Automorphism Groups of Geometrically Represented Graphs

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Computer Science Institute of Charles University,
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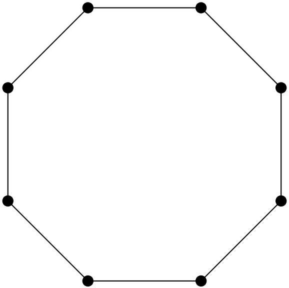
Automorphism Groups of Graphs

An automorphism of a graph X is a permutation π of the vertices such that $xy \in E(X)$ if and only if $\pi(x)\pi(y) \in E(X)$.



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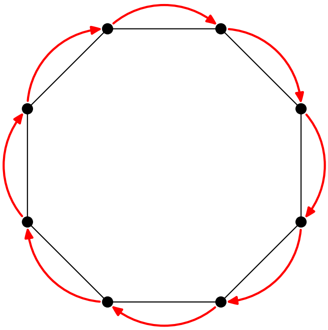

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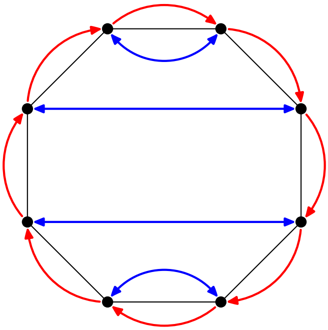

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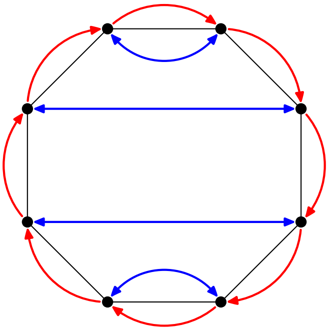
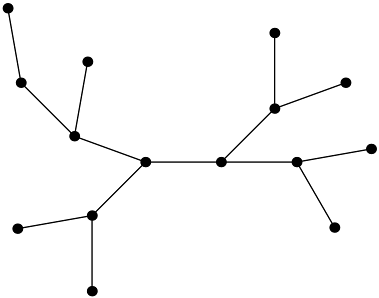


Theorem (Frucht): For each group G there exits a graph X such that $G \cong \mathrm{Aut}(X)$.

Trees (TREE)



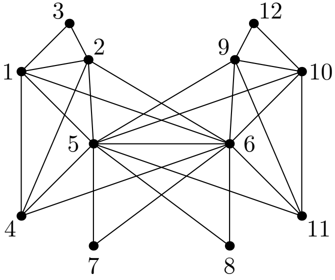
Probably, the first class of graphs, whose automorphism groups were studied are trees. Jordan (1869) gave a characterization of $\mathrm{Aut}(\mathsf{TREE})$ in terms of group products (we will see later).

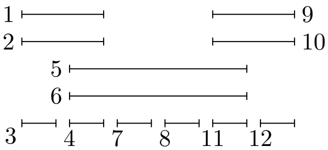
$$\mathrm{Aut}(\mathcal{C}) = \{\mathrm{Aut}(X) \colon X \in \mathcal{C}\}$$

Interval (INT) and Proper (PINT) Graphs

Interval graph is an intersection graph of intervals of the real line.

A proper interval graph is an interval graph that has an intersection representation with no interval properly containing another.





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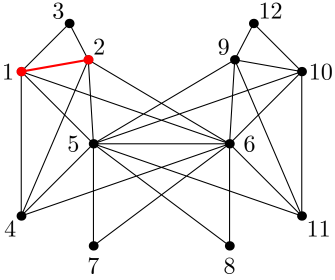

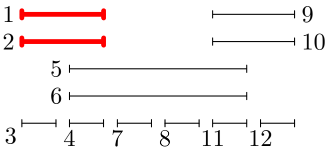

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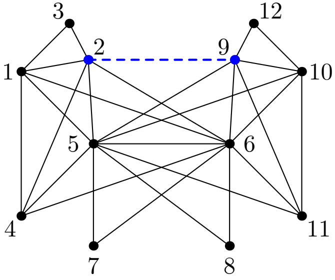
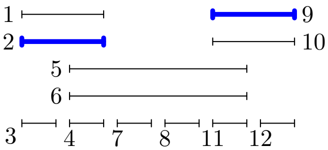
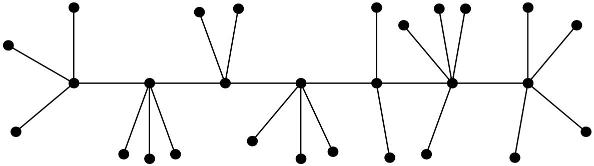




Caterpillars (CATERPILLAR)



Caterpillars are trees with leaves attached to one central path.

It is known that CATERPILLAR = TREE $\cap$ INT.

