# Automorphism Groups of Geometrically Represented Graphs

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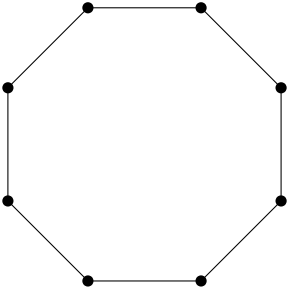
# Automorphism Groups of Graphs

An automorphism of a graph $X$ is a permutation $\pi$ of the vertices such that $xy \in E(X)$ if and only if $\pi(x)\pi(y) \in E(X)$.



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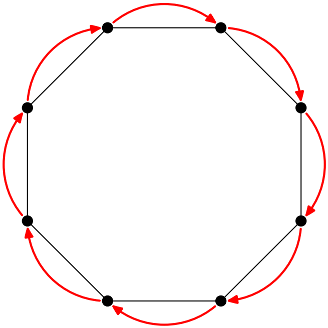

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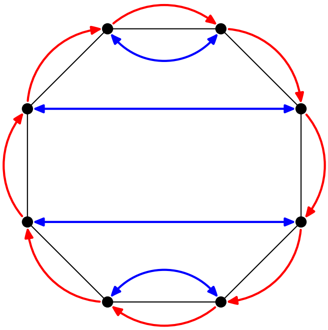

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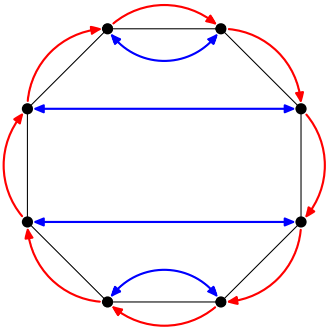
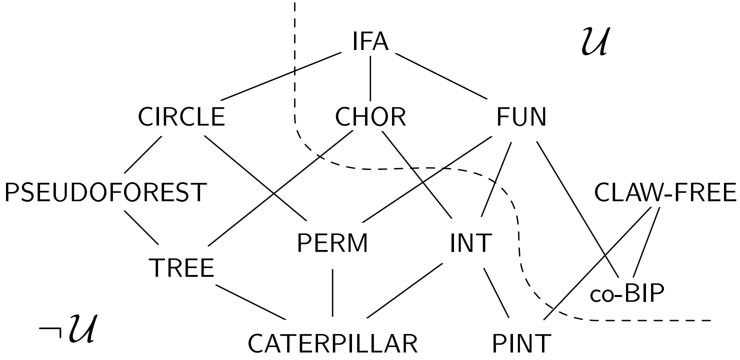


Theorem (Frucht): For each group $G$ there exits a graph $X$ such that $G \cong \mathrm{Aut}(X)$.

# Geometric Intersection Graphs

If $\mathcal{C}$ is a class of graphs, then $\mathrm{Aut}(\mathcal{C}) = \{\mathrm{Aut}(X) \colon X \in \mathcal{C}\}$.



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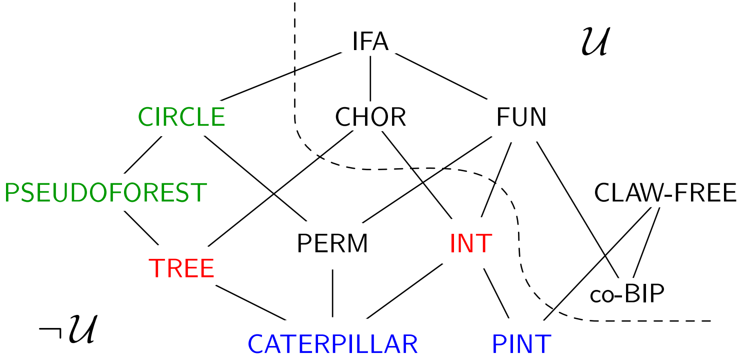


Our Result:

(i) $\mathrm{Aut}(\text{INT}) = \mathrm{Aut}(\text{TREE})$

(ii) $\mathrm{Aut}(\text{connected PINT}) = \mathrm{Aut}(\text{CATERPILLAR})$

(iii) $\mathrm{Aut}(\text{CIRCLE}) = \mathrm{Aut}(\text{PSEUDOFOREST})$

# Interval Graphs

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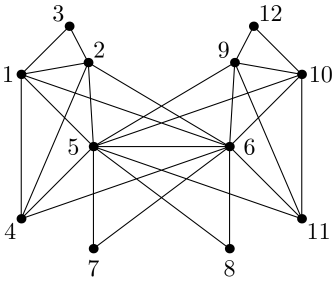
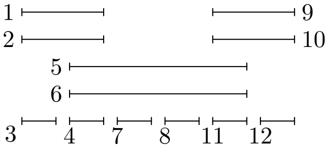
- Interval representation of a graph $X$ is a set $\{I_x \colon x \in V(X)\}$ such that each $I_x$ is an interval on the real line and $xy \in E(X)$ if and only if $I_x \cap I_y \neq \emptyset$.
- A graph $X$ is an interval graph if and only if it has an interval representation.



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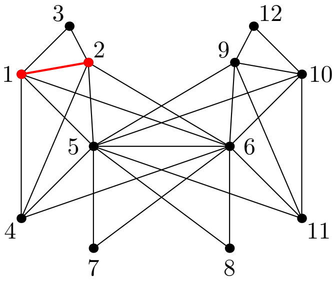


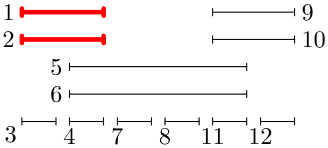

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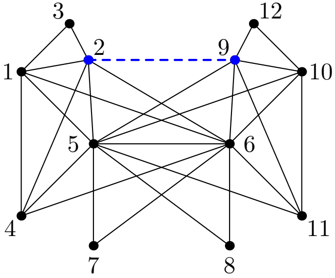
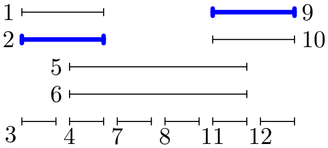
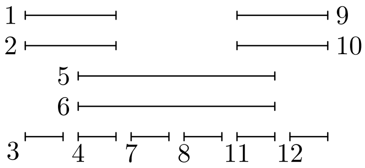


# Characterization of Interval Graphs

Theorem (Fulkerson and Gross): A graph $X$ is an interval graph if and only if there exists ordering of the maximal cliques such that for every $x \in V(X)$ the maximal cliques containing $x$ appear consecutively.



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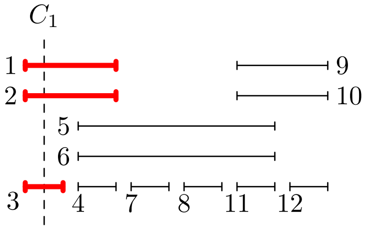

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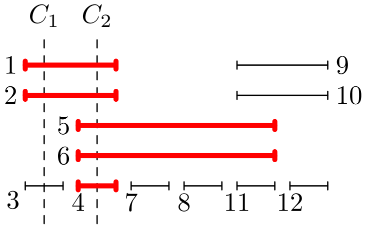

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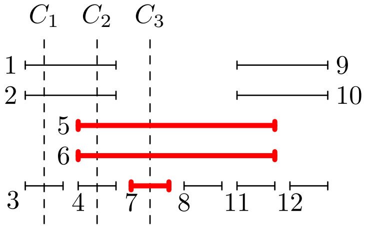

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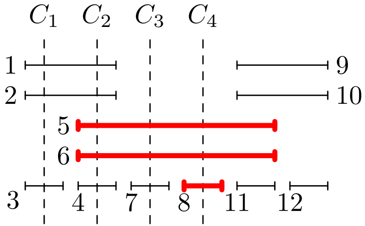

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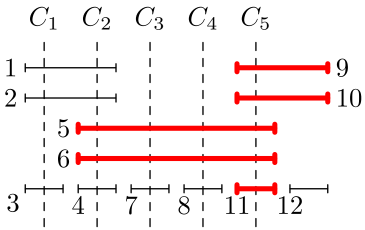

