Automorphism Groups of Geometrically Represented Graphs

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Many interesting and important classes of intersection graphs are obtained by restricting the sets $\mathcal{R}_x$ to some geometric objects.

Map Theory

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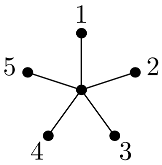

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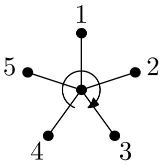

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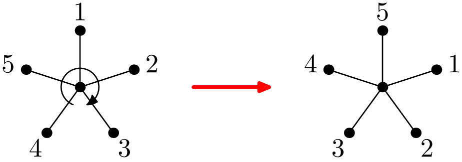

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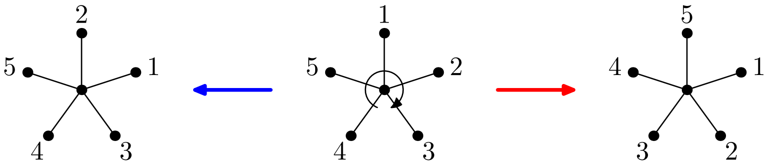

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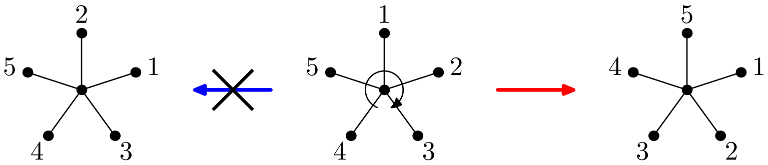

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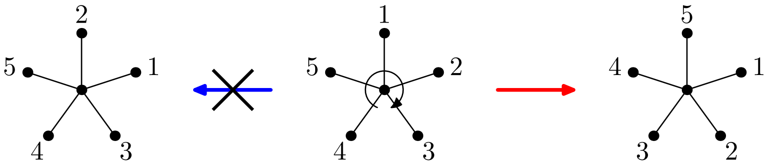

For every automorphism $\pi \in \mathrm{Aut}(X)$, one of the following holds:

- π is an automorphism of the map $\mathcal{M}$,
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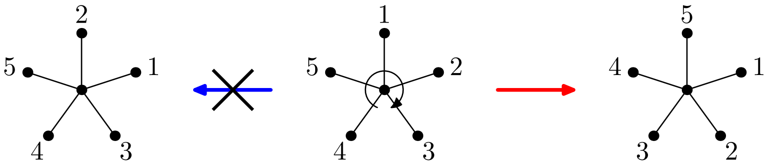

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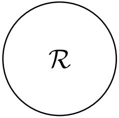
Typically, the automorphism group of the map $\mathrm{Aut}(\mathcal{M})$ is not complicated and it is a subgroup of $\mathrm{Aut}(X)$.

Action of the Group $\mathrm{Aut}(X)$ on $\mathfrak{Rep}(X)$

For every $X \in \mathcal{C}$, the group $\mathrm{Aut}(X)$ acts on the set of its geometric intersection representations $\mathfrak{Rep}(X)$.



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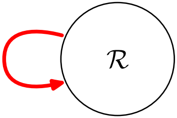

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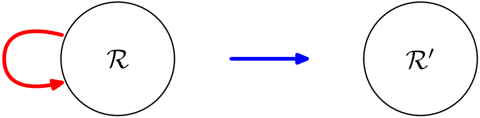

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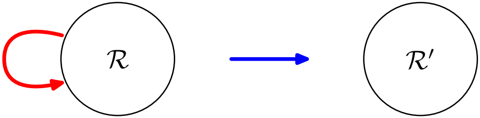
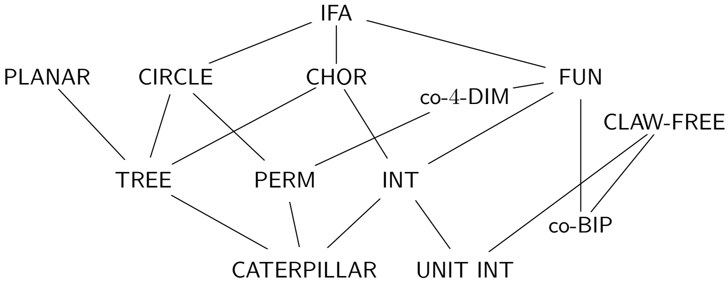


The stabilizer of a representation $\mathcal{R}$ is denoted by $\mathrm{Aut}(\mathcal{R})$ and it contains the automorphisms of the representation $\mathcal{R}$.

To understand the morphisms between the individual representations, we need to understand the structure of all geometric representations.

If this structure is strong enough, we can understand the morphisms between representations and determine $\mathrm{Aut}(X)$.

Studied Classes of Graphs



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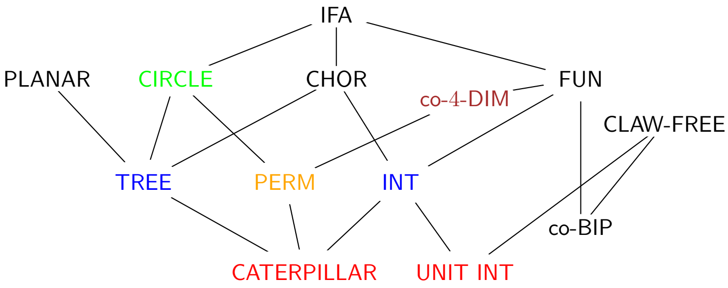

Our Results:

(i) $\mathrm{Aut}(\text{INT}) = \mathrm{Aut}(\text{TREE})$; PQ-trees.