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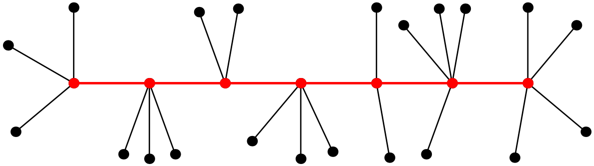

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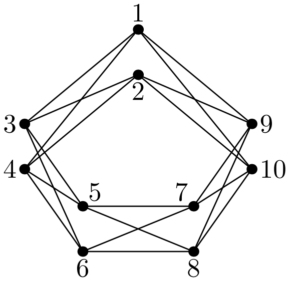
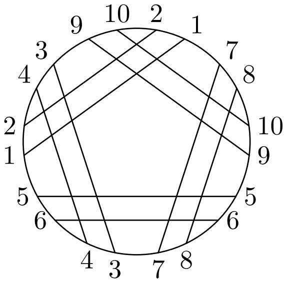
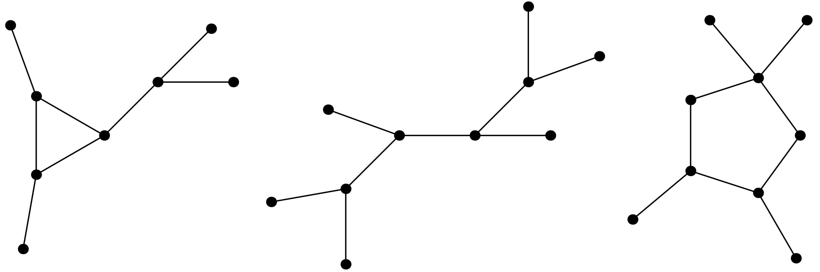
Circle Graphs (CIRCLE)

Circle graphs are intersection graphs of chords of a circle.





Pseudoforests (PSEUDOFOREST)



Each connected component contains at most one cycle.

It is easy to see that PSEUDOFOREST $\subsetneq$ CIRCLE.

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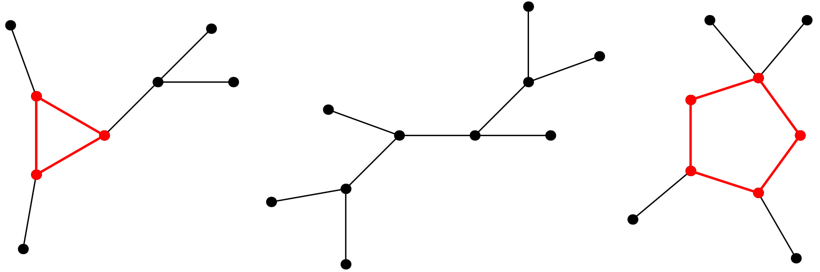

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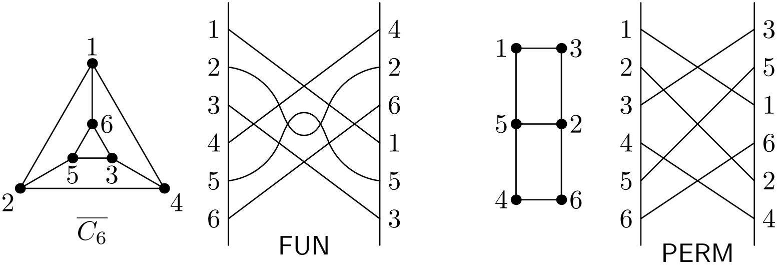
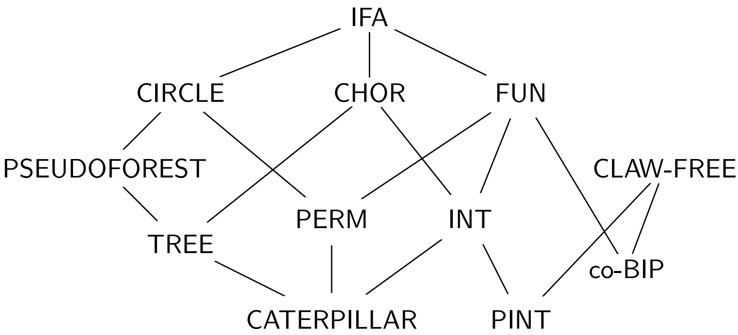
Function (FUN) and Permutation (PERM) Graphs

Function graphs are intersection graphs of continuous real-valued functions on the interval $[0,1]$.

We get the permutation graphs by considering only linear functions.



Inclusions of the Relevant Graph Classes



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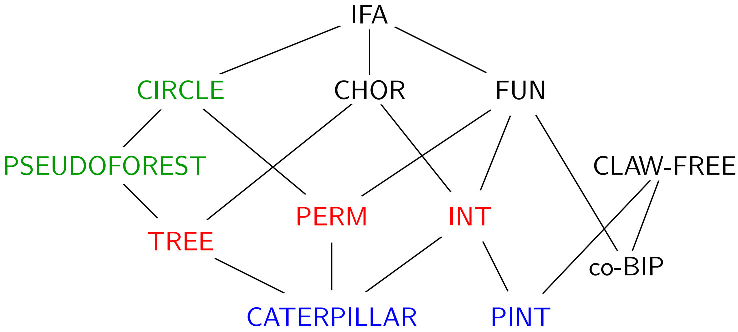

Our Results:

(i) $\mathrm{Aut}(\text{INT}) = \mathrm{Aut}(\text{TREE}) = \mathrm{Aut}(\text{PERM})$
(Hanlon – 1982; Booth and Colbourn – 1981)

(ii) $\mathrm{Aut}(\text{connected PINT}) = \mathrm{Aut}(\text{CATERPILLAR})$

(iii) $\mathrm{Aut}(\text{CIRCLE}) = \mathrm{Aut}(\text{PSEUDOFOREST})$

(iv) We present a general approach for working with $\mathrm{Aut}(\mathcal{C})$.

