Automorphism Groups of Interval Graphs

Peter Zeman
joint work with Pavel Klavík

Faculty of Mathematics and Physics,
Charles University in Prague

ATCAGC 2014

The Automorphism Group of a Disconnected Graph

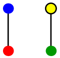
- If a graph G has n pairwise nonisomorphic connected components $G_1, \ldots, G_n$, then

$$\mathrm{Aut}(G) = \mathrm{Aut}(G_1) \times \cdots \times \mathrm{Aut}(G_n).$$

The Automorphism Group of a Disconnected Graph

- If a graph G has n pairwise nonisomorphic connected components $G_1, \ldots, G_n$, then

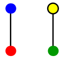
$$\mathrm{Aut}(G) = \mathrm{Aut}(G_1) \times \cdots \times \mathrm{Aut}(G_n).$$

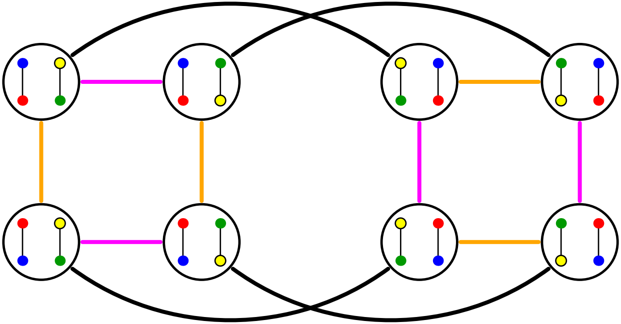
- $\mathrm{Aut}\Big($  $\Big) \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

The Automorphism Group of a Disconnected Graph

- If a graph G has n pairwise nonisomorphic connected components $G_1, \dots, G_n$, then

$$\mathrm{Aut}(G) = \mathrm{Aut}(G_1) \times \cdots \times \mathrm{Aut}(G_n).$$

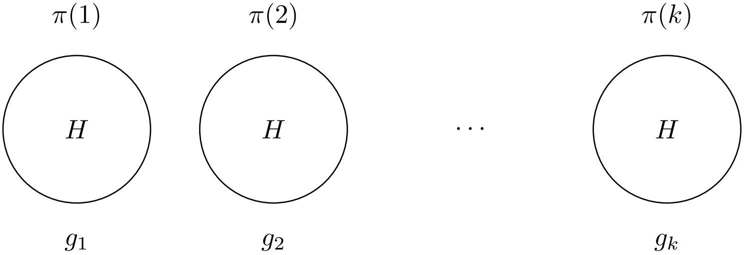
- $\mathrm{Aut}\Big($  $\Big) \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$



Wreath product

- If a graph G contains k copies of H, then the automorphism group of G is isomorphic to $\mathrm{Aut}(H) \wr \mathbb{S}_k$, where

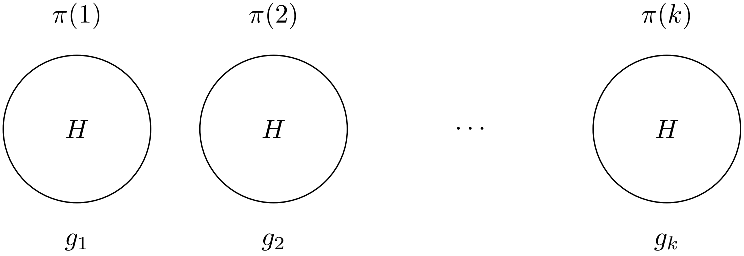
$$\mathrm{Aut}(H) \wr \mathbb{S}_k = \left\{ (g_1, \ldots, g_k, \pi) \colon g_i \in \mathrm{Aut}(H), \pi \in \mathbb{S}_k \right\}.$$



Wreath product

- If a graph G contains k copies of H, then the automorphism group of G is isomorphic to $\mathrm{Aut}(H) \wr \mathbb{S}_k$, where

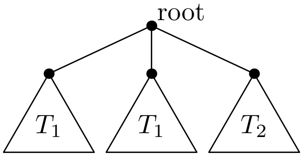
$$\mathrm{Aut}(H) \wr \mathbb{S}_k = \{(g_1, \ldots, g_k, \pi) \colon g_i \in \mathrm{Aut}(H), \pi \in \mathbb{S}_k\}.$$



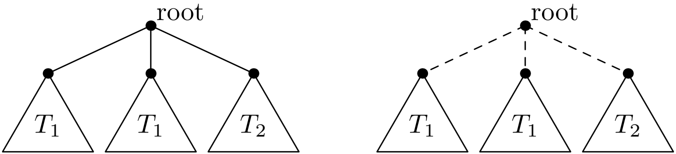
- If a graph G contains k_i copies of G_i for $i = 1, \ldots, n$, then the automorphism group of G is isomorphic to

$$\mathrm{Aut}(G_1) \wr \mathbb{S}_{k_1} \times \cdots \times \mathrm{Aut}(G_n) \wr \mathbb{S}_{k_n}.$$

