has a representable basis B. This is used in the key Proposition 4.2.2 bellow. We construct the triangular configuration $\Delta_B^{\mathcal{C}}$ as follows. First, for every basis vector b we construct a triangular configuration Δ_b^{C}. Then, the triangular configuration $\Delta_B^{\mathcal{C}}$ is the union of $\Delta_b^{\mathcal{C}}$, $b \in B$.

Triangular configuration $\Delta_b^{\mathcal{C}}$

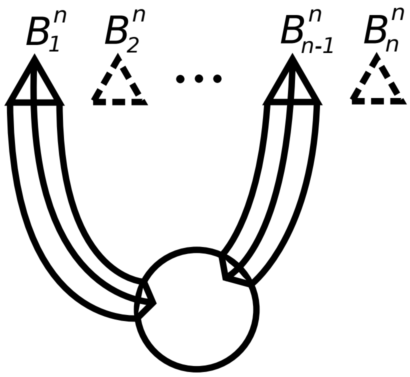

Figure 4.3: $\Delta_b^{\mathcal{C}}$ represents a basis vector $(b^1, 0, \ldots, b^{n-1}, 0)$ of $\mathcal{C}$.

Proposition 4.2.2. *Let $\mathbb{F}$ be a field. Let $a_1, a_2, \ldots, a_k$ be a subset of distinct non-zero elements of a cyclic subgroup G of the additive group of $\mathbb{F}$. Let n be a positive integer. Then there exists a triangular configuration $\mathcal{M}$ such that* $\dim \ker \mathcal{M} = 1$ *and there exists a vector $v \in \ker \mathcal{M}$ having $a_1, a_2, \ldots, a_k$ among its entries. Moreover, the vector v contains each entry a_i at least n times and v is non-zero on all entries.*

The proof is postponed to Section 4.2.7. We recall that b is a basis vector of a representable basis B and that the length of $\mathcal{C}$ is n. Let $b = (b^1, b^2, \ldots, b^n)$ and let $a_1, a_2, \ldots, a_k$ be all different entries of b. Let $\mathcal{M}(b)$ and $v(b) = (v^t)_{t \in T(\mathcal{M}(b))}(b)$ be a triangular configuration and a vector provided by Proposition 4.2.2. Next, we construct $\Delta_b^{\mathcal{C}}$. We proceed in three steps. In the first step, the construction starts from $B^n \cup \mathcal{M}(b)$. See Section 4.2.1 for definition of B^n.

In the second step, we make the following tunnels. Let $J(b)$ be the set of indices of non-zero entries of b. Let g be an injection $g : J(b) \mapsto T(\mathcal{M}(b))$ such that $\forall j \in J(b), v(b)^{g(j)} = b^j$. We note that g exists since the multiplicity of each a_i in b is at most n. For each $j \in J(b)$ we create link by tunnel from the triangle $g(j)$ to the triangle B^n_j.

In the third step, we remove the triangles $g(j)$, $j \in J(b)$, from $\Delta^{\mathcal{C}}_b$. An example of $\Delta^{\mathcal{C}}_b$ for $b = (b^1, 0, \dots, b^{n-1}, 0)$ is depicted in Figure 4.3.

Proposition 4.2.3. *The dimension of* $\ker \Delta^{\mathcal{C}}_b$ *is equal to* 1*. Moreover, there exists a generator* $u(b)$ *of* $\ker \Delta^{\mathcal{C}}_b$ *such that* $u(b)^{B^n_j} = b^j$ *for* $j = 1, \dots, n$ *and* $u(b)^t \neq 0$ *for all* $t \in T(\Delta^{\mathcal{C}}_b - B^n)$*.*

Proof. By Proposition 4.2.2, the dimension of $\ker \mathcal{M}(b)$ is 1. The triangular configuration $\Delta^{\mathcal{C}}_b$ is obtained from $\mathcal{M}(b) \cup B^n$ by linking some disjoint triangles of B^n by tunnel to some triangles of $\mathcal{M}(b)$ and by removing triangles of $\mathcal{M}(b)$ that are linked by a tunnel. Hence, the dimension of $\ker \Delta^{\mathcal{C}}_b$ is 1.

Let u be a vector from $\ker \mathcal{M}(b)$ given by Proposition 4.2.2. If entry b^j is non-zero, in construction of $\mathcal{M}(b)$ we link the triangle B^n_j with a triangle t of $T(\mathcal{M}(b))$ such that $u^t = b_j$ and then we remove triangle t. Therefore by Proposition 4.2.1, we can extend vector u to $u(b)$ such that $u(b)^{B^n_j} = b^j$. If entry b^k is zero, then triangle B^n_j is isolated. Therefore, $u(b)^{B^n_j} = 0$.

The vector u given by Proposition 4.2.2, is non-zero on all entries. By Proposition 4.2.1, all entries indexed by triangles of tunnels linked to $\mathcal{M}(b)$ are non-zero. Hence, $u(b)^t \neq 0$ for all $t \in T(\Delta^{\mathcal{C}}_b - B^n)$. $\square$

Construction of $\Delta^{\mathcal{C}}_B$

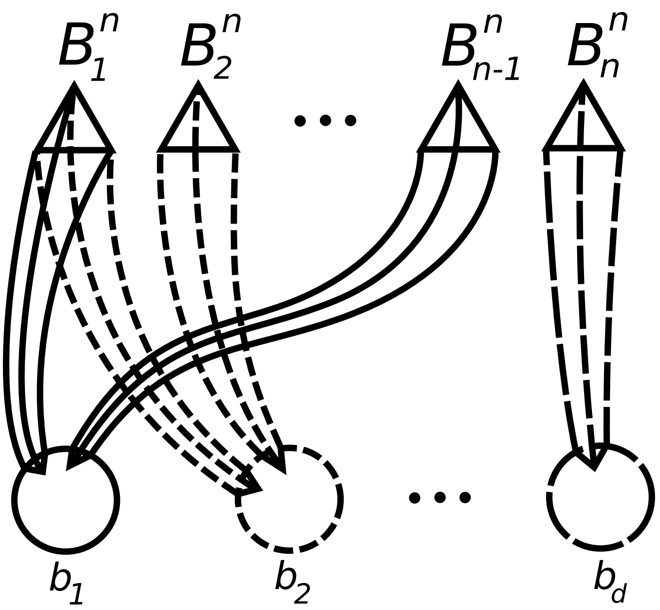

Figure 4.4: An example of a triangular configuration $\Delta^{\mathcal{C}}_B$, where $B = \{b_1, \dots, b_d\}$.

Triangular configurations $\Delta^{\mathcal{C}}_b$, $b \in B$, share only triangles of B^n. Hence, $\mathcal{A}(\Delta^{\mathcal{C}}_b) \cap \mathcal{A}(\Delta^{\mathcal{C}}_{b'}) = \mathcal{A}(B_n)$ holds for $b \neq b'$; $b, b' \in B$.

The triangular configuration $\Delta_B^{\mathcal{C}}$ is the union of $\Delta_b^{\mathcal{C}}$, $b \in B$. An example of a triangular configuration $\Delta_B^{\mathcal{C}}$ is depicted in Figure 4.4.