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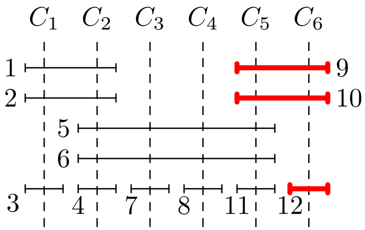

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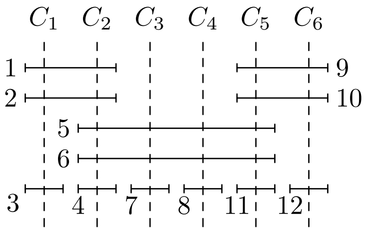

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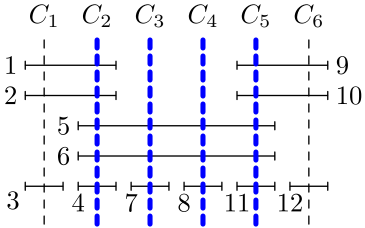

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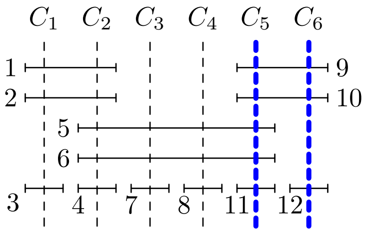
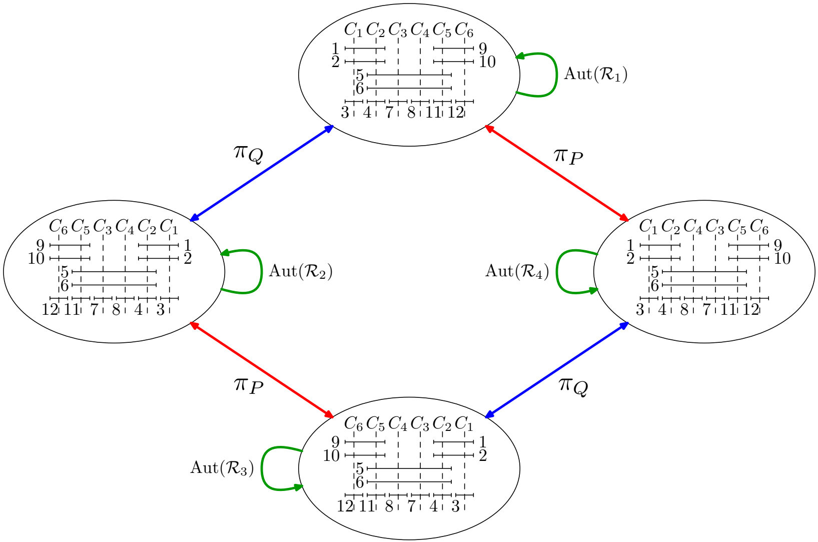
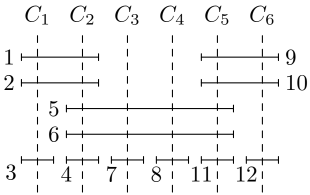
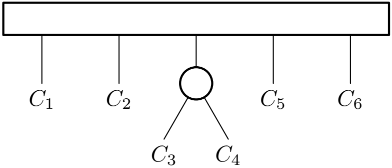


# The Action of $\mathrm{Aut}(X)$ on $\mathfrak{Rep}/\!\sim$



# PQ-trees (Booth and Lueker)



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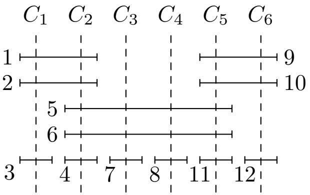


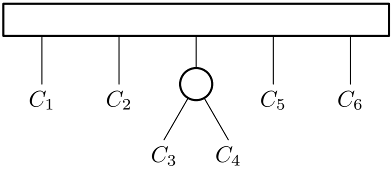


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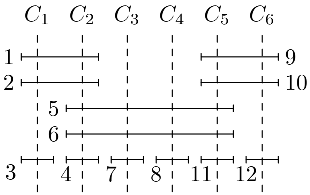