Automorphism Groups of Trees

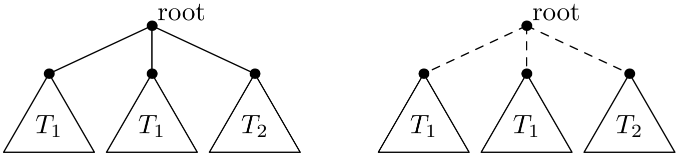


Automorphism Groups of Trees



$$\mathrm{Aut}(T_1) \wr \mathbb{S}_2 \times \mathrm{Aut}(T_2)$$

Automorphism Groups of Trees



$$\mathrm{Aut}(T_1) \wr \mathbb{S}_2 \times \mathrm{Aut}(T_2)$$

Theorem (Jordan, 1869)

The finite group Γ is isomorophic to the automorphism group of a finite tree if and only if $\Gamma \in \mathcal{T}$, where the class $\mathcal{T}$ of finite groups is defined inductively as follows:

(a) $\{1\} \in \mathcal{T}$,

(b) *if* $\Gamma_1, \Gamma_2 \in \mathcal{T}$ *then* $\Gamma_1 \times \Gamma_2 \in \mathcal{T}$,

(c) *if* $\Gamma \in \mathcal{T}$ *and* $n \geq 2$ *then* $\Gamma \wr \mathbb{S}_n \in \mathcal{T}$.

For interval graphs we show that we need to add an operation (d).

Interval Graphs

Let $I_1, \ldots, I_n$ be intervals on a real line. The corresponding interval graph G is the intersection graph of those intervals.

Interval Graphs

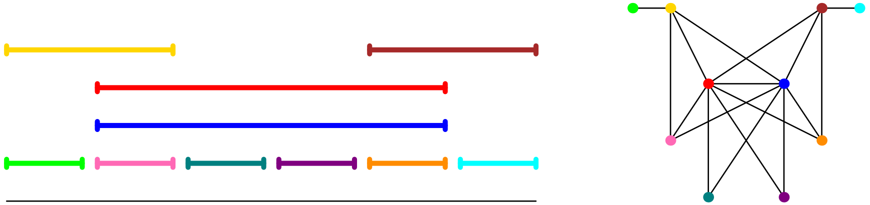
Let $I_1, \ldots, I_n$ be intervals on a real line. The corresponding interval graph G is the intersection graph of those intervals.

- $V(G) = \{I_1, \ldots, I_n\}$.
- $\{I_x, I_y\} \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$.

Interval Graphs

Let $I_1, \ldots, I_n$ be intervals on a real line. The corresponding interval graph G is the intersection graph of those intervals.

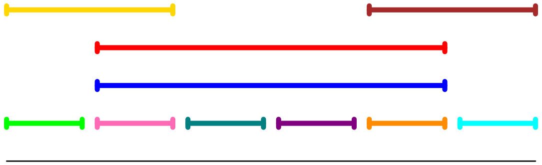
- $V(G) = \{I_1, \ldots, I_n\}$.
- $\{I_x, I_y\} \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$.

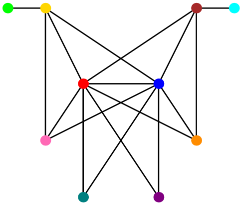


Interval Graphs

Let $I_1, \ldots, I_n$ be intervals on a real line. The corresponding interval graph G is the intersection graph of those intervals.

- $V(G) = \{I_1, \ldots, I_n\}$.
- $\{I_x, I_y\} \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$.





Colbourn and Kellogg found (1981) a linear time algorithm for finding a set of generators of the automorphism group of an interval graph.

Characterization of interval graphs

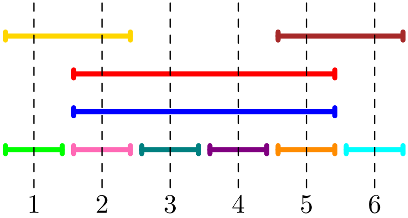
Theorem (Fulkerson and Gross)

A graph G is an interval graph if and only if there exists and ordering of the maximal cliques such that for every vertex $v \in V(G)$, the cliques containing v appear in it consequtively.

Characterization of interval graphs

Theorem (Fulkerson and Gross)

A graph G is an interval graph if and only if there exists and ordering of the maximal cliques such that for every vertex $v \in V(G)$, the cliques containing v appear in it consequtively.