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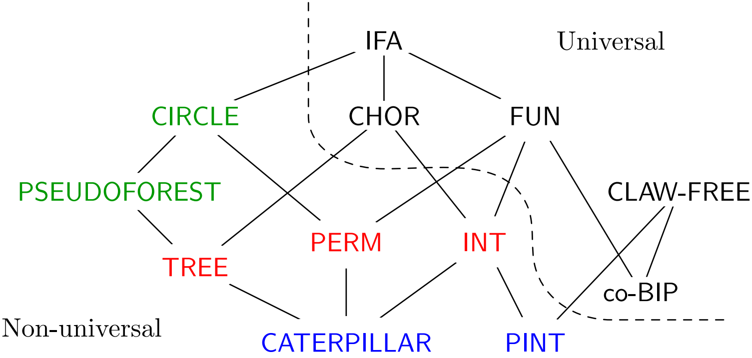

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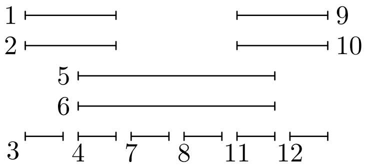
$\mathrm{Aut}(\mathsf{INT})$

Characterization of Interval Graphs

Theorem (Fulkerson and Gross): A graph X is an interval graph if and only if there exists ordering of the maximal cliques such that for every $x \in V(X)$ the maximal cliques containing x appear consecutively.



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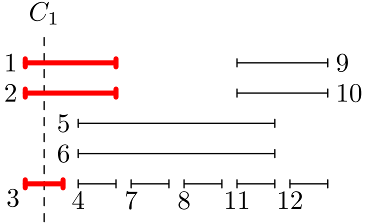

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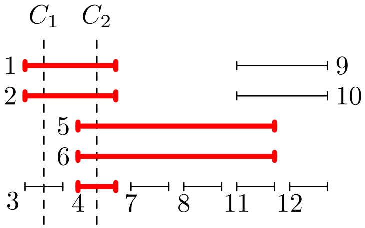

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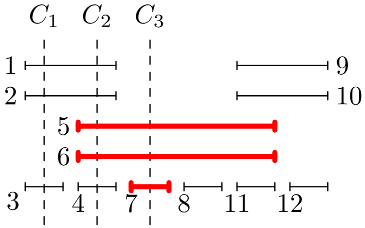

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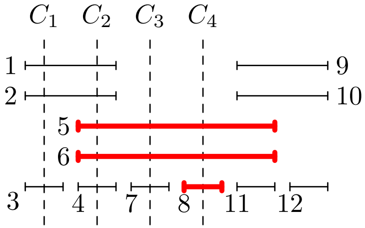

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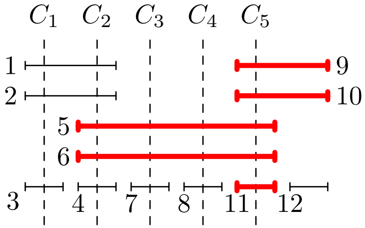

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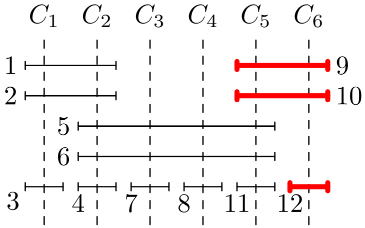

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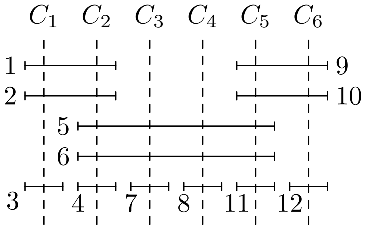

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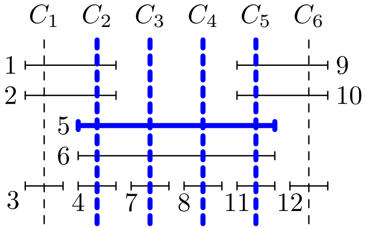

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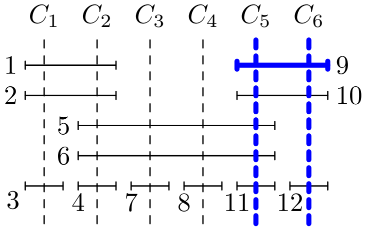
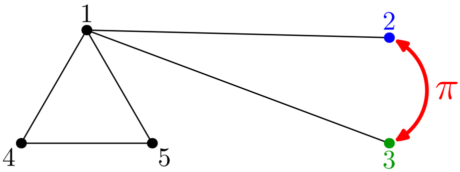
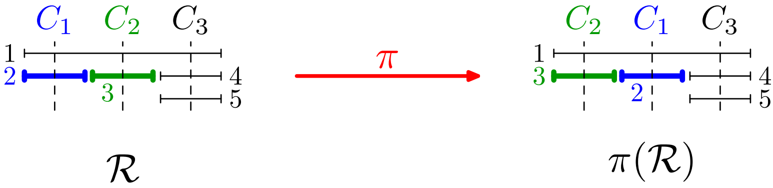
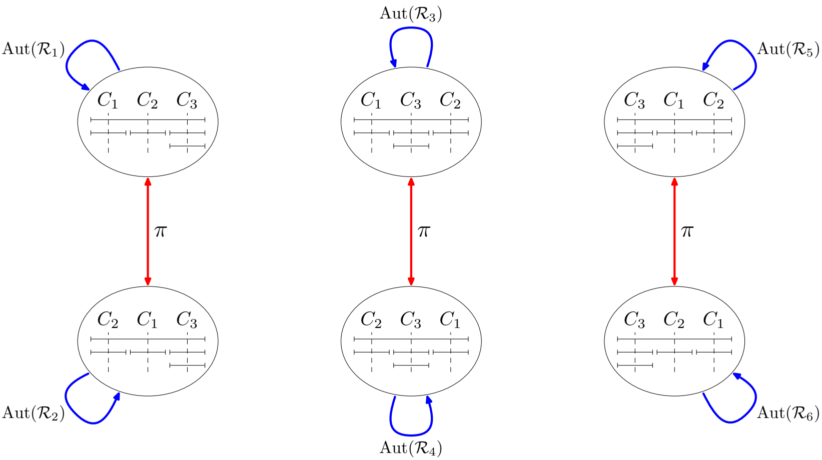