(ii) $\mathrm{Aut}(\text{connected UNIT INT}) = \mathrm{Aut}(\text{CATERPILLAR})$; PQ-trees.

(iii) A characterization of $\mathrm{Aut}(\text{PERM})$; modular trees.

(iv) Universality of the class 4-DIM.

(v) Characterization of $\mathrm{Aut}(\text{CIRCLE})$; split trees.

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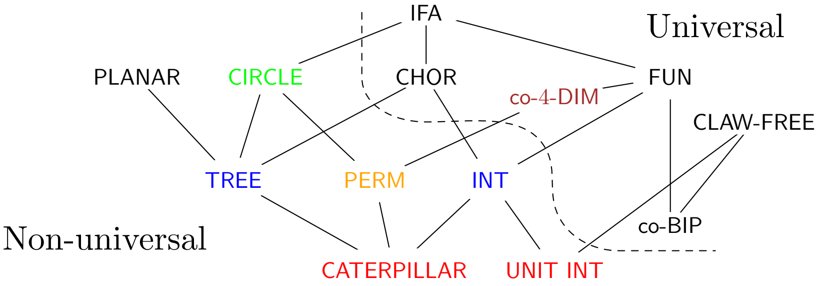

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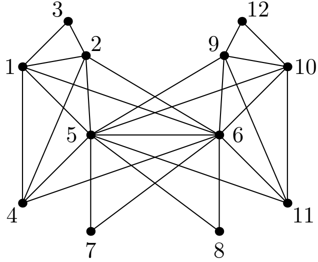
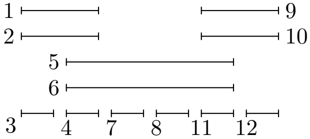
$\mathrm{Aut}(\mathsf{INT})$

Interval (INT) and unit interval graphs (UNIT INT)

Interval graphs are intersection graphs of intervals of the real line.

Unit interval graphs are interval graphs which have a representation with every interval of length one.





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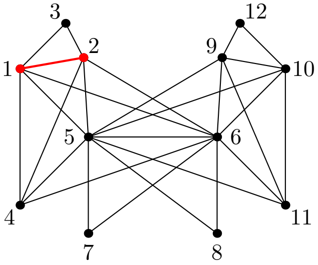

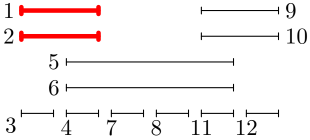

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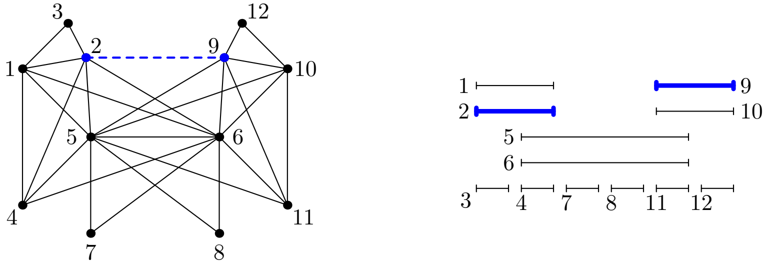
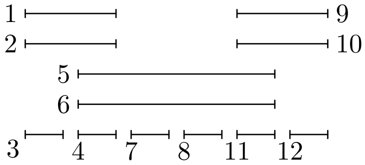


A Characterization of Interval Graphs

Theorem (Fulkerson a Gross): A graph X is an interval graph if and only if there exists an ordering of its maximal cliques such that for each vertex $x \in V(X)$, the maximal cliques containing x appear consecutively.



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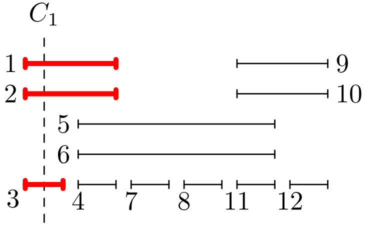

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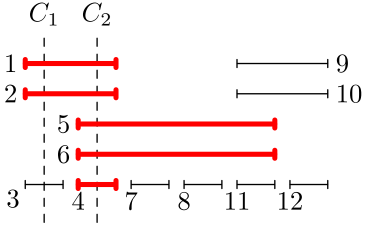

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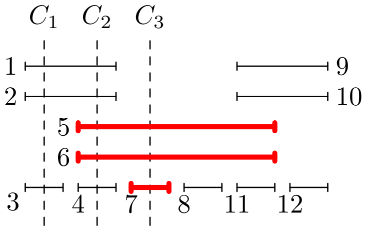

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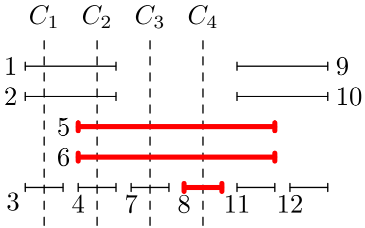

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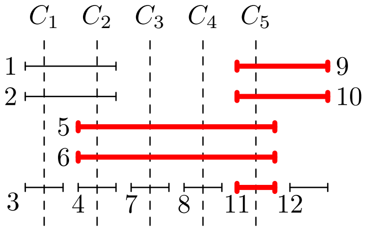