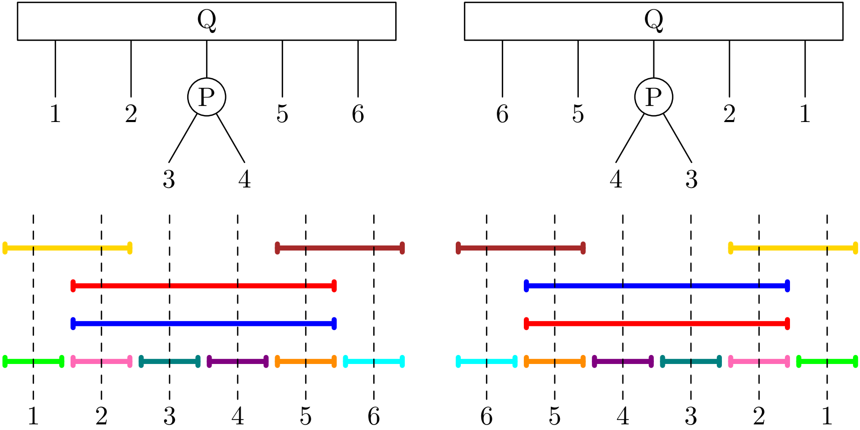
Restricting conditions for the ordering are $\{1, 2\}$, $\{5, 6\}$ and $\{2, 3, 4, 5\}$.

PQ-trees

Booth and Lueker (1976) invented PQ-trees for a more general purpose and used them to design a linear time algorithm for recognizing interval graphs.

PQ-trees

Booth and Lueker (1976) invented PQ-trees for a more general purpose and used them to design a linear time algorithm for recognizing interval graphs.



Automorphisms of PQ-trees

Two PQ-trees T and T' are equivalent if one can be obtained from the other by applying the following two equivalence transformations:

- Arbitrarily permute the children of a P-node.
- Reverse the children of a Q-node.

Automorphisms of PQ-trees

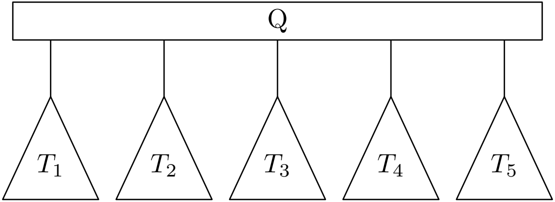
Two PQ-trees T and T' are equivalent if one can be obtained from the other by applying the following two equivalence transformations:

- Arbitrarily permute the children of a P-node.
- Reverse the children of a Q-node.

If ε represents a sequence of equivalence transformations, $\varepsilon \in \mathrm{Aut}(T)$ if there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(T_\varepsilon)$ is T.

Automorphism groups of PQ-trees

If we consider only PQ-trees with no Q-node, we get the same automorphism groups as for trees.



If T_1 is isomorphic to T_5 and T_2 is isomorphic to T_4, then reversing the ordering of $T_1, \ldots, T_5$ is an automorphism of T.

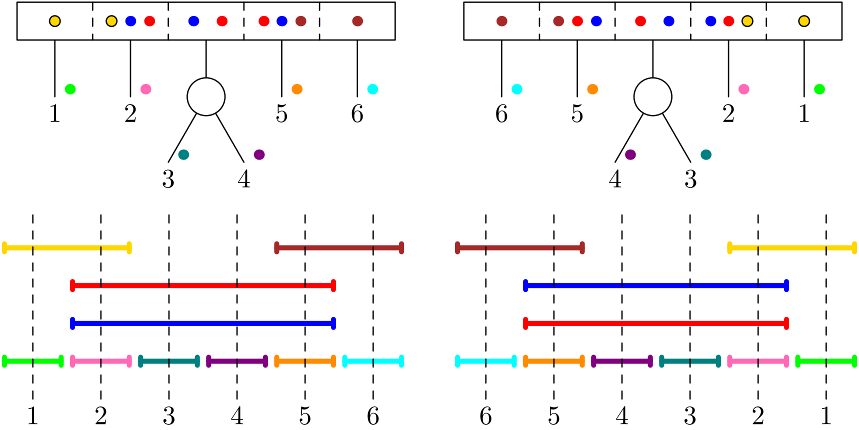
$$\begin{aligned}\mathrm{Aut}(T) &= (\mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_5)) \rtimes \mathbb{Z}_2 \\ &= \{(t_1, \ldots, t_5, z) \colon t_i \in \mathrm{Aut}(T_i), z \in \mathbb{Z}_2\}\end{aligned}$$

MPQ-trees

Korte and Möhring used MPQ-trees (1989) to desing a more simple recognition algorithm for interval graphs.

MPQ-trees

Korte and Möhring used MPQ-trees (1989) to desing a more simple recognition algorithm for interval graphs.



Automorphisms of MPQ-trees

Two MPQ-trees T and T' are equivalent if one can be obtained from the other by applying the equivalence transformations and reordering the sections with a node preserving permutation.

If ε represents a sequence of equivalence transformations and ν is a node preserving permutation, then $(\varepsilon, \nu) \in \mathrm{Aut}(T)$ if there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(T_{\varepsilon,\nu})$ is T.

Automorphism groups of MPQ-trees

- If T is a MPQ-tree for an interval graph G, then
$$\mathrm{Aut}(T) \cong \mathrm{Aut}(G).$$
- $\mathrm{Aut}(T) = E \times N$, where E is the automorphism group of the corresponding PQ-tree and N is a direct product of symmetric groups.

Theorem

The finite group Γ is isomorophic to the automorphism group of a finite interval graph if and only if $\Gamma \in \mathcal{