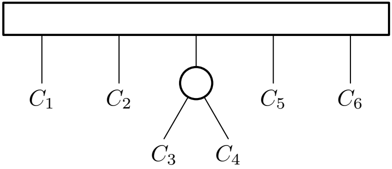


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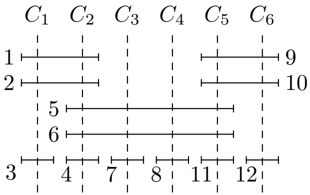


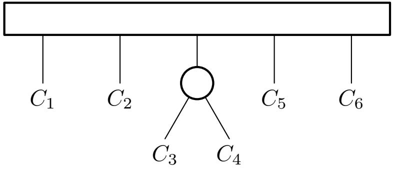


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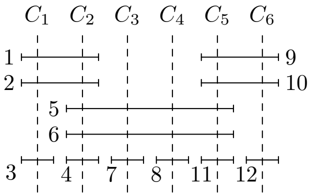
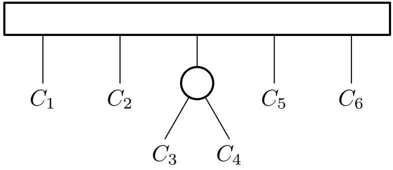
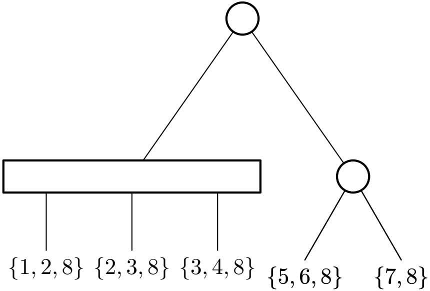



- A PQ-tree has two types of internal nodes: P-nodes and Q-nodes.
- The leaves of a PQ-tree correspond to the maximal cliques of an interval graph.
- There are two equivalence transformations: arbitrary permutation of the children of a P-node, and a reversal of the children of a Q-node.
- Each interval graph can be represented by a PQ-tree such that all equivalent PQ-trees represent all possible interval representations.

# Automorphisms of PQ-trees

Each symmetric equivalence transformation of a PQ-tree $T$ is an automorphism of $T$. Automorphisms of $T$ form a group.



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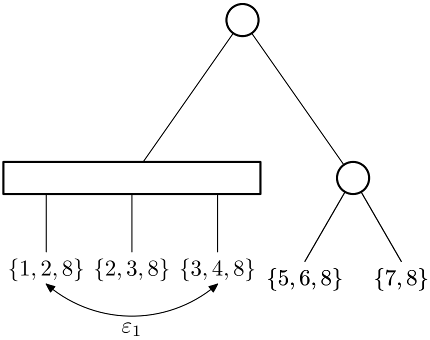

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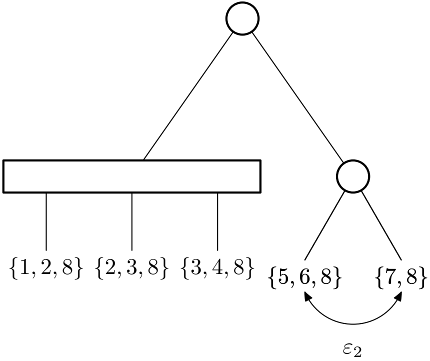

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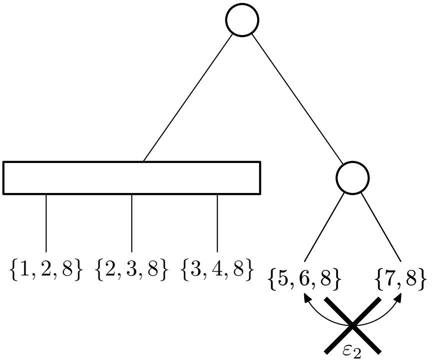

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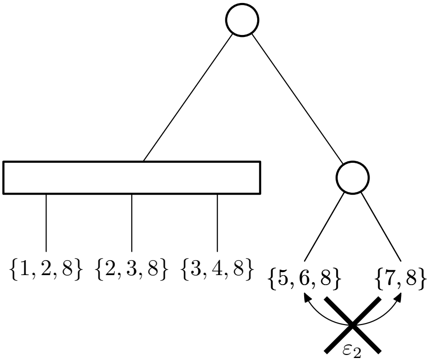
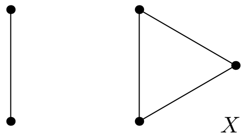



Proposition: If $T$ is a PQ-tree representing an interval graph $X$, then $\mathrm{Aut}(T) \cong \mathrm{Aut}(X)/\mathrm{Aut}(\mathcal{R})$.