Morphisms Induced by an Automorphism $\pi \in \mathrm{Aut}(X)$





Action of $\mathrm{Aut}(X)$ on $\mathfrak{Rep}$ (part 1)



We can use $\mathrm{Aut}(\mathcal{R})$ and $\mathrm{Aut}(X)/\mathrm{Aut}(\mathcal{R})$ to determine $\mathrm{Aut}(X)$.

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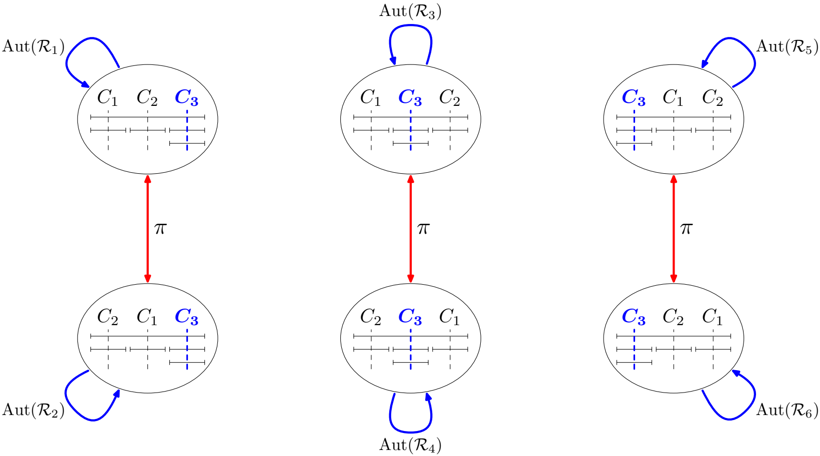

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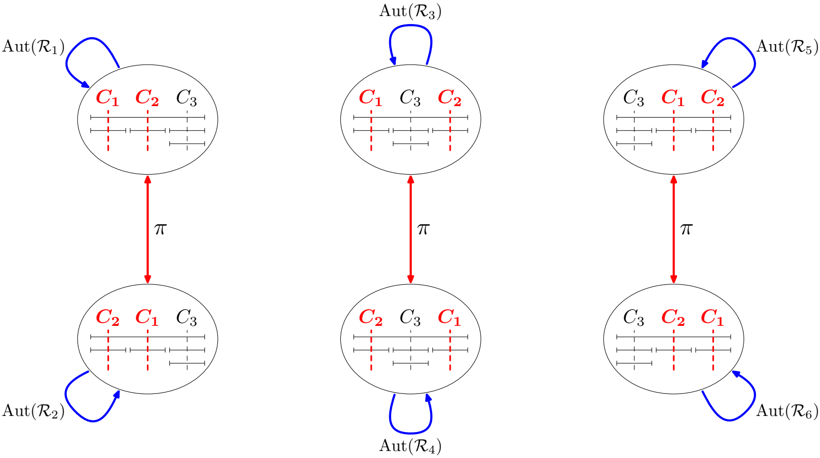

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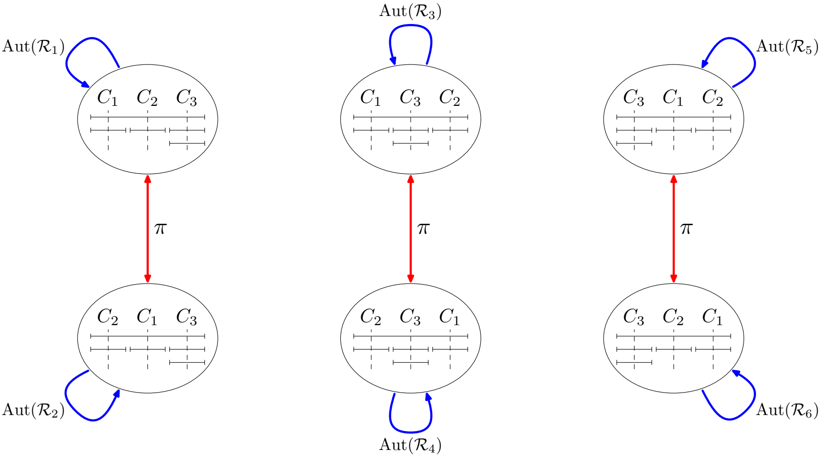