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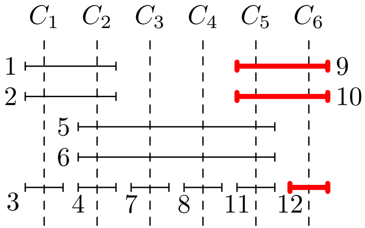

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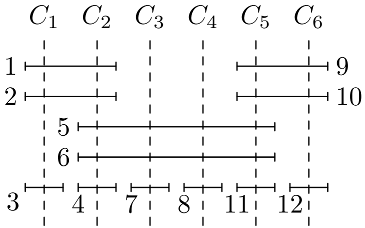

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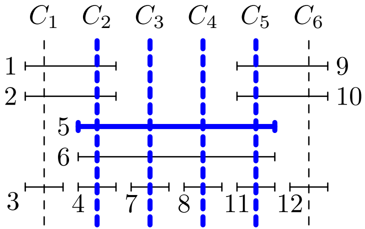

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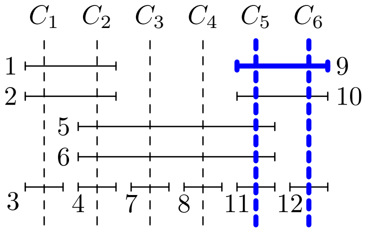
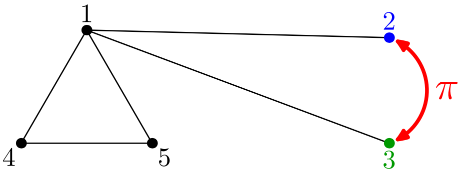
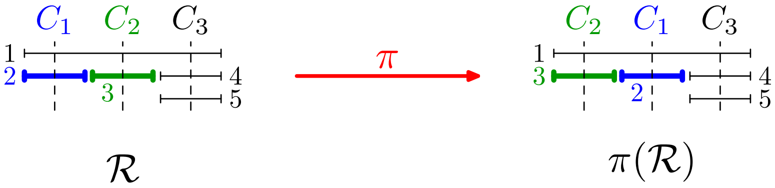
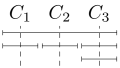


Morphisms Induced by an Automorphism $\pi \in \mathrm{Aut}(X)$





The Action of $\mathrm{Aut}(X)$ on $\mathfrak{Rep}(X)$



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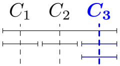

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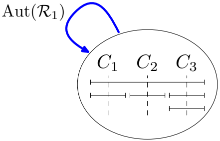

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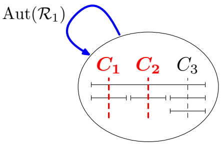

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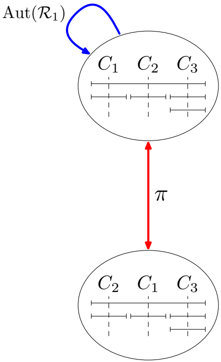

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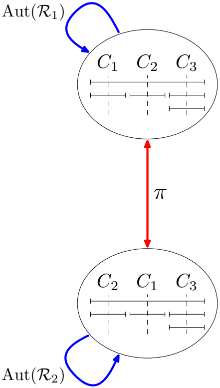

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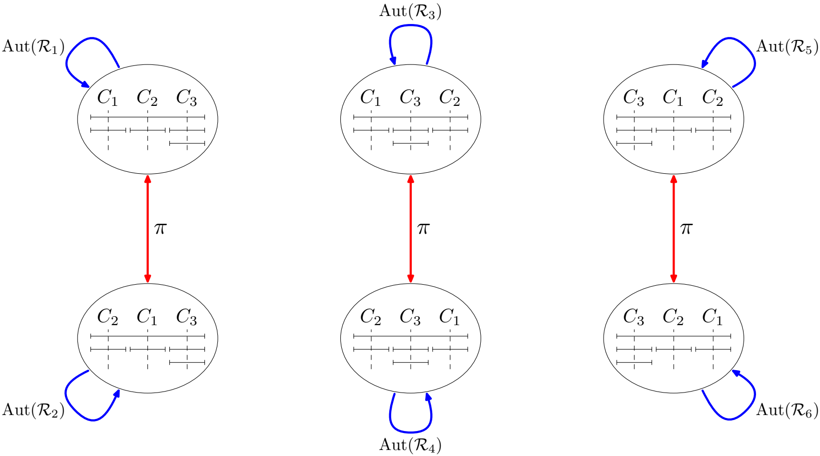

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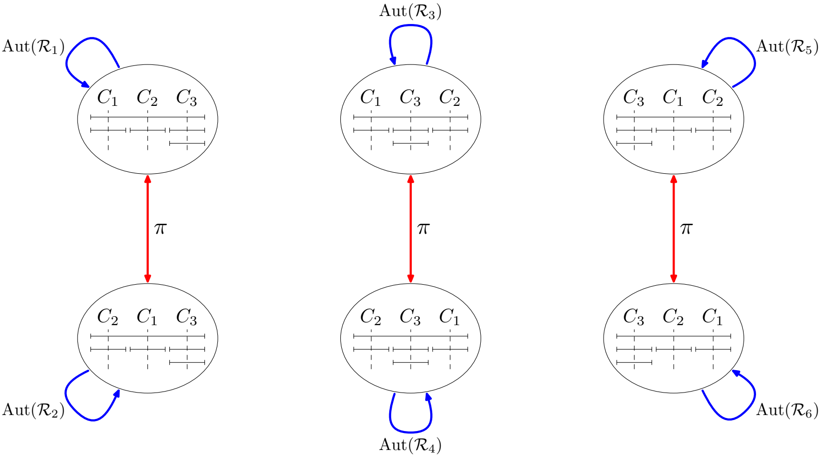
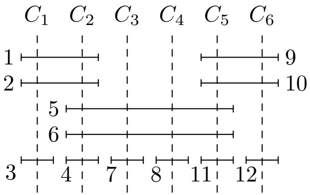
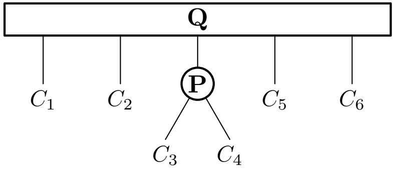