We define the following vectors of $\ker \Delta_B^{\mathcal{C}}$. The vector $\mathrm{gen}(\Delta_b^{\mathcal{C}})$ is defined as $\mathrm{gen}(\Delta_b^{\mathcal{C}})^t := u(b)^t$ for $t \in T(\Delta_b^{\mathcal{C}})$ and $\mathrm{gen}(\Delta_b^{\mathcal{C}})^t := 0$ for $t \in T(\Delta_B^{\mathcal{C}} - \Delta_b^{\mathcal{C}})$. As a corollary of Proposition 4.2.3 we get:

Corollary 4.2.1. *Each entry of vector* $\mathrm{gen}(\Delta_b^{\mathcal{C}})$ *is non-zero on each entry indexed by a triangle of* $\Delta_b^{\mathcal{C}} - B^n$. *Moreover, the vectors* $\mathrm{gen}(\Delta_b^{\mathcal{C}})$, $b \in B$, *are linearly independent.* □

Definition 4.2.1. We define a linear mapping $f\colon \mathcal{C} \mapsto \ker \Delta_B^{\mathcal{C}}$ in the following way: Let c be a codeword of $\mathcal{C}$ and let $c = \sum_{b\in B} \alpha_b b$ be the unique expression of c, where $\alpha_b \in \mathbb{F}$. We define $f(c) := \sum_{b\in B} \alpha_b \,\mathrm{gen}(\Delta_b^{\mathcal{C}})$. The entries of $f(c)$ are indexed by the triangles of $\Delta_B^{\mathcal{C}}$. Let $R = \{1, \dots, n\}$ be the set of coordinates of $\mathcal{C}$. We define an injection $\mu : R \mapsto T(\Delta)$ as: $\mu(i) = B_i^n$ for $i = 1, \dots, n$.

Proposition 4.2.4. *Denote* $\left|T\left(\Delta_B^{\mathcal{C}}\right)\right|$ *by* m. *Let* $c = (c^1, \dots, c^n)$ *and*

$$f(c) = \left(f(c)^{B_1^n}, \dots, f(c)^{B_n^n}, f(c)^{n+1}, \dots, f(c)^m\right).$$

Then $f(c)^{B_j^n} = c^j$ *for all* $j = 1, \dots, n$ *and all* $c \in \mathcal{C}$.

Proof.

$$f(c)^{B_j^n} = \sum_{b\in B} \alpha_b \,\mathrm{gen}(\Delta_b^{\mathcal{C}})^{B_j^n}.$$

By Proposition 4.2.3 and by definition of $\mathrm{gen}(\Delta_b^{\mathcal{C}})$,

$$\sum_{b\in B} \alpha_b \,\mathrm{gen}(\Delta_b^{\mathcal{C}})^{B_j^n} = \sum_{b\in B} \alpha_b b^{B_j^n} = c^j.$$

□

Corollary 4.2.2. *The linear mapping* f *is injective.* □

Lemma 4.2.1. *For each non-zero vector* w *of* $\ker \Delta_B^{\mathcal{C}}$ *there exists* $b \in B$ *and* $\gamma_b \neq 0$ *such that* $w^t = \gamma_b \,\mathrm{gen}(\Delta_b^{\mathcal{C}})^t$ *for all* $t \in T(\Delta_b^{\mathcal{C}} - B^n)$.

Proof. The kernel $\ker B^n$ is $\emptyset$, since the triangles of B^n are disjoint. Therefore, every non-zero vector $w \in \ker \Delta^{\mathcal{C}}$ has a non-zero element $w^t \neq 0$ indexed by a triangle $t \in T(\Delta_b^{\mathcal{C}} - B^n)$ for some $b \in B$.

Let j be an index such that $b^j \neq 0$. Let t_1, t_2, t_3 be three triangles of $\Delta_b^{\mathcal{C}}$ touching edges of B_j^n. Then by proposition 4.2.1, $w^{t_1} = w^{t_2} = w^{t_3}$. By Proposition 4.2.3, the dimension of $\Delta_b^{\mathcal{C}}$ is 1. Hence, there is a non-zero scalar γ_b such that $w^t = \gamma_b \,\mathrm{gen}(\Delta_b^{\mathcal{C}})^t$ for all $t \in T(\Delta_b^{\mathcal{C}} - B^n)$.

□

Theorem 4.2.2. *Let* $\mathcal{C}$ *be a linear code with a representable basis* B *and let* $\Delta_B^{\mathcal{C}}$ *be the triangular configuration from this section. Then* $\mathcal{C} = \ker \Delta_B^{\mathcal{C}}/S$, *where* S *is a set of indices. Moreover, the linear mapping* f *defined above is a bijection between the linear code* $\mathcal{C}$ *and* $\ker \Delta_B^{\mathcal{C}}$ *which maps minimal codewords to minimal codewords.*

Proof. By Proposition 4.2.4, the code $\mathcal{C}$ equals $\ker \Delta_B^{\mathcal{C}}/S$, where S is the set of triangles of $\Delta_B^{\mathcal{C}} - B^n$.

By Corollary 4.2.2, the mapping f is injective. It remains to be proven that $\dim \mathcal{C} = \dim \ker \Delta_B^{\mathcal{C}}$. We show that every codeword w of $\ker \Delta_B^{\mathcal{C}}$ is in the linear span of $\{f(b)|b \in B\}$. Let $B(w)$ be the following set of basis vectors $\{b \in B | \exists t \in T(\Delta_b^{\mathcal{C}} - B^n)$ such that $w^t \neq 0\}$. By Lemma 4.2.1, the set $B(w)$ is not empty. By Lemma 4.2.1, vector $w - \sum_{b \in B(w)} \gamma_b \operatorname{gen}(\Delta_b^{\mathcal{C}})$ is non-zero only on coordinates indexed by triangles of B^n. Since $\ker B^n = \emptyset$, vector $w - \sum_{b \in B(w)} \gamma_b \operatorname{gen}(\Delta_b^{\mathcal{C}})$ is 0. Hence, the vector w is in the span of $\{f(b_1), \dots, f(b_d)\}$.

Finally, we show that f maps minimal codewords to minimal codewords. Recall a partial order on $\mathcal{C}$. Let $r = (r^1, \dots, r^n), s = (s^1, \dots, s^n)$ be codewords of $\mathcal{C}$. Then $r \preceq s$ if $r^i \neq 0$ implies $s^i \neq 0$ for all $i = 1, \dots, n$. A codeword s is *minimal* if $r \preceq s$ implies $r = s$ for all r. Let s be a minimal codeword. Suppose on the contrary that $f(s)$ is not a minimal codeword of $\ker \Delta_B^{\mathcal{C}}$. Then $f(r) \prec f(s)$ for some codeword r. Therefore, for all $j = 1, \dots, n$; $f(r)^{B_j^n} \neq 0$ implies that $f(s)^{B_j^n} \neq 0$. By Proposition 4.2.4, $r^j = f(r)^{B_j^n}$ and $s^j = f(s)^{B_j^n}$, for all $j = 1, \dots, n$. Hence, $r^j \neq 0$ implies that $s^j \neq 0$, for all $j = 1, \dots, n$. Thus, $r \prec s$. This is a contradiction. □

4.2.6 Balanced triangular configuration $\Delta_B^{\mathcal{C}}$

A triangular configuration $\Delta_B^{\mathcal{C}}$ is *balanced* if there is an integer e such that $\left|T\left(\Delta_b^{\mathcal{C}}\right)\right| - w(b) = e$ for all $b \in B$. This e is denoted by $e(\Delta_B^{\mathcal{C}})$. A code $\mathcal{C}$ is *even* if all codewords have an even weight. The following proposition is straightforward.