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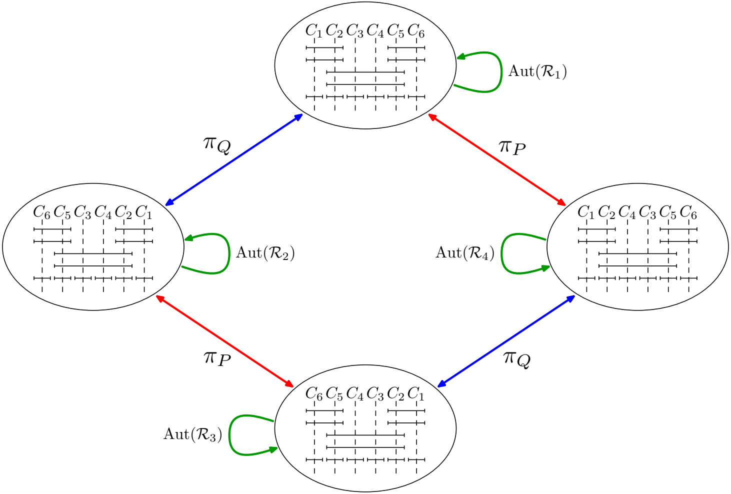
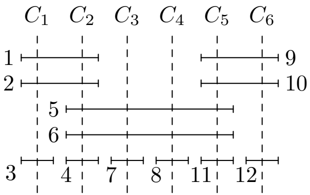
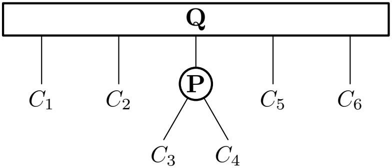
Action of $\mathrm{Aut}(X)$ on $\mathfrak{Rep}$ (part 2)



In this case, the action of $\mathrm{Aut}(X)$ on $\mathfrak{Rep}$ is transitive.

PQ-trees (Booth and Lueker)





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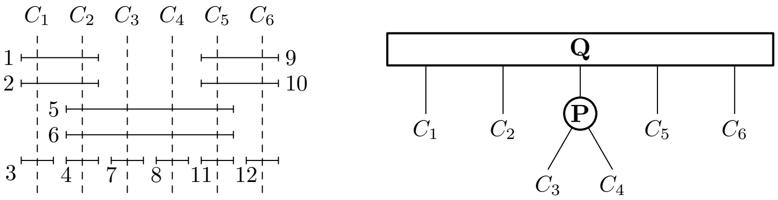

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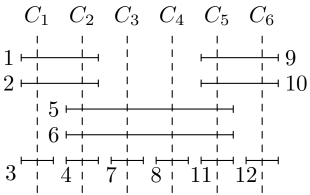

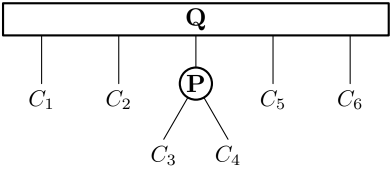

- Two types of internal nodes: P-nodes, Q-nodes. The leaves correspond to the maximal cliques of an interval graph.
- Two equivalence transformations: arbitrary permutation of the children of a P-node, reversal of the children of a Q-node.

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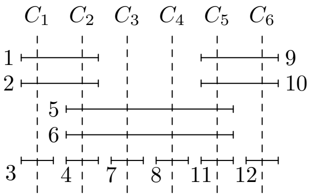

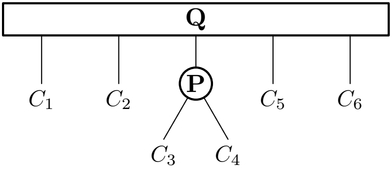

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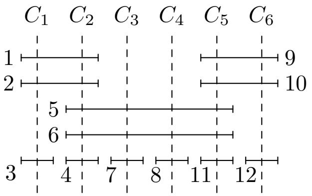

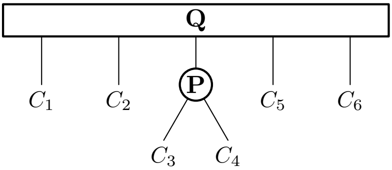

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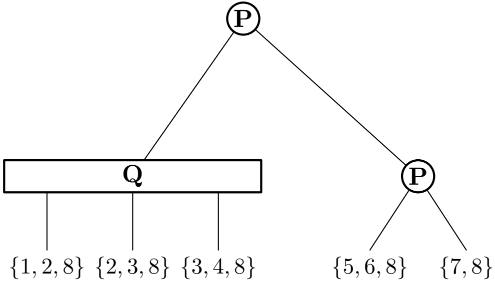
- All possible consecutive orderings of the maximal are obtained by applying the equivalence transformations.

Automorphisms of PQ-trees

Each symmetric equivalence transformation of a PQ-tree T is an automorphism of T. Automorphisms of T form a group.