From $\mathrm{Aut}(\mathcal{R})$ and $\mathrm{Aut}(X)/\mathrm{Aut}(\mathcal{R})$, we can determine $\mathrm{Aut}(X)$.

PQ-trees (Booth and Lueker)





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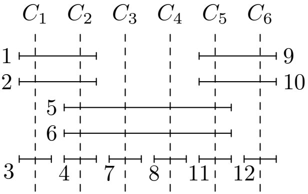
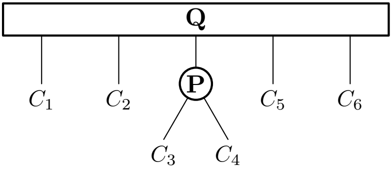




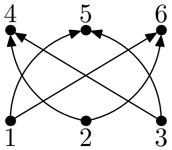
Theorem:

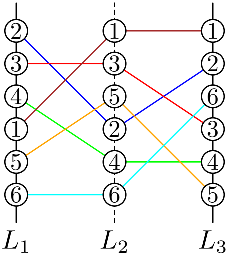
$$\begin{aligned} \mathrm{Aut}(\mathsf{INT}) &= \mathrm{Aut}(\mathsf{TREE}) \\ \mathrm{Aut}(\mathsf{UNIT\ INT}) &= \mathrm{Aut}(\mathsf{CATERPILLAR}) \end{aligned}$$

Comparability graphs

Comparability graphs

Comparability graphs (COMP) are graphs whose edges can be transitively oriented ($x \to y$ a $y \to z \implies x \to z$).





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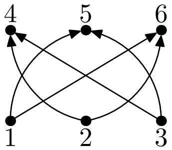
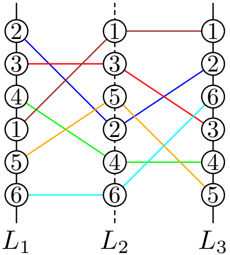




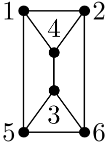
We denote the class of comparability graphs of dimension at most k by k-DIM. We obtain an infinite hierarchy of graph classes:

$$1\text{-DIM} \subsetneq 2\text{-DIM} \subsetneq \cdots \subsetneq k\text{-DIM} \subsetneq \cdots \subsetneq \text{COMP}.$$

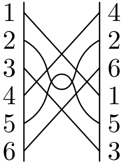
Function and Permutation Graphs

Function graphs (FUN) are intersection graphs of continuous real-valued functions defined on the interval $[0, 1]$.

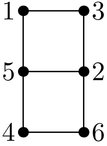
We get the permutation graphs (PERM) as intersection graphs of linear functions.

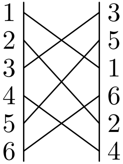


$\overline{C_6}$



FUN



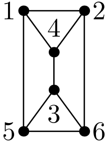


PERM

Function and Permutation Graphs

Function graphs (FUN) are intersection graphs of continuous real-valued functions defined on the interval $[0,1]$.

We get the permutation graphs (PERM) as intersection graphs of linear functions.



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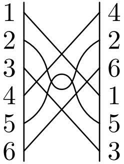

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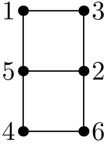
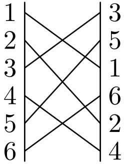




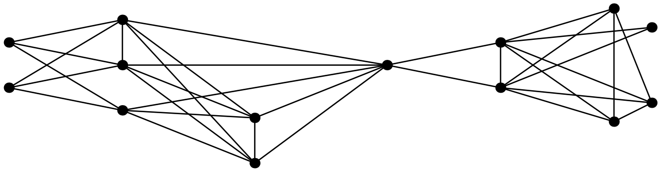
PERM

The following relations are well-known:

$$\begin{aligned} \text{FUN} &= \text{co-COMP}, \\ \text{PERM} &= \text{2-DIM} = \text{COMP} \cap \text{co-COMP}. \end{aligned}$$

Modular Decomposition and the Modular Tree

A set of vertices $M \subseteq V(X)$ is called a module of X if every vertex $x \notin M$ is either adjacent to all the vertices in M, or none of them.