Proposition 4.2.5. $e(\Delta_B^{\mathcal{C}}) = \left|T\left(\Delta_B^{\mathcal{C}} - B^n\right)\right| / \dim \mathcal{C}$ □

Proposition 4.2.6. *Let C be an even linear code with a representable basis $B = \{b_1, \dots, b_d\}$. Let n be an integer. Then there exists a balanced triangular configuration $\Delta_B^{\mathcal{C}}$ such that $n < e(\Delta_B^{\mathcal{C}})$.*

Proof. Let $\Delta_B^{\mathcal{C}}$ be the triangular configuration from Section 4.2.5. Let $k_i = \left|T\left(\Delta_{b_i}^{\mathcal{C}}\right)\right| - w(b_i)$ for $i = 1, \dots, d$. We suppose that $k_1 \geq k_2 \geq \dots \geq k_d$, if not, we rename the indices. Every k_i is even, since $\mathcal{C}$ is even.

We expand the parts of $\Delta_B^{\mathcal{C}}$ by the following algorithm. First, we define two steps. Let Δ be a triangular configuration. Step A: We choose a triangle of Δ and subdivide it in the way depicted in Figure 4.6. This step increases the number of triangles of Δ by 6. Step B: We choose two triangles of Δ connected by an edge of degree 2 and subdivide them in the way depicted in Figure 4.7. This step increases the number of triangles of Δ by 4.

We initialize the set I to $\{1\}$, then we apply the following procedure while the set $I \neq \{1, \dots, d\}$.

Let i be the smallest index not in I. We repeatedly apply step A to $\Delta_{b_i}^{\mathcal{C}} - B^n$ until $k_i \leq k_{i-1} - 6$. Then, if $k_i = k_{i-1} - 4$, we apply step B to $\Delta_{b_i}^{\mathcal{C}} - B^n$. If $k_i = k_{i-1} - 2$, we apply step B to $\Delta_{b_j}^{\mathcal{C}} - B^n$ for every $j \in I$, and step A to $\Delta_{b_i}^{\mathcal{C}} - B^n$. Then, we put the index i to I and repeat these steps.

Note that, we can apply step B to $\Delta_{b_j}^{\mathcal{C}} - B^n$ only if it contains two triangles connected by an edge of degree 2. If triangular configuration $\Delta_{b_j}^{\mathcal{C}} - B^n$ does not contain an edge of degree 2, we apply step A to $\Delta^{\mathcal{C}}_{b_i} - B^n$ for every $i = 1, \ldots, d$. Then, $\Delta^{\mathcal{C}}_{b_j} - B^n$ contains two triangles connected by an edge of degree 2.

After this procedure, we have a balanced triangular configuration $\Delta^{\mathcal{C}}_B$. If $e(\Delta^{\mathcal{C}}_B) < n$, we repeatedly apply step A on $\Delta^{\mathcal{C}}_{b_i} - B^n$ for every $i = 1, \ldots, d$ unless $e(\Delta^{\mathcal{C}}_B) > n$.

$\square$

Let c be a codeword of $\mathcal{C}$ and let $c = \sum_{b \in B} \alpha_b b$ be the unique expression of c, where $\alpha_b \in \mathbb{F}$. The *degree* of c with respect to a basis B is defined to be the number of non-zero α_b's. The degree is denoted by $d(c)$.

Proposition 4.2.7. *Let $\mathcal{C}$ be an even finite linear code over a field $\mathbb{F}$ with a representable basis B and let $\Delta^{\mathcal{C}}_B$ be a balanced triangular configuration provided by Proposition 4.2.6 and let $f : \mathcal{C} \mapsto \ker \Delta^{\mathcal{C}}_B$ be the linear mapping from Definition 4.2.1. Then $w(f(c)) = w(c) + d(c)e(\Delta^{\mathcal{C}}_B)$ for every codeword $c \in \mathcal{C}$.*

Proof. Write c as $\sum_{b \in B} \alpha_b b$. Then, $f(c) = \sum_{b \in B} \alpha_b \operatorname{gen}(\Delta^{\mathcal{C}}_b)$. Let I be the set of basis vectors b such that $\alpha_b \neq 0$. By Corollary 4.2.1, vector $f(c)$ has non-zero elements indexed by triangles of $\Delta^{\mathcal{C}}_b - B^n$ for all $b \in I$. The number of these triangles is $d(c)e(\Delta^{\mathcal{C}}_B)$, since $\left|T\left(\Delta^{\mathcal{C}}_b - B^n\right)\right| = e(\Delta^{\mathcal{C}}_B)$ for all $b \in I$ and $|I| = d(c)$. By Proposition 4.2.4, $f(c)^{B^n_j} = c^j$ for $j = 1, \ldots, n$. Hence, the number of non-zero coordinates indexed by triangles of B^n is $w(c)$. Therefore, $w(f(c)) = w(c) + d(c)e(\Delta^{\mathcal{C}}_B)$. $\square$

4.2.7 Proof of Proposition 4.2.2

Oriented triangular sphere $\vec{\mathcal{S}^m}$

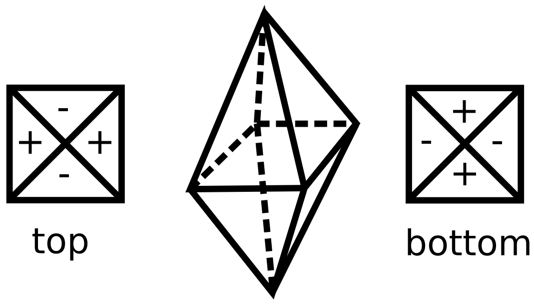

Figure 4.5: Oriented triangular sphere $\vec{\mathcal{S}^8}$.

In this section we give a proof of Proposition 4.2.2. The oriented triangular sphere $\vec{\mathcal{S}^m}$ is a triangulation of a 2-dimensional sphere by m triangles, such that there is an assignment of sign '+' and '−' to triangles and every edge is incident with one '+' triangle and one '−' triangle. An example is depicted in Figure 4.5 for $m = 8$.

Proposition 4.2.8. *Let $\mathbb{F}$ be a field and let a be a non-zero element of $\mathbb{F}$. Then* $\ker \vec{\mathcal{S}^m} = \operatorname{span}(\{(a, -a, a, -a, \ldots, a, -a)\})$. $\square$

Proposition 4.2.9. *Let k, l be non-negative integers. Then there exists the oriented triangular sphere $\vec{\mathcal{S}^m}$ with $m = 8 + 6l + 4k$ triangles.*

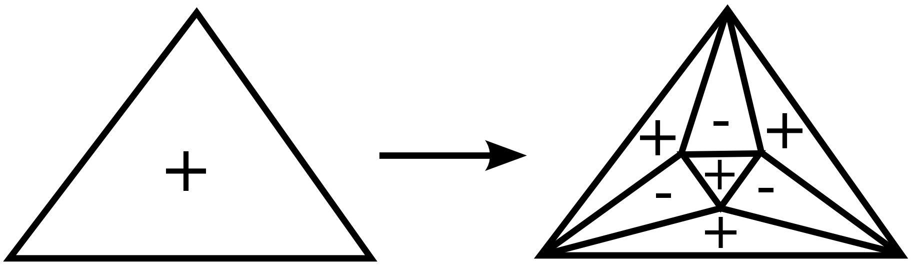

Figure 4.6: Triangle subdivision

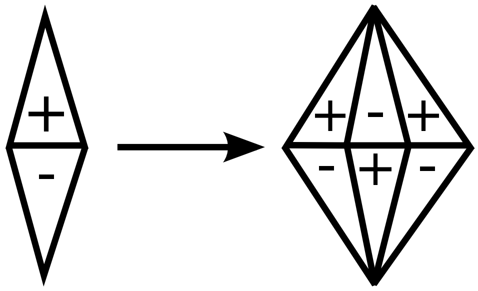

Figure 4.7: Triangles subdivision

Proof. We construct the desired sphere $\vec{\mathcal{S}^m}$ from the sphere $\vec{\mathcal{S}^8}$, depicted in Figure 4.5, by sequentially subdividing triangles of $\vec{\mathcal{S}^8}$ in the way depicted in Figure 4.6 or 4.7. These subdivisions increase the number of triangles by 4 or by 6.