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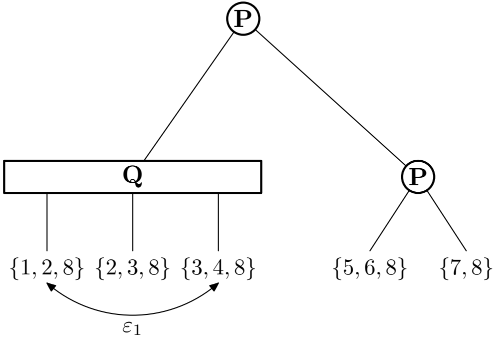

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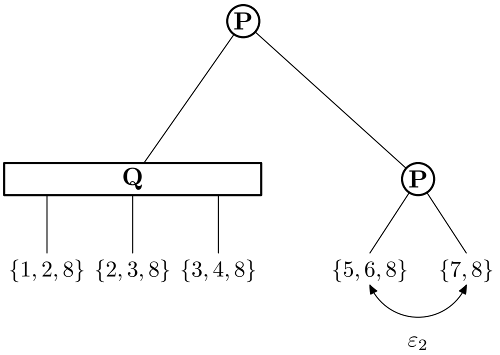

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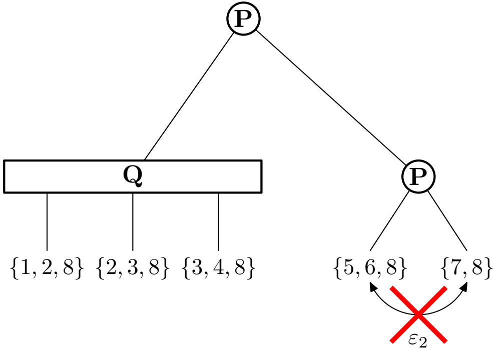

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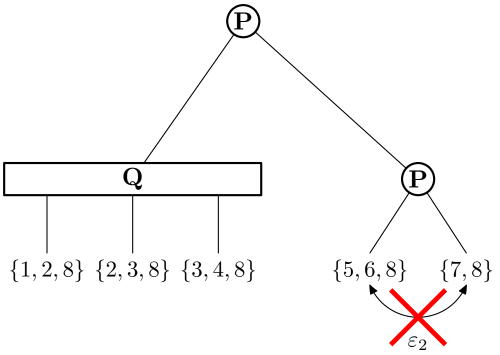
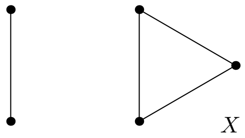


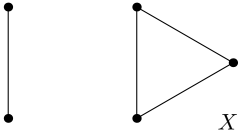
Proposition: If T is a PQ-tree representing an interval graph X, then $\mathrm{Aut}(T) \cong \mathrm{Aut}(X)/\mathrm{Aut}(\mathcal{R})$.

Group Products

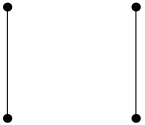
Automorphism Groups of Disconnected Graphs



Automorphism Groups of Disconnected Graphs

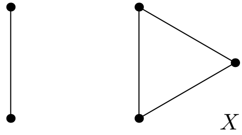


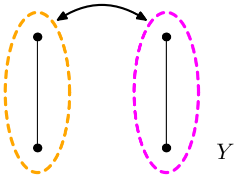
X

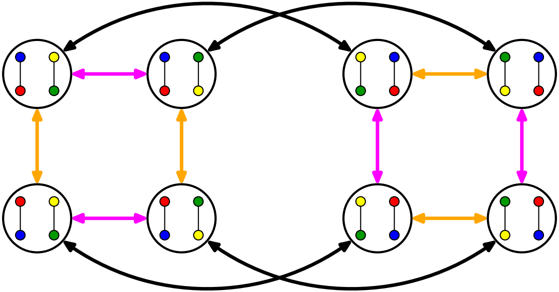


Y

Automorphism Groups of Disconnected Graphs







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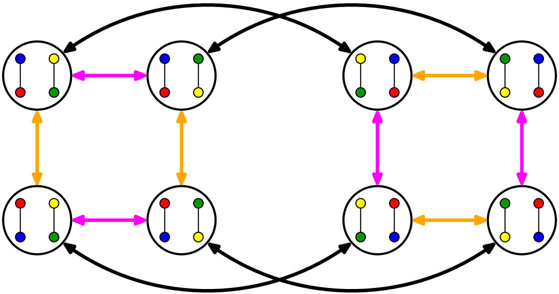

$$\mathrm{Aut}(Y) = (\mathbb{S}_2 \times \mathbb{S}_2) \rtimes \mathbb{S}_2 = \mathbb{S}_2 \wr \mathbb{S}_2$$

Automorphism Groups of Trees (Jordan)

Theorem: If graph X contains k_i copies of a graph X_i, then

$$\mathrm{Aut}(X) = \mathrm{Aut}(X_1) \wr \mathbb{S}_{k_1} \times \cdots \times \mathrm{Aut}(X_{k_n}) \wr \mathbb{S}_{k_n}.$$

Theorem (Jordan, 1869): The class $\mathrm{Aut}(\mathsf{TREE})$ can be described inductively:

(a) $\{1\} \in \mathrm{Aut}(\mathsf{TREE})$.

(b) If $G_1, G_2 \in \mathrm{Aut}(\mathsf{TREE})$, then $G_1 \times G_2 \in \mathrm{Aut}(\mathsf{TREE})$.