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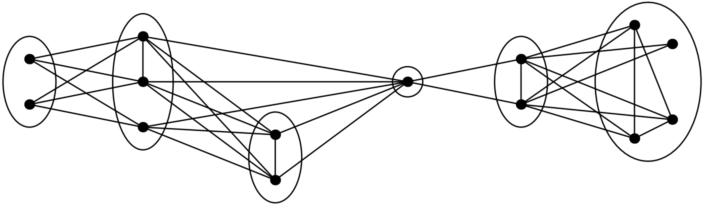

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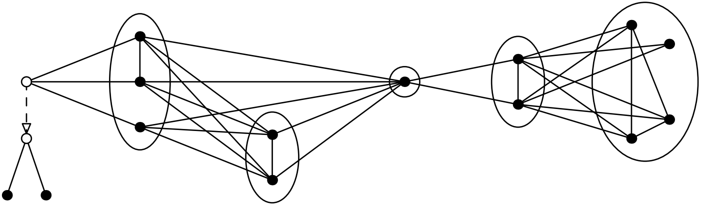

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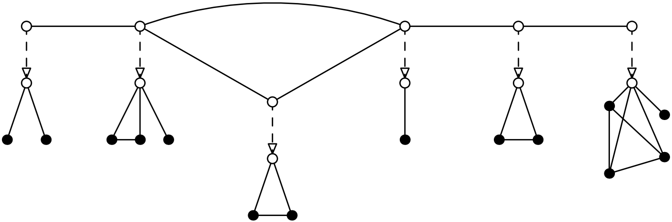

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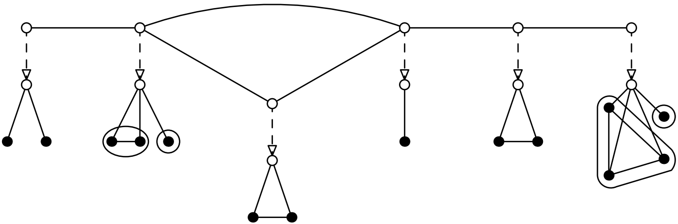

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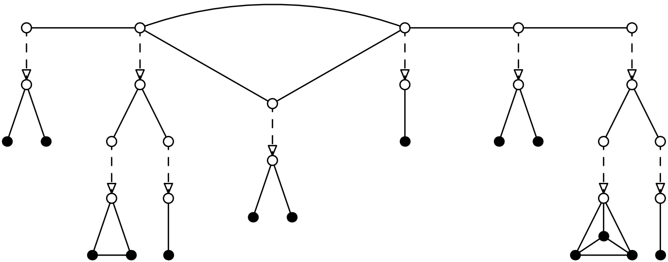

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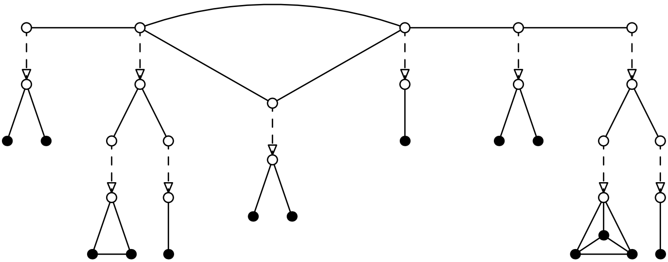
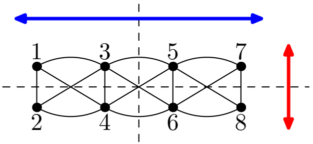
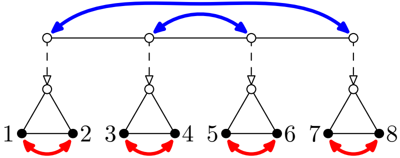


The decomposition stops on prime graphs (graphs having only trivial modules) and degenerate graphs (a clique or an independent set). Gallai proved that the resulting modular tree is unique.

Automorphisms of a Modular Tree

Automorphism of a modular tree are automorphisms of the graph preserving the types of vertices and edges.





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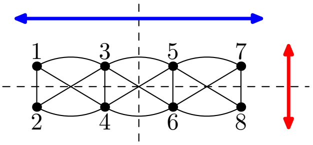

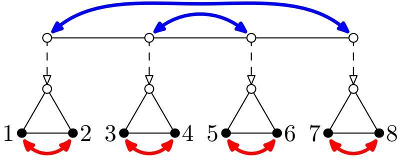

Lemma: If T is a modular tree of X, then $\mathrm{Aut}(T) \cong \mathrm{Aut}(X)$.

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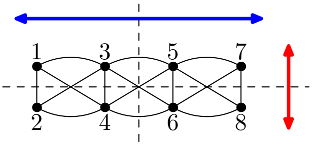

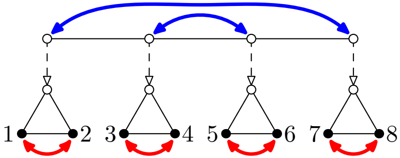

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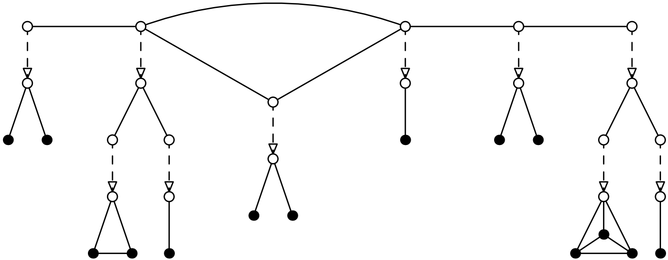
A Recursive Formula:

$$\mathrm{Aut}(T) \cong \big(\mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_k)\big) \rtimes \mathrm{Aut}(R).$$

Modular Trees and Comparability Graphs

Gallai proved the following:

- If two modules M_1 and M_2 are adjacent, then either $M_1 \to M_2$, or $M_2 \to M_1$.
- A graph X is a comparability graph if and only if every node of its modular tree is a comparability graph.
- Every prime comparability graph has at most two transitive orientations.