$\square$

Oriented triangular multisphere $\vec{\mathcal{M}}^M_{n_1,n_2,\dots,n_k}$

In this subsection we construct the oriented triangular configuration which we call oriented triangular multisphere. We note that an important property of the oriented triangular multisphere which we use is that it has an even number of triangles. In the construction of oriented triangular multisphere we proceed in four steps.

Step 1. Let $n_1, n_2, \dots, n_k$ be distinct positive integers and let M be an integer. We start with oriented triangular configuration

$$\vec{\mathcal{M}}_1 := \vec{\mathcal{S}}_1 \cup t_{12} \cup t'_{12} \cup \vec{\mathcal{S}}_2 \cup t_{23} \cup t'_{23} \cup \dots \cup t_{(k-1)k} \cup t'_{(k-1)k} \cup \vec{\mathcal{S}}_k$$

where $t_{i(i+1)}$, $t'_{i(i+1)}$ are empty triangles, $i = 1, \dots, k-1$; and $\vec{\mathcal{S}}_j$ is oriented triangular sphere $\vec{\mathcal{S}^m}$ (see Section 4.2.7) such that $m > 4n_i$ and $m > 2M$ for every $i = 1, \dots, k$ and every $j = 1, \dots, k$. If k equals 1, the triangular multisphere $\vec{\mathcal{M}}^M_{n_1}$ is $\vec{\mathcal{S}}_1$. Recall that an empty triangle is a set of three edges forming a boundary of a triangle.

Step 2. We make the following links between the triangles of $\vec{\mathcal{M}}_1$. For every $i = 1, \dots, k-1$; we choose n_{i+1} different triangles of $\left[\vec{\mathcal{S}}_i\right]_-$ (for this notation see Section 4.2.2) and create the link by the tunnel from empty triangle $t_{i(i+1)}$ to each chosen triangle. Then, we choose n_i different triangles of $\left[\vec{\mathcal{S}}_{(i+1)}\right]_+$ and create

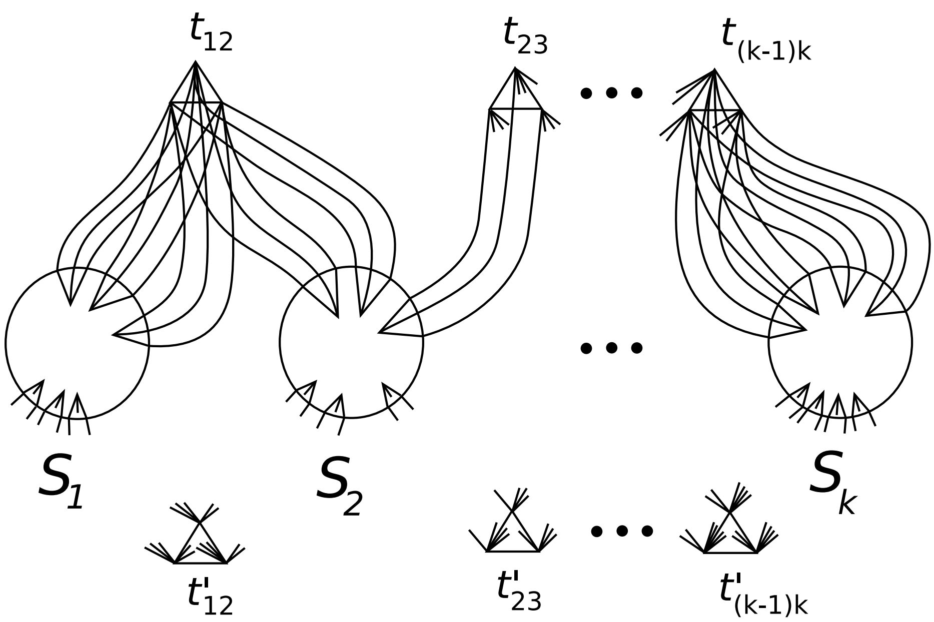

Figure 4.8: Triangular multisphere $\mathcal{M}^M_{2,3,1,\ldots,4,2}$

the link by the tunnel from each chosen triangle to empty triangle $t_{i(i+1)}$. Then, we delete the triangles of $\left[\vec{\mathcal{S}}_i\right]_-$ and $\left[\vec{\mathcal{S}}_{(i+1)}\right]_+$ that we linked with a tunnel from $\vec{\mathcal{M}}_1$. We denote the resulting triangular configuration by $\vec{\mathcal{M}}_2$.

Step 3. To achieve the even number of triangles of the multisphere we make the following links between the triangles of $\vec{\mathcal{M}}_2$. For every $i = 1, \ldots, k-1$; we choose n_{i+1} different triangles of $\left[\vec{\mathcal{S}}_i\right]_-$ and create the link by the tunnel from empty triangle $t'_{i(i+1)}$ to each chosen triangle. Then, we choose n_i different triangles of $\left[\vec{\mathcal{S}}_{(i+1)}\right]_+$ and create the link by the tunnel from each chosen triangle to empty triangle $t'_{i(i+1)}$. Finally, we delete the triangles of $\left[\vec{\mathcal{S}}_i\right]_-$ and $\left[\vec{\mathcal{S}}_{(i+1)}\right]_+$ that we linked with a tunnel from $\vec{\mathcal{M}}_2$. The resulting triangular configuration is oriented triangular multisphere $\vec{\mathcal{M}}^M_{n_1,n_2,\ldots,n_k}$.

Step 4. For every $i = 1, \ldots, k-1$, we denote by $\vec{T}_i$ the triangular configuration consisting of all the tunnels linked to oriented triangular sphere $\vec{\mathcal{S}}_i$ in steps 2,3. We denote the set of triangles $\left[\vec{\mathcal{S}}_i\right]_+ \cup \left[\vec{T}_i\right]_+$ by $\left[\vec{\mathcal{M}}^M_{n_1,n_2,\ldots,n_k}\right]_{+i}$ and the set of triangles $\left[\vec{\mathcal{S}}_i\right]_- \cup \left[\vec{T}_i\right]_-$ by $\left[\vec{\mathcal{M}}^M_{n_1,n_2,\ldots,n_k}\right]_{-i}$. An example of $\mathcal{M}^M_{n_1,n_2,\ldots,n_k}$ is depicted in Figure 4.8.