# Group Products

# Automorphism Groups of Disconnected Graphs



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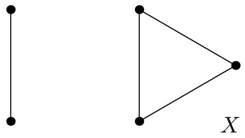


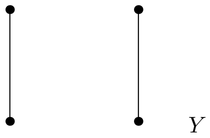

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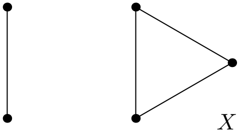
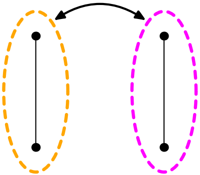
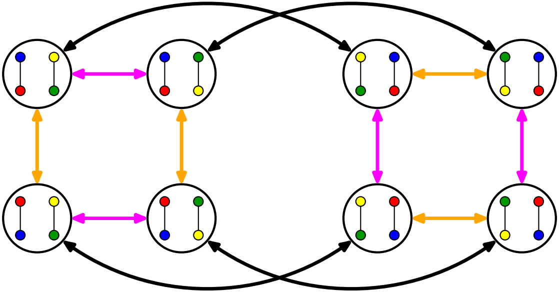


$X$



$Y$



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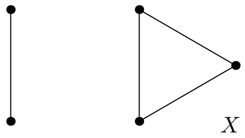
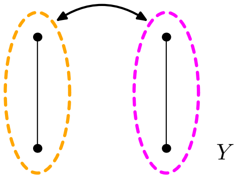
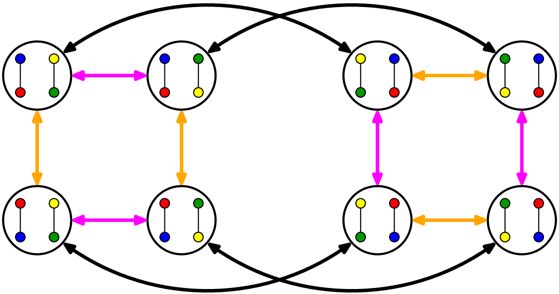


$$\mathrm{Aut}(Y) = (\mathbb{S}_2 \times \mathbb{S}_2) \rtimes \mathbb{S}_2 = \mathbb{S}_2 \wr \mathbb{S}_2$$

# Automorphism Groups of Trees (Jordan)

Theorem: If graph $X$ contains $k_i$ copies of a graph $X_i$, then

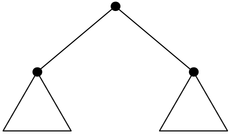
$$\mathrm{Aut}(X) = \mathrm{Aut}(X_1) \wr \mathbb{S}_{k_1} \times \cdots \times \mathrm{Aut}(X_{k_n}) \wr \mathbb{S}_{k_n}.$$

Theorem (Jordan, 1869): A group $G \in \mathrm{Aut}(\mathsf{TREE})$ if and only if $G \in \mathcal{T}$, where $\mathcal{T}$ is defined inductively as follows:

(a) $\{1\} \in \mathcal{T}$.

(b) If $G_1, G_2 \in \mathcal{T}$, then $G_1 \times G_2 \in \mathcal{T}$.

(c) If $G \in \mathcal{T}$ and $n \geq 2$, then $G \wr \mathbb{S}_n \in \mathcal{T}$.



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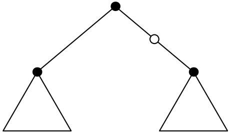

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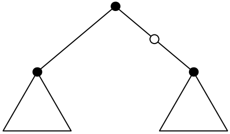

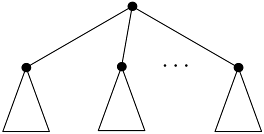

# Automorphism Groups of Trees (Jordan)

Theorem: If graph $X$ contains $k_i$ copies of a graph $X_i$, then

$$\mathrm{Aut}(X) = \mathrm{Aut}(X_1) \wr \mathbb{S}_{k_1} \times \cdots \times \mathrm{Aut}(X_{k_n}) \wr \mathbb{S}_{k_n}.$$

Theorem (Jordan, 1869): A group $G \in \mathrm{Aut}(\mathsf{TREE})$ if and only if $G \in \mathcal{T}$, where $\mathcal{T}$ is defined inductively as follows:

(a) $\{1\} \in \mathcal{T}$.