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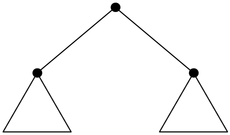

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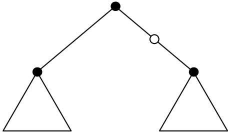

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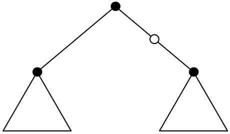

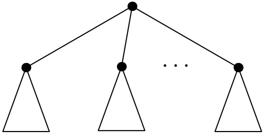

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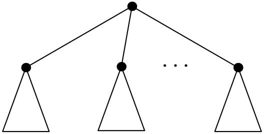

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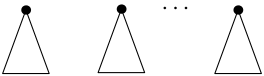

Automorphism Groups of Interval Graphs

Theorem: The class $\mathrm{Aut}(\mathsf{INT})$ can be described inductively:

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(d) If $G_1, G_2, G_3 \in \mathrm{Aut}(\mathsf{INT})$ and $G_1 \cong G_3$, then

$$(G_1 \times G_2 \times G_3) \rtimes \mathbb{Z}_2 \in \mathrm{Aut}(\mathsf{INT}).$$

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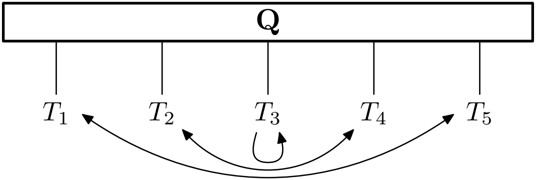
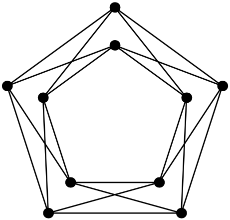


$\mathrm{Aut}(\mathsf{CIRCLE})$

Split Decomposition



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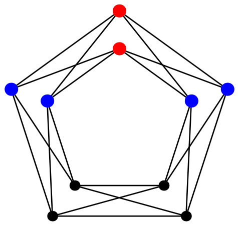

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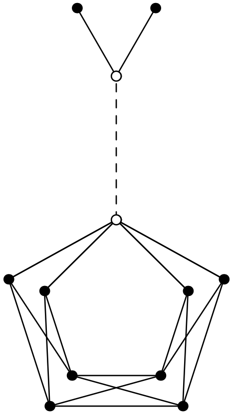

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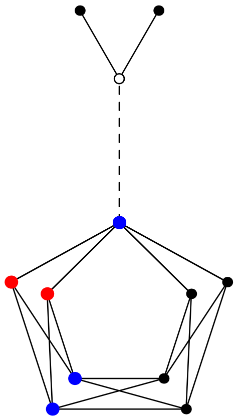

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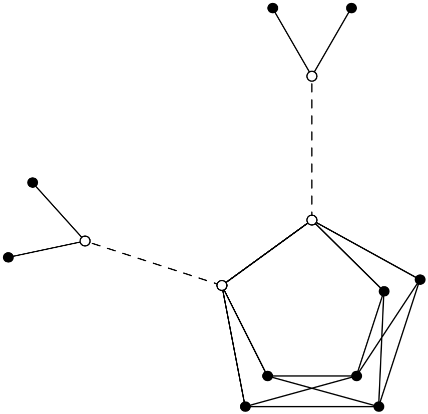

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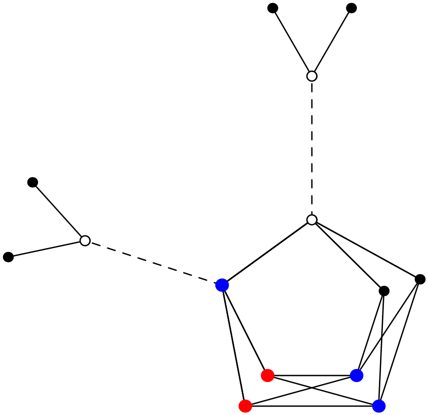

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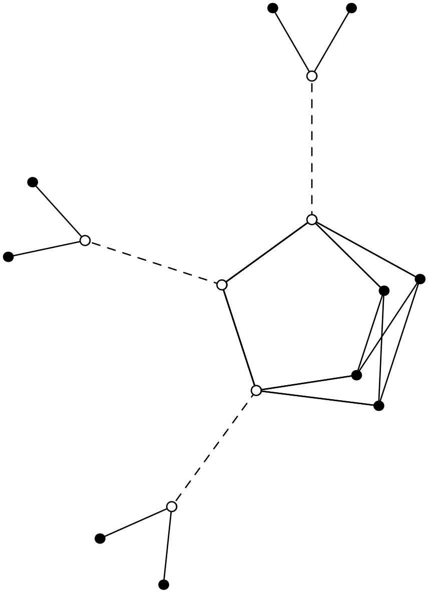

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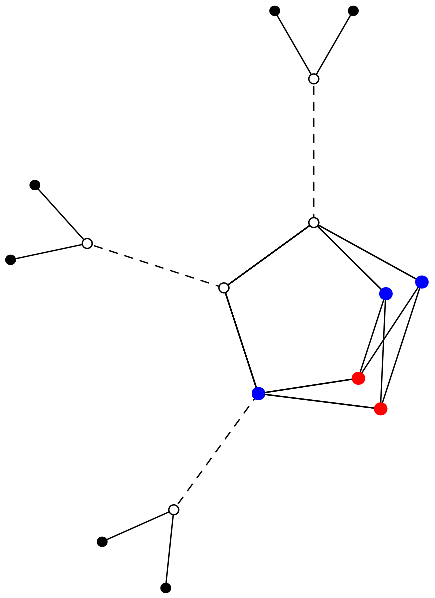