Proof of Proposition 4.2.2. Let g be a generator of G and let n_i be such that $a_i = n_i \times g = \overbrace{g + g + \cdots + g}^{n_i}$. We show that the desired configuration $\vec{\mathcal{M}}$ is oriented triangular multisphere $\vec{\mathcal{M}}^M_{n_1,n_2,\ldots,n_k}$; where $M = n$. First, we need to show that there is a vector $v \in \ker \vec{\mathcal{M}}$ with entries containing the elements $a_1, a_2, \ldots, a_k$. We construct such vector v by setting coordinates indexed by the triangles of $\left[\vec{\mathcal{M}}\right]_{+i}$ (recall step 4 above) to a_i and coordinates indexed by the triangles of

$\left[\vec{\mathcal{M}}\right]_{-i}$ to $-a_i$ for $i = 1, \ldots, k$.

Now, we show that the vector v belongs to $\ker \vec{\mathcal{M}}$. Let e be an edge different from the edges of empty triangles $t_{i(i+1)}$ and $t'_{i(i+1)}$. Then the edge e is incident with two triangles and the equation indexed by e is $a_i - a_i = 0$. Let e be an edge of an empty triangle $t_{i(i+1)}$ or $t'_{i(i+1)}$. Then the edge e is incident with n_i triangles from $\left[\vec{\mathcal{M}}\right]_{-(i+1)}$ and n_{i+1} triangles from $\left[\vec{\mathcal{M}}\right]_{+i}$. So, the equation indexed by e is

$$n_{i+1} \times a_i - n_i \times a_{i+1} = n_{i+1} \times (n_i \times g) - n_i \times (n_{i+1} \times g) = 0.$$

Hence, the vector v belongs to $\ker \vec{\mathcal{M}}$.

A *triangle path* is a sequence of triangles $t_1, \ldots, t_k$ such that t_i and t_{i+1} have a common edge, for every $i = 1, \ldots, k-1$. Next, we prove a claim.

Claim 4.2.3. *Let e be an edge of $\vec{\mathcal{M}}$. Let $t_1, \ldots, t_d$ be triangles incident with e in any order. Let v be a vector from $\ker \vec{\mathcal{M}}$. Then the entries of v indexed by $t_2, \ldots, t_d$ are determined by the entry indexed by t_1.*

Proof of Claim 4.2.3. If e is neither an edge of an empty triangle $t_{i(i+1)}$ nor $t'_{i(i+1)}$, then the edge e has degree 2 and the lemma follows.

Suppose that e is an edge of an empty triangle $t_{i(i+1)}$ or $t'_{i(i+1)}$. The entries indexed by the triangles that belong to $\left[\vec{\mathcal{M}}\right]_{-i}$ have the same value, since they are connected by a triangle path such that each inner edge of the triangle path has degree 2 in $\vec{\mathcal{M}}$. The same holds for the entries indexed by the triangles of $\left[\vec{\mathcal{M}}\right]_{+(i+1)}$. Without loss of generality we can suppose that t_1 belongs to $\left[\vec{\mathcal{M}}\right]_{-i}$ and let t_l be an element of $\{t_2, \ldots, t_d\}$ that belongs to $\left[\vec{\mathcal{M}}\right]_{+(i+1)}$. Let v_1 be the entry of v indexed by t_1 and let v_l be an entry indexed by t_l. If $v_1 \notin G$, we choose an appropriate scalar $\alpha \in \mathbb{F}$ such that $\alpha v_1 \in G$ and set $v := \alpha v$. Then the following equation holds

$$n_{i+1} \times v_1 = n_i \times v_l. \tag{4.1}$$

We show that there is only one solution v_l. We use the fact that every cyclic subgroup G of the additive group of the field $\mathbb{F}$ has a prime or an infinite order. In the case of an infinite order, Equation 4.1 has only one solution v_l. In the case of a prime order, since the integers n_i and n_{i+1} do not divide the group order and by Lagrange's theorem, Equation 4.1 has only one solution v_l. **End of proof of Claim 4.2.3.**

Finally, we prove the proposition. Since there is a triangle path between any two triangles of $\vec{\mathcal{M}}$ and by the above lemma, the dimension of the kernel $\ker \vec{\mathcal{M}}$ is 1. From the definition of the vector v, all entries of v are non-zero. ☐

4.3 Weight enumerator

In this section, we state the connection between the weight enumerators of finite linear codes and the weight enumerators of their triangular representations.

We define the *extended weight enumerator* (with respect to a fixed basis) by

$$W_{\mathcal{C}}^k(x) := \sum_{\substack{c \in \mathcal{C} \\ d(c)=k}} x^{w(c)}.$$

If a code $\mathcal{C}$ has dimension d, then

$$W_{\mathcal{C}}(x) = \sum_{k=0}^{d} W_{\mathcal{C}}^{k}(x).$$

Recall that a basis B is representable if for each $b \in B$ each entry of b belongs to a cyclic subgroup $G(b)$ of the additive group of the field $\mathbb{F}$. A code $\mathcal{C}$ is even if all codewords have an even weight.

Proposition 4.3.1. *Let $\mathcal{C}$ be an even finite linear code with a representable basis B and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular configuration provided by Proposition 4.2.6. Then*

$$W^{k}_{\ker \Delta_B^{\mathcal{C}}}(x) = W_{\mathcal{C}}^{k}(x)x^{ke(\Delta_B^{\mathcal{C}})}.$$

Proof. Let f be the mapping from Definition 4.2.1. For every codeword c of degree k of $\mathcal{C}$ there is codeword $f(c)$ of degree k of $\ker \Delta_B^{\mathcal{C}}$. By Proposition 4.2.7, $w(f(c)) = w(c) + ke(\Delta_B^{\mathcal{C}})$. Therefore,

$$W^{k}_{\ker \Delta_B^{\mathcal{C}}}(x) = \sum_{\substack{f(c)\in\ker \Delta_B^{\mathcal{C}} \\ d(f(c))=k}} x^{w(f(c))} = \sum_{\substack{c\in\mathcal{C} \\ d(c)=k}} x^{w(c)+ke(\Delta_B^{\mathcal{C}})} = W_{\mathcal{C}}^{k}(x)x^{ke(\Delta_B^{\mathcal{C}})}.$$

□

Proposition 4.3.2. *Let $\mathcal{C}$ be an even finite linear code of length n with a representable basis B and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular configuration provided by Proposition 4.2.6. Then the inequality $ke(\Delta_B^{\mathcal{C}}) \leq w(d) \leq ke(\Delta_B^{\mathcal{C}}) + n$ holds for every codeword d of degree k of $\ker \Delta_B^{\mathcal{C}}$.*

Proof. Let f be the mapping from Definition 4.2.1. By Proposition 4.2.7, $w(d) = w(f^{-1}(d)) + ke(\Delta_B^{\mathcal{C}})$. Since $0 \leq w(f^{-1}(d)) \leq n$ for every $d \in \ker \Delta_B^{\mathcal{C}}$, the inequality $ke(\Delta_B^{\mathcal{C}}) \leq w(d) \leq ke(\Delta_B^{\mathcal{C}}) + n$ holds. □