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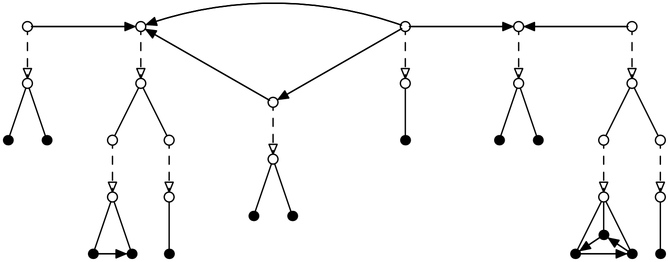
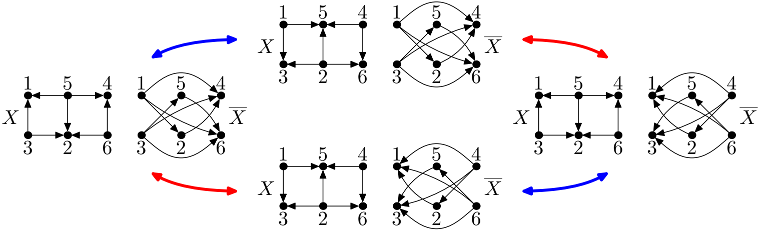


The Action of $\mathrm{Aut}(X)$ on the Transitive Orientations

Every automorphism of X induces a permutation of its transitive orientations.

If X is a permutation graph, then the group $\mathrm{Aut}(X)$ acts on the pairs $(\rightarrow, \overline{\rightrightarrows})$, where $\rightarrow$ is a transitive orientation of X and $\overline{\rightrightarrows}$ is a transitive orientation of $\overline{X}$.



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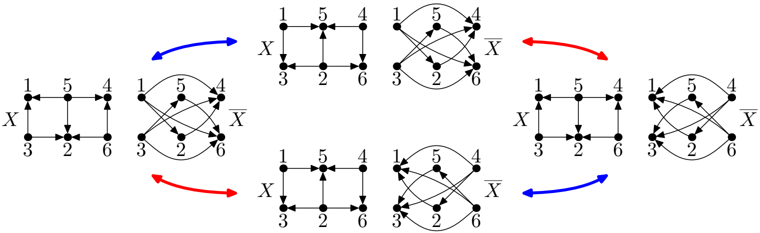
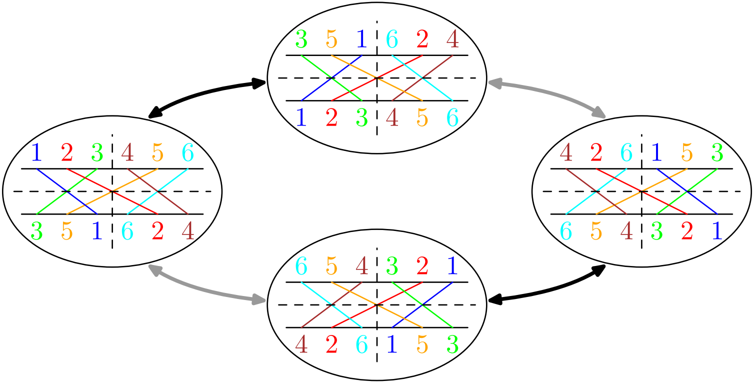


The automorphism group of a prime permutation graph is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The Action of $\mathrm{Aut}(X)$ on $\mathfrak{Rep}(X)$



Lemma: The action of $\mathrm{Aut}(X)$ on the representations of a permutation graph X has only trivial stabilizers. The structure of all representations is captured by the modular tree.

A Characterization of $\mathrm{Aut}(\mathsf{PERM})$

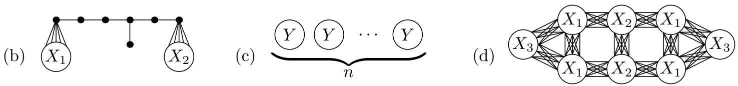
Theorem: The groups in $\mathrm{Aut}(\mathsf{PERM})$ can be described inductively:

(a) $\{1\} \in \mathrm{Aut}(\mathsf{PERM})$.

(b) $G_1, G_2 \in \mathrm{Aut}(\mathsf{PERM}) \implies G_1 \times G_2 \in \mathrm{Aut}(\mathsf{PERM})$.

(c) $G \in \mathrm{Aut}(\mathsf{PERM}) \implies G \wr \mathbb{S}_n \in \mathrm{Aut}(\mathsf{PERM})$.

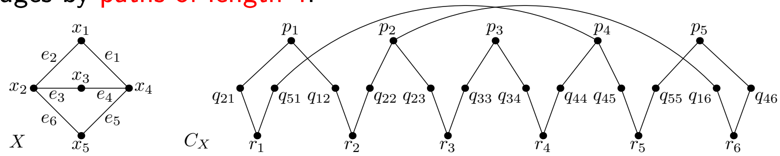
(d) $G_1, G_2, G_3 \in \mathrm{Aut}(\mathsf{PERM}) \implies$
$$\left(G_1^4 \times G_2^2 \times G_3^2\right) \rtimes \mathbb{Z}_2^2 \in \mathrm{Aut}(\mathsf{PERM}).$$



$\mathrm{Aut}(\textsf{4-DIM})$

A Construction for Bipartite Graphs

For a bipartite graph X with $V = (A, B)$ we construct a comparability graph C_X of dimension 4 such that we replace the edges by paths of length 4.