(b) If $G_1, G_2 \in \mathcal{T}$, then $G_1 \times G_2 \in \mathcal{T}$.

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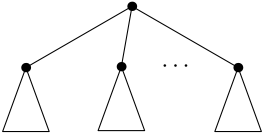

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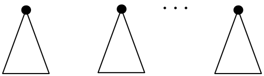

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Theorem: A group $G \in \mathrm{Aut}(\mathsf{INT})$ if and only if $G \in \mathcal{I}$, where $\mathcal{I}$ is defined inductively as follows:

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(d) If $G_1, G_2, G_3 \in \mathcal{I}$ and $G_1 \cong G_3$, then

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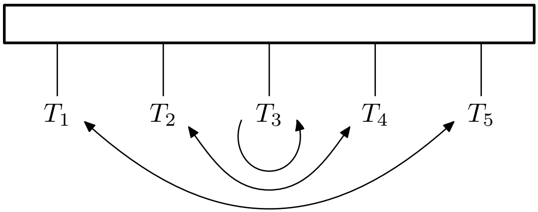

# Circle Graphs

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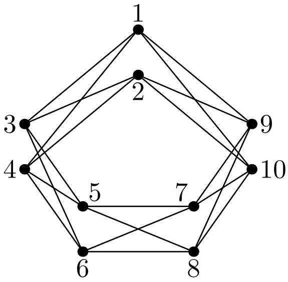
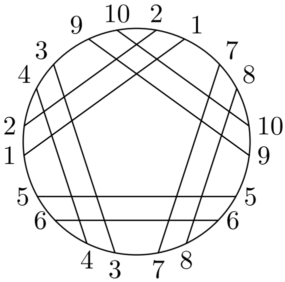
- Circle representation of a graph $X$ is a set $\{C_x : x \in V(X)\}$ such that each $C_x$ is a chord of a circle and $xy \in E(X)$ if and only if $C_x \cap C_y \neq \emptyset$.
- A graph $X$ is a circle graph if and only if it has an circle representation.



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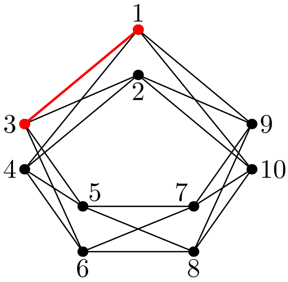


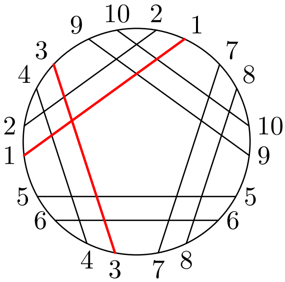

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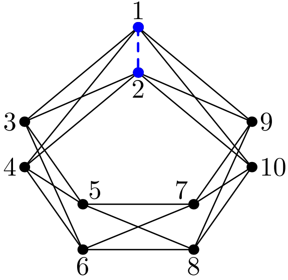
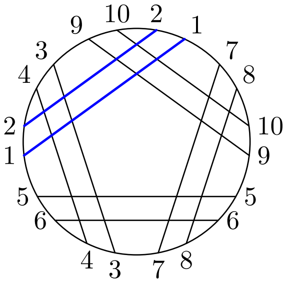
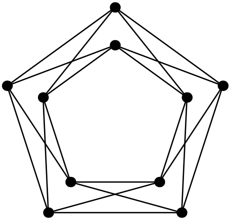


# Split Decomposition



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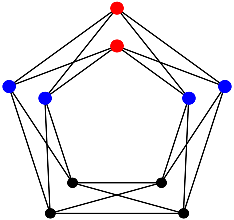