Corollary 4.3.1. *Let $\mathcal{C}$ be an even finite linear code of dimension d and length n with a representable basis B and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular configuration provided by Proposition 4.2.6 such that $n < e(\Delta_B^{\mathcal{C}})$. Denote $e(\Delta_B^{\mathcal{C}})$ by e. Let $\sum_{i=0}^{de+n} a_i x^i$ be the weight enumerator of $\ker \Delta_B^{\mathcal{C}}$. Then*

$$W^{k}_{\ker \Delta_B^{\mathcal{C}}}(x) = \sum_{i=ke}^{ke+n} a_i x^i. \quad \square$$

Theorem 4.3.1. *Let $\mathcal{C}$ be an even finite linear code of dimension d and length n with a representable basis B and let $\Delta_B^{\mathcal{C}}$ be a balanced triangular configuration provided by Proposition 4.2.6 such that $n < e(\Delta_B^{\mathcal{C}})$. Denote $e(\Delta_B^{\mathcal{C}})$ by e. Let $\sum_{i=0}^{de+n} a_i x^i$ be the weight polynomial of $\ker \Delta_B^{\mathcal{C}}$. Then*

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{de+n} a_i x^{i \bmod e}.$$

Proof. The inequality $w(c) \leq n$ holds for every codeword $c \in \mathcal{C}$. Let f be the mapping from Definition 4.2.1. By Proposition 4.2.7, $w(f(c)) = w(c) + d(c)e$ for every codeword c of $\mathcal{C}$. Since $n < e$, the following equality holds.

$$w(f(c)) \bmod e = (w(c) + d(c)e) \bmod e = w(c).$$

Hence,

$$W_{\mathcal{C}}(x) = \sum_{i=0}^{de+n} a_i x^{i \bmod e}.$$

□

4.3.1 Proof of Theorem 4.1.2

The *double code*, denoted by $\mathcal{C}^2$, of a linear code $\mathcal{C}$ of length n is the code

$$\mathcal{C}^2 = \left\{ \left(c^1, \ldots, c^n, c^1, \ldots, c^n\right) : c \in C \right\}.$$

Proposition 4.3.3. *Let $\mathcal{C}$ be a linear code and let $\mathcal{C}^2$ be its double code. Then, the double code $\mathcal{C}^2$ is even and the code $\mathcal{C}$ is a punctured code of its double code $\mathcal{C}^2$ and there is a linear bijection between $\mathcal{C}$ and $\mathcal{C}^2$ that maps minimal codewords to minimal codewords and $W_{\mathcal{C}}(x) = W_{\mathcal{C}^2}(x^{\frac{1}{2}})$.* □

Proof of Theorem 4.1.2. Let $\mathcal{C}^2$ be the double code of $\mathcal{C}$. The code $\mathcal{C}^2$ is even. Let B^2 be the basis $\{(b^1, \ldots, b^n, b^1, \ldots, b^n) | b \in B\}$ of $\mathcal{C}^2$. The basis B^2 is representable. Let $\Delta_{B^2}^{\mathcal{C}^2}$ be a balanced triangular configuration provided by Proposition 4.2.6 such that $e(\Delta_{B^2}^{\mathcal{C}^2}) > n_2$, where $n_2 = 2n$ is the length of $\mathcal{C}^2$. We denote $e(\Delta_{B^2}^{\mathcal{C}^2})$ by e. By Theorem 4.2.2, the code $\mathcal{C}^2$ is equal to $\ker \Delta_{B^2}^{\mathcal{C}^2}/S'$ for some set of indices S' and there exists a linear bijection $f' : \mathcal{C}^2 \mapsto \ker \Delta_{B^2}^{\mathcal{C}^2}$ which maps minimal codewords to minimal codewords. By Proposition 4.3.3, the code $\mathcal{C}$ is equal to $\mathcal{C}^2/S''$ where $S'' = \{n+1, \ldots, 2n\}$ and there is a linear bijection $f'' : \mathcal{C} \mapsto \mathcal{C}^2$ which maps minimal codewords to minimal codewords. Therefore, the code $\mathcal{C}$ is equal to $\ker \Delta_{B^2}^{\mathcal{C}^2}/(S' \cup S'')$ and there is a linear bijection $f : \mathcal{C} \mapsto \ker \Delta_{B^2}^{\mathcal{C}^2}$ which maps minimal codewords to minimal codewords. Hence, the code $\mathcal{C}$ is triangular representable and $\Delta_{B^2}^{\mathcal{C}^2}$ is its triangular representation. We denote $\Delta_{B^2}^{\mathcal{C}^2}$ by Δ.

By Proposition 4.2.5, $e = |S'|/\dim \mathcal{C}^2$. Let $S(\ker \Delta, \mathcal{C}) = S' \cup S''$. The cardinality $|S''|$ is equal to n. Hence, $e = (|S(\ker \Delta, \mathcal{C})| - n)/\dim \mathcal{C}^2 = (|S(\ker \Delta, \mathcal{C})| - n)/\dim \mathcal{C}$.

If the code $\mathcal{C}$ is finite, the formula for the weight enumerator follows from Theorem 4.3.1 and from Proposition 4.3.3.

□

Proof of Corollary 4.1.1. The additive group of $GF(p)$, where p is a prime, is cyclic. In the case of rationals, we multiply the basis vectors by a sufficiently large integer, so that all vectors are integral. Hence, all elements of all basis vectors belong to the cyclic group of integers. Then, we use Theorem 4.1.2. □

4.3.2 Multivariate weight enumerator

Let $\mathcal{C}$ be a linear code of length n. Let R be the set of coordinates of $\mathcal{C}$. The *assignment of variables to coordinates* is a mapping λ from R to the set of indices of variables $\{1, \ldots, k\}$. The *multivariate weight enumerator* of $\mathcal{C}$ is

$$W_{\mathcal{C}}^{\lambda}(x_1, \ldots, x_k) = \sum_{c \in \mathcal{C}} \prod_{\substack{i=1 \\ c_i \neq 0}}^{n} x_{\lambda(i)}.$$

Theorem 4.3.2. *Let $\mathcal{C}$ be a finite linear code over a field $\mathbb{F}$, $\mathcal{C} \subseteq \mathbb{F}^n$. Let λ be an assignment of variables. If $\mathcal{C}$ is triangular representable, then there exists a triangular configuration Δ and an injection $\mu : \{1, \ldots, 2n\} \mapsto T(\Delta)$ such that: if*

$$\sum_{\substack{i_1+i_2+\cdots+i_k \leq m \\ i_1 \geq 0, i_2 \geq 0, \ldots, i_k \geq 0}} a_{i_1 i_2 \ldots i_k} x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k}$$

is the multivariate weight enumerator $W_{\ker \Delta}^{\lambda'}(x_1, \ldots, x_k)$ of $\ker \Delta$ with the assignment of variables λ' defined: $\lambda'(t) := \lambda(\mu^{-1}(t))$ if $t \in \mu(\{1, \ldots, n\})$ and $\lambda'(t) := \lambda(\mu^{-1}(t) - n)$ if $t \in \mu(\{n+1, \ldots, 2n\})$ and $\lambda'(t) := k$ if $t \notin \mu(\{1, \ldots, 2n\})$. Then

$$W_{\mathcal{C}}^{\lambda}(x_1, \ldots, x_k) = \sum_{\substack{i_1+i_2+\cdots+i_k \leq m \\ i_1 \geq 0, i_2 \geq 0, \ldots, i_k \geq 0}} a_{i_1 i_2 \ldots i_k} x_1^{(i_1/2)} x_2^{(i_2/2)} \ldots x_{k-1}^{(i_{k-1}/2)} x_k^{((i_k \bmod e)/2)},$$

where $e = (|S(\ker \Delta, \mathcal{C})| - n)/\dim \mathcal{C}$.