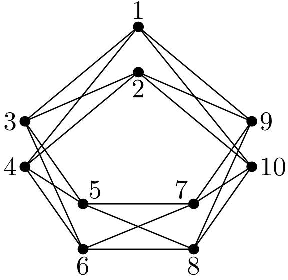
The linear ordering L_1, L_2, L_3, L_4 (the sets P_A, P_B, Q_A, Q_B depend on A and B, respectively) are defined as follows:

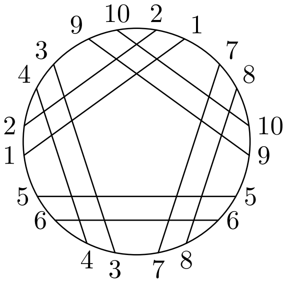
$$\begin{aligned}
L_1 &= \langle p_i : p_i \in P_A \rangle \langle r_k q_{ik} : q_{ik} \in Q_A, \uparrow k \rangle \langle I_i : p_i \in P_B, \uparrow i \rangle, \\
L_2 &= \langle p_i : p_i \in P_A \rangle \langle r_k q_{ik} : q_{ik} \in Q_A, \downarrow k \rangle \langle I_i : p_i \in P_B, \downarrow i \rangle, \\
L_3 &= \langle p_j : p_j \in P_B \rangle \langle r_k q_{jk} : q_{jk} \in Q_B, \uparrow k \rangle \langle I_i : p_i \in P_A, \uparrow i \rangle, \\
L_4 &= \langle p_j : p_j \in P_B \rangle \langle r_k q_{jk} : q_{jk} \in Q_B, \downarrow k \rangle \langle I_i : p_i \in P_A, \downarrow i \rangle.
\end{aligned}$$

$\mathrm{Aut}(\mathsf{CIRCLE})$

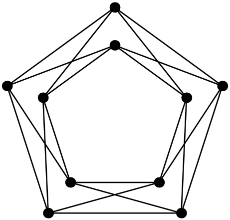
Circle graphs (CIRCLE)

Circle graphs are intersection graphs of chords of a circle.





Split Decomposition and Split Trees



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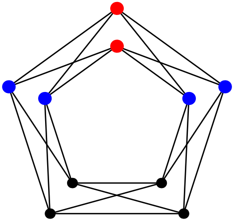

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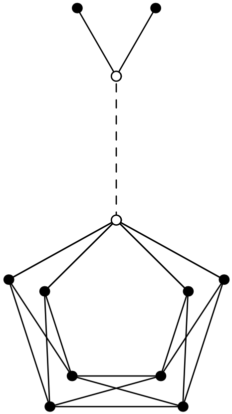

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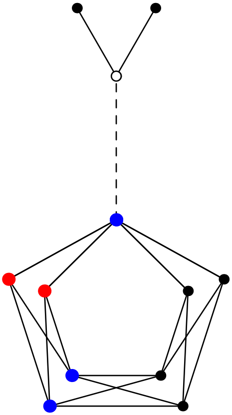

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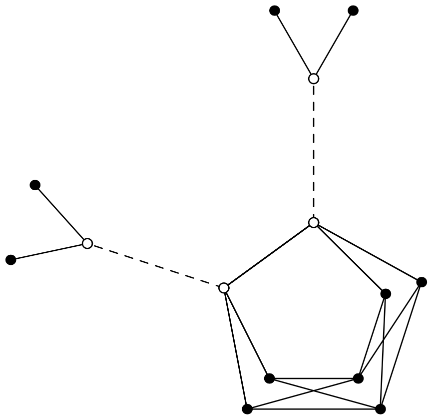

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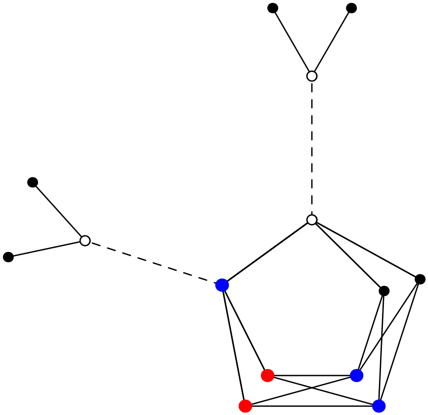

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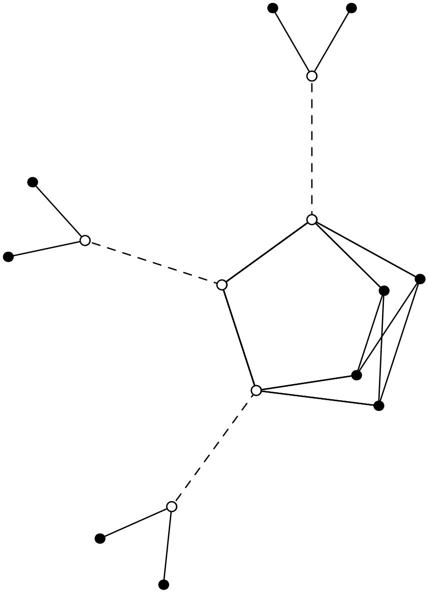

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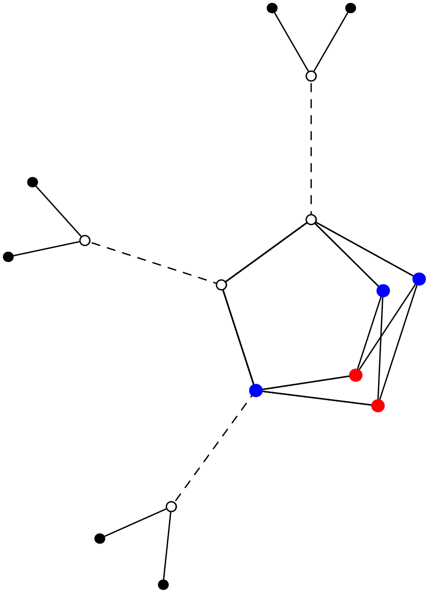