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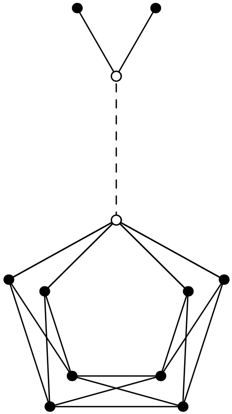

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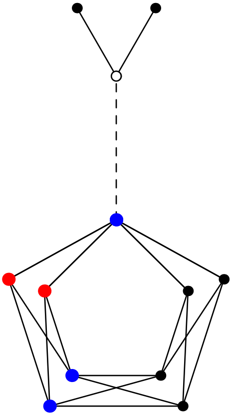

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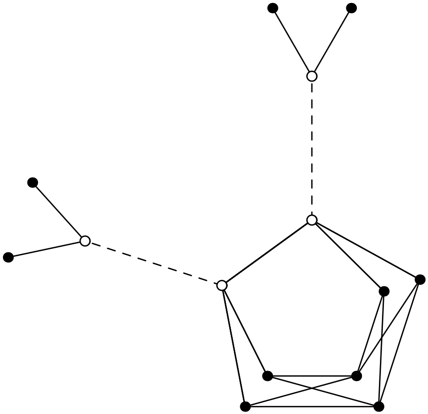

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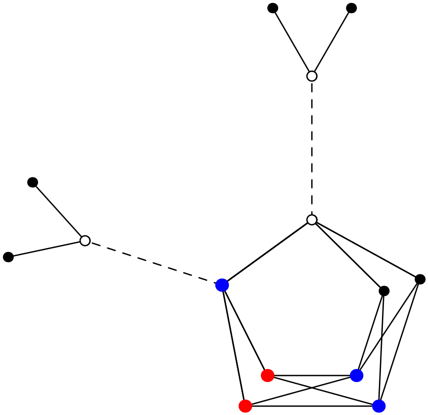

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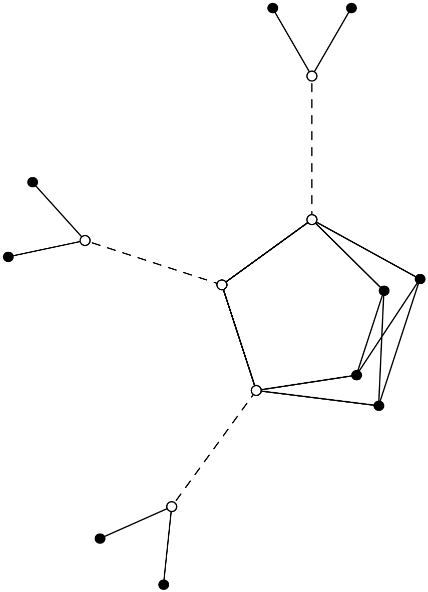

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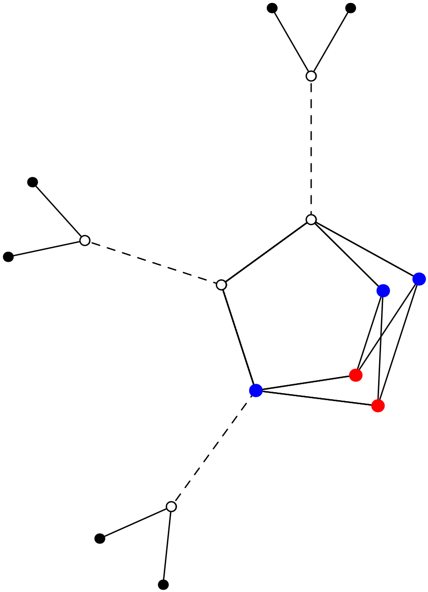

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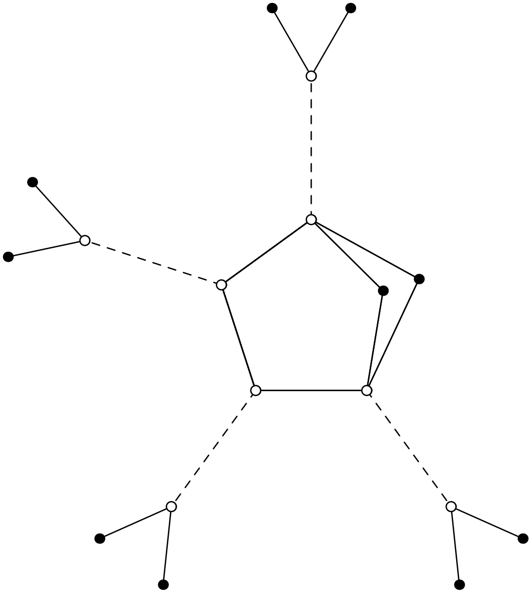