Proof. By Theorem 4.1.3, the code $\mathcal{C}$ has a representable basis B. Let $\mathcal{C}^2$ be the double code of $\mathcal{C}$. Let B^2 be the basis $\{(b^1, \ldots, b^n, b^1, \ldots, b^n) | b \in B\}$ of $\mathcal{C}^2$. The code $\mathcal{C}^2$ is even and the basis B^2 is representable. Let $\Delta_{B^2}^{\mathcal{C}^2}$ be a balanced triangular configuration provided by Proposition 4.2.6 such that $e(\Delta_{B^2}^{\mathcal{C}^2}) > n_2$, where $n_2 = 2n$ is the length of $\mathcal{C}^2$. The desired configuration Δ is $\Delta_{B^2}^{\mathcal{C}^2}$. We denote $e(\Delta)$ by e. By Theorem 4.2.2, the code $\mathcal{C}^2$ is equal to $\ker \Delta / S'$ for some set of indices S'. By Proposition 4.3.3, the code $\mathcal{C}$ is equal to $\mathcal{C}^2 / S''$ where $S'' = \{n+1, \ldots, 2n\}$. Therefore, the code $\mathcal{C}$ is equal to $\ker \Delta / (S' \cup S'')$. By Proposition 4.2.5, $e = |S'| / \dim \mathcal{C}^2$. Let $S(\ker \Delta, \mathcal{C}) = S' \cup S''$. The cardinality $|S''|$ is equal to n. Hence, $e = (|S(\ker \Delta, \mathcal{C})| - n)/\dim \mathcal{C}^2 = (|S(\ker \Delta, \mathcal{C})| - n)/\dim \mathcal{C}$.

We define an assignment of variables of $\mathcal{C}^2$ in the following way: $\lambda''(i) := \lambda''(n+i) := \lambda(i)$ for $i = 1, \ldots, n$. Let $\mu : \{1, \ldots, 2n\} \mapsto T(\Delta)$ be the injection from Definition 4.2.1. Then, $S' = T(\Delta) \setminus \mu(\{1, \ldots, 2n\})$.

The weight of the variable indexed by j in a codeword c with respect to an assignment of variables λ is the number of nonzero coordinates of c assigned to variable j. We denote this weight by $w_j^{\lambda}(c)$. Let $f : \mathcal{C}^2 \mapsto \ker \Delta$ be the mapping from Definition 4.2.1. By Proposition 4.2.4, $f(c)^t = c^{\mu^{-1}(t)}$ for $t \in \mu^{-1}(\{1, \ldots, 2n\})$ and $c \in \mathcal{C}^2$. For $j = 1, \ldots, k-1$ we have $w_j^{\lambda'}(f(c)) = w_j^{\lambda''}(c)$, since $\lambda'(t) = k \neq j$ for all $t \in S' = T(\Delta) \setminus \mu(\{1, \ldots, 2n\})$ and $\lambda'(t) = \lambda''(\mu^{-1}(t))$ for $t \in \mu^{-1}(\{1, \ldots, 2n\})$. The Hamming weight of c satisfies $w(c) = \sum_{j=1}^{k} w_j^{\lambda}(c)$ for arbitrary λ. By Proposition 4.2.7, $w(f(c)) = w(c) + d(c)e$ for every codeword c of $\mathcal{C}^2$. From this equation we subtract equations $w_j^{\lambda'}(f(c)) = w_j^{\lambda''}(c)$ for $j =$

$1, \ldots, k-1$. Hence, $w_k^{\lambda'}(f(c)) = w_k^{\lambda''}(c) + d(c)e$. The inequality $w(c) \leq 2n$ holds for every codeword $c \in \mathcal{C}^2$. Since $2n < e$, the following equality holds.

$$w_k^{\lambda'}(f(c)) \bmod e = (w_k^{\lambda''}(c) + d(c)e) \bmod e = w_k^{\lambda''}(c).$$

The weights of codewords of a code and its double code satisfy

$$2w_j^{\lambda}((c^1, \ldots, c^n)) = w_j^{\lambda''}((c^1, \ldots, c^n, c^1, \ldots, c^n))$$

for every $j = 1, \ldots, k$. Hence,

$$W_{\mathcal{C}}^{\lambda}(x_1, \ldots, x_k) = \sum_{\substack{i_1+i_2+\cdots+i_k \leq m \\ i_1 \geq 0, i_2 \geq 0, \ldots, i_k \geq 0}} a_{i_1 i_2 \ldots i_k} x_1^{(i_1/2)} x_2^{(i_2/2)} \ldots x_{k-1}^{(i_{k-1}/2)} x_k^{((i_k \bmod e)/2)},$$

$\square$

Theorem 4.3.3. *Let G be a connected graph and let n be the number of edges of G. Let $E = \{e_1, \ldots, e_n\}$ be the set of edges of G and let $w : E \mapsto \{w_1, \ldots, w_k\}$ be weights of edges of G. Let q be a prime. Then there exists triangular configuration Δ and an injection $\mu : \{1, \ldots, 2n\} \mapsto T(\Delta)$ such that if*

$$\sum_{\substack{i_1+i_2+\cdots+i_k \leq m \\ i_1 \geq 0, i_2 \geq 0, \ldots, i_k \geq 0}} a_{i_1 i_2 \ldots i_k} x_1^{i_1} \ldots x_{k-1}^{i_{k-1}} x_k^{i_k}$$

is the multivariate weight enumerator $W_{\ker \Delta}^{\lambda'}(x_1, \ldots, x_k)$ of $\ker \Delta$ over $GF(q)$ with the assignment of variables λ' defined: $\lambda'(t) := i$ if $t \in \mu(\{1, \ldots, n\})$ and $w(e_{\mu^{-1}(t)}) = w_i$ or $t \in \mu(\{n+1, \ldots, 2n\})$ and $w(e_{\mu^{-1}(t)-n}) = w_i$, and $\lambda'(t) := k$ if $t \notin \mu(\{1, \ldots, 2n\})$), then $P^q(G, x)$ equals

$$\frac{q}{\prod_{uv \in E} x^{-w(uv)}} \sum_{\substack{i_1+i_2+\cdots+i_k \leq m \\ i_1 \geq 0, i_2 \geq 0, \ldots, i_k \geq 0}} a_{i_1 i_2 \ldots i_k} x^{-w_1(i_1/2)} \ldots x^{-w_{k-1}(i_{k-1}/2)} x^{-w_k((i_k \bmod e)/2)},$$

where e is a positive integer linear in number of edges of G.