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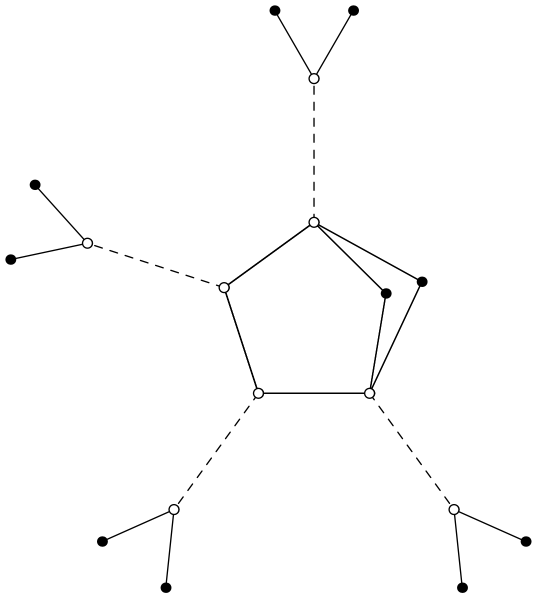

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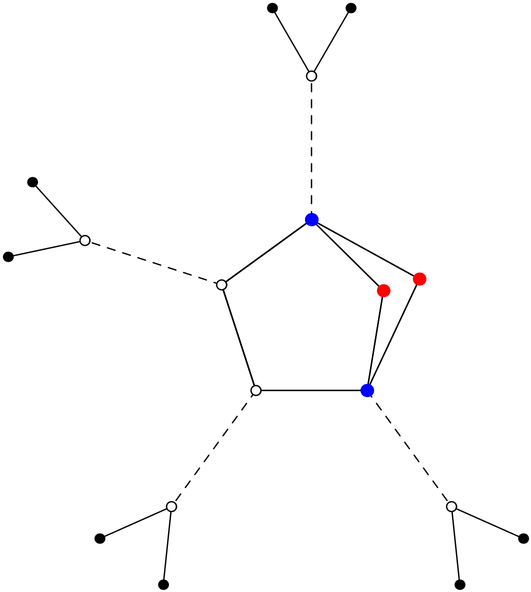

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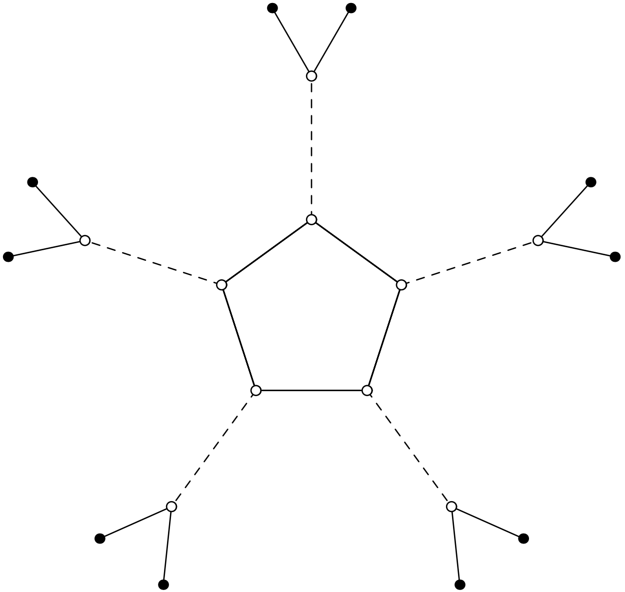


A Characterization of Aut(connected CIRCLE)

Theorem: Let $\mathcal{S}$ be a class of group defined as follows:

(a) $\{1\} \in \mathcal{S}$.

(b) $G_1, G_2 \in \mathcal{S} \implies G_1 \times G_2 \in \mathcal{S}$.

(c) $G \in \mathcal{S} \implies G \wr \mathbb{S}_n \in \mathcal{S}$.

(d) $G_1, G_2, G_3, G_4 \in \mathcal{S} \implies (G_1^4 \times G_2^2 \times G_3^2 \times G_4^2) \rtimes \mathbb{Z}_2^2 \in \mathcal{S}$.

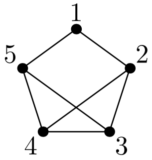
Then the class Aut(connected CIRCLE) can be defined inductively:

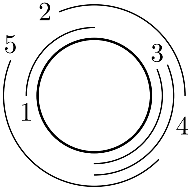
(e) $G \in \mathcal{S} \implies$
$G^m \rtimes \mathbb{Z}_n \in \mathrm{Aut}$(connected CIRCLE), for $n \neq 2$.

(f) $G_1, G_2 \in \mathcal{S} \implies$
$(G_1 \times G_2^2) \rtimes \mathbb{D}_n \in \mathrm{Aut}$(connected CIRCLE), for $n \geq 3$.

Open Problems

Problem: What are the automorphism groups of circular-arc graphs?





Conjecture: The automorphism groups of comparability graphs of dimension 3 are universal.

Problem: We have shown that the automorphism groups of trees are the same as the automorphism groups of interval graphs. Is this also true for the endomorphism monoids?

Thank you!