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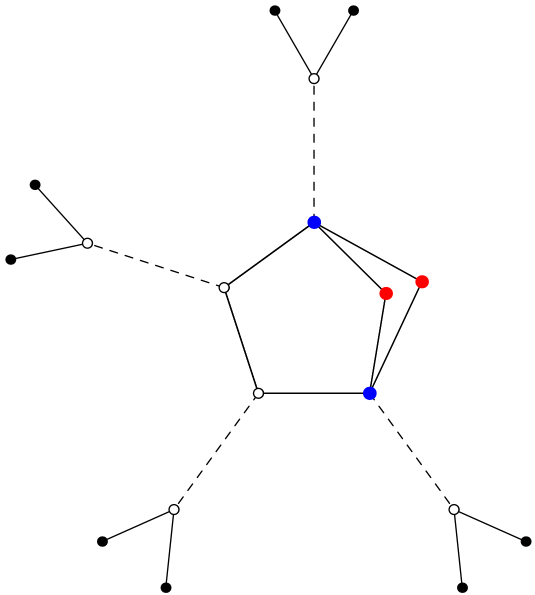

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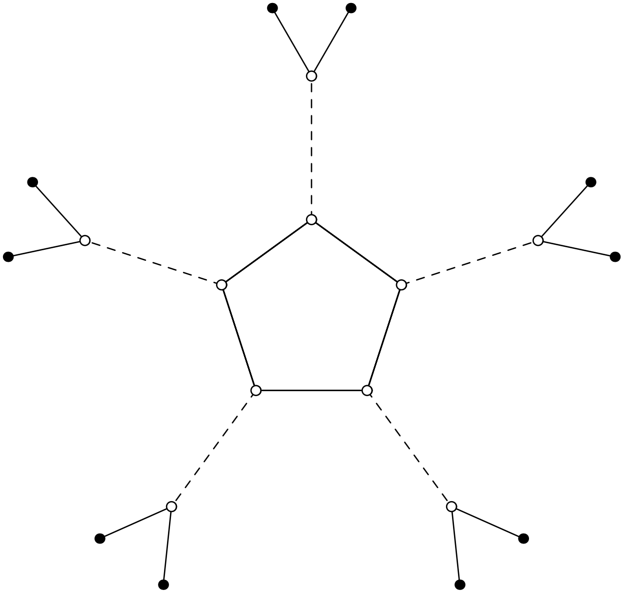


# Automorphism Groups of Circle Graphs

It is clear that $\mathrm{Aut}(\mathsf{PSEUDOFOREST}) \subseteq \mathrm{Aut}(\mathsf{CIRCLE})$ since each pseudoforest is a circle graph.

We prove that $\mathrm{Aut}(\mathsf{PSEUDOTREE}) =$

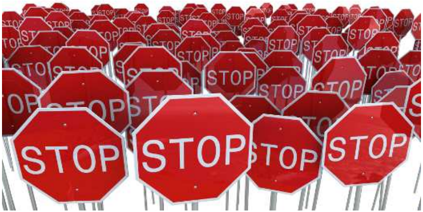
$$\bigcup_{n \geq 1} \mathrm{Aut}(\mathsf{TREE}) \rtimes \mathbb{D}_n \cup \mathrm{Aut}(\mathsf{TREE}) \rtimes \mathbb{Z}_n.$$

Finally, we prove that each connected circle graph $X$ has $\mathrm{Aut}(X) \in \mathrm{Aut}(\mathsf{PSEUDOTREE})$; we use the split decomposition.

# Open Problems

- What are the automorphism groups of circular-arc graphs?
- What is the precise relationship between universal graph classes and GI-complete graph classes?

# Thank you!




$\mathbb{D}_8 \wr \mathbb{S}_\infty$