Proof. The proof follows from the following calculations.

$$\prod_{uv \in E} x^{-w(uv)} P^q(G, x) = \sum_{s: V \mapsto \{0, \ldots, q-1\}} \prod_{uv \in E} x^{\delta(s(u), s(v)) w(uv) - w(uv)}$$

Let $cut(s)$ be the set of edges uv of G such that $s(u) \neq s(v)$. Then

$$\sum_{s: V \mapsto \{0, \ldots, q-1\}} \prod_{uv \in E} x^{\delta(s(u), s(v)) w(uv) - w(uv)} = \sum_{s: V \mapsto \{0, \ldots, q-1\}} \prod_{uv \in cut(s)} x^{-w(uv)}.$$

The following part of this proof is taken from Galluccio and Loebl [8] and modified for the multivariate enumerator. Let z be a vector from $GF(q)^{|V(G)|}$ defined as $z_v := s(v)$. Let O_G be an oriented incidence matrix of G. Then, $e \in cut(s)$ if and only if $(O_g^T z)_e \neq 0$. Hence,

$$\sum_{s: V \mapsto \{0, \ldots, q-1\}} \prod_{uv \in cut(s)} x^{-w(uv)} = \sum_{z \in GF(q)^{|V(G)|}} \prod_{\substack{(O^T z)_{uv} \neq 0 \\ uv \in E(G)}} x^{-w(uv)}.$$

Let $\mathcal{C}$ equals $\{O_G^T z | z \in GF(q)\}$. Let us define an equivalence on $GF(q)^{|V(G)|}$ by $w \equiv z$ if $O_G^T w = O_G^T z$. Observe that each equivalence class consists of q elements since $O_G^T w = O_G^T z$ if and only if $z - w$ is a constant vector. Hence,

$$\sum_{z \in GF(q)^{|V(G)|}} \prod_{\substack{(O^T z)_{uv} \neq 0 \\ uv \in E(G)}} x^{-w(uv)} = q \sum_{c \in \mathcal{C}} \prod_{\substack{c_{uv} \neq 0 \\ uv \in E(G)}} x^{-w(uv)} = q \sum_{c \in \mathcal{C}} \prod_{\substack{c_{uv} \neq 0 \\ uv \in E(G)}} x^{-w_{\lambda(uv)}}$$
$$= q W_{\mathcal{C}}^{\lambda}(x^{-w_1}, \ldots, x^{-w_k}),$$

where λ is the assignment of variables defined: $\lambda(uv) := i$ if $w(uv) = w_i$, for $i = 1, \ldots, k$. By Corollary 4.1.1, $\mathcal{C}$ is triangular representable. By Theorem 4.3.2, $qW_{\mathcal{C}}^{\lambda}(x^{-w_1}, \ldots, x^{-w_k})$ equals

$$q \sum_{\substack{i_1 + i_2 + \cdots + i_k \leq m \\ i_1 \geq 0, i_2 \geq 0, \ldots, i_k \geq 0}} a_{i_1 i_2 \ldots i_k} x^{-w_1(i_1/2)} \ldots x^{-w_{k-1}(i_{k-1}/2)} x^{-w_k((i_k \bmod e)/2)},$$

where $e = (|S(\ker \Delta, \mathcal{C})| - |E(G)|)/\dim \mathcal{C}$.

$\square$

Theorem 4.1.4 follows from the above theorem.

4.4 Triangular non-representability

In this section we prove a sufficient condition for triangular non-representability. We will show that for non-prime fields every construction of triangular representation fails on very weak condition that a linear code and its triangular representation have to have the same dimension.

Proof of Theorem 4.1.3. First, we observe that the field $\mathbb{F}$ contains a proper subfield $\mathbb{P}$. We use the fact that every field contains a prime subfield $\mathbb{P}$ isomorphic to rationals or $GF(q)$, where q is a prime, and the fact that every two elements of a prime field belong to a common cyclic subgroup of the additive group of the prime field.

We can view the field $\mathbb{F}$ as a vector space over $\mathbb{P}$. This space can have an infinite dimension. An element f of $\mathbb{F}$ is equal to $(f^1, f^2, \ldots)$, where f^i's are elements of $\mathbb{P}$. We identify the vectors $(f^1, 0, 0, \ldots)$ that have only first non-zero coordinate with the subfield $\mathbb{P}$. Let $f = (f^1, f^2, f^3, \ldots)$ be an element of $\mathbb{F}$. We define two projections on the vector space $\mathbb{F}$ over $\mathbb{P}$:

$$[f]^1 := \left(f^1, 0, 0, \ldots\right)$$

and

$$[f]^{2+} := \left(0, f^2, f^3, \ldots\right).$$

Note that, $[f]^1 \in \mathbb{P}$ for all $f \in \mathbb{F}$, and $[f]^{2+} = 0$ or $[f]^{2+} \in \mathbb{F} \setminus \mathbb{P}$ for all $f \in \mathbb{F}$.

For a contradiction suppose that the linear code $\mathcal{C}$ is triangular representable. We say that a vector of $\mathcal{C}$ is bad if it has two entries which belong to no common cyclic subgroup. Let Δ be a triangular representation of $\mathcal{C}$. Let B be a basis of $\mathcal{C}$ with the minimum number of bad vectors. Let $b \in B$ be a bad vector. We recall that each basis B has a bad vector by assumptions of the theorem.

The bad vector b contains two entries p, f which belong to no common cyclic subgroup. We can suppose that p belongs to $\mathbb{P}$, otherwise we choose a non-zero scalar α such that $\alpha p \in \mathbb{P}$ and replace b by αb in basis B. Then, the entry f belongs to $\mathbb{F} \setminus \mathbb{P}$.

Let v be a vector from $\ker \Delta$. We define two projections:

$$[v]^1 := ([v^1]^1, [v^2]^1, [v^3]^1, \dots)$$

and

$$[v]^{2+} := ([v^1]^{2+}, [v^2]^{2+}, [v^3]^{2+}, \dots).$$

Since every element of the incidence matrix of Δ is 0 or 1, the projections $[v]^1$ and $[v]^{2+}$ of every vector $v \in \ker \Delta$ belong to $\ker \Delta$.

Since the linear code $\mathcal{C}$ is a punctured code of $\ker \Delta$ and the codes $\mathcal{C}$ and $\ker \Delta$ have the same number of codewords, we can define mapping g from $\mathcal{C}$ to $\ker \Delta$ such that c is a punctured codeword of $g(c)$ for every $c \in \mathcal{C}$. The mapping g is linear and bijective.

The set $B_\Delta := \{g(b) | b \in B\}$ is a basis of $\ker \Delta$. The equation $g(b) = [g(b)]^1 + [g(b)]^{2+}$ holds. Since $g(b)$ contains both entries p and f, both vectors $[g(b)]^1$ and $[g(b)]^{2+}$ are non-zero. The vectors $[g(b)]^1$ and $g[(b)]^{2+}$ are linear independent. Hence, one of the vectors $[g(b)]^1$ or $[g(b)]^{2+}$ does not belong to $\operatorname{span}(B_\Delta \setminus \{g(b)\})$. We denote this vector by $g(b')$.

Hence, the set $\{g(b')\} \cup B_\Delta \setminus \{g(b)\}$ is a basis of $\ker \Delta$ and the set $B' := \{b'\} \cup B \setminus \{b\}$ is a basis of $\mathcal{C}$. Now, basis B' has smaller number of bad vectors than B, a contradiction with the minimality of B.

□

Proof of Corollary 4.1.2. We know that the field $\mathbb{F}$ contains a proper subfield $\mathbb{P}$. Let p be an element of $\mathbb{P}$ and let f be an element of $\mathbb{F} \setminus \mathbb{P}$. By Theorem 4.1.3, the linear code $C = \operatorname{span}(\{(f, p)\})$ over $\mathbb{F}$ is not triangular representable. □

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