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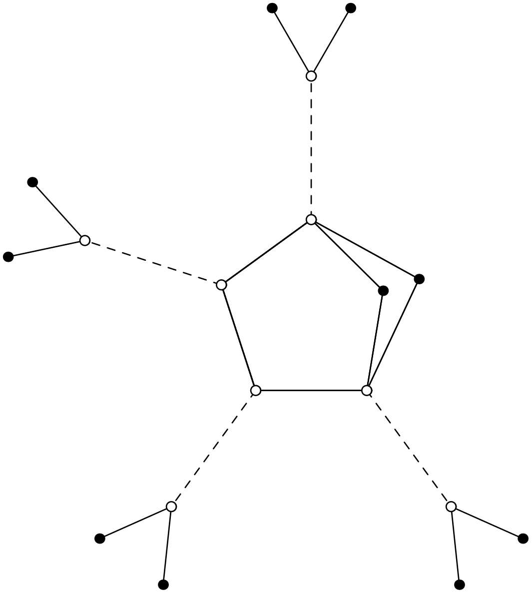

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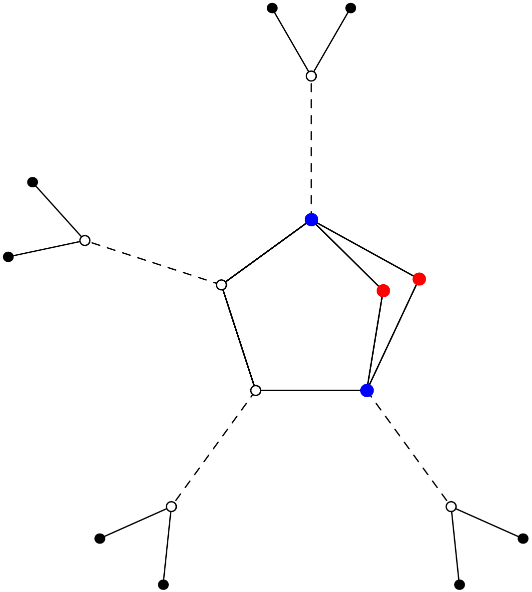

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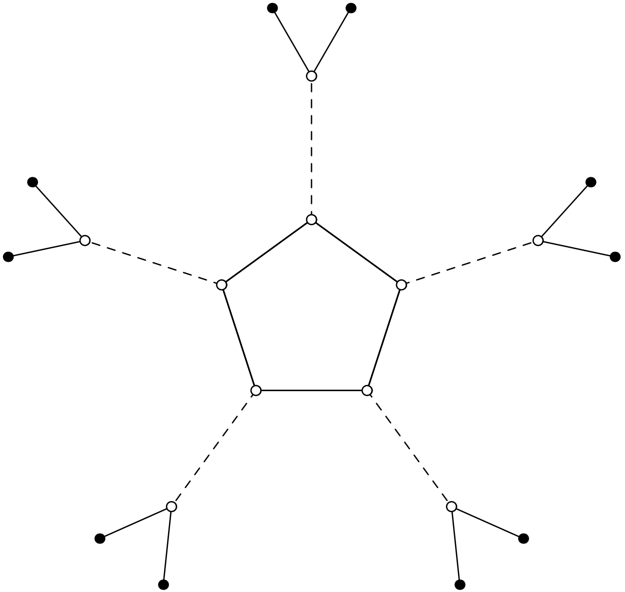


Automorphism Groups of Circle Graphs

It is clear that $\mathrm{Aut}(\mathsf{PSEUDOFOREST}) \subset \mathrm{Aut}(\mathsf{CIRCLE})$ since each pseudoforest is a circle graph.

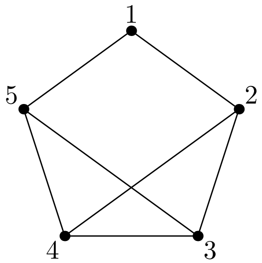
We prove that $\mathrm{Aut}(\mathsf{PSEUDOTREE}) =$

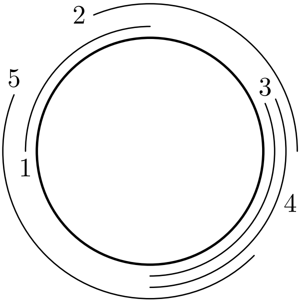
$$\bigcup_{n \geq 1} \mathrm{Aut}(\mathsf{TREE}) \rtimes \mathbb{D}_n \cup \mathrm{Aut}(\mathsf{TREE}) \rtimes \mathbb{Z}_n.$$

Finally, we prove that each connected circle graph X has $\mathrm{Aut}(X) \in \mathrm{Aut}(\mathsf{PSEUDOTREE})$; we use the split decomposition.

Open Problems

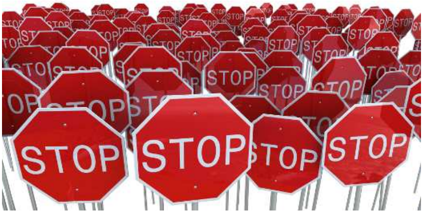
What are the automorphism groups of circular-arc graphs?





What is the precise relationship between universal graph classes and GI-complete graph classes?

Thank you!



$$\mathbb{D}_8 \wr \mathbb{S}_\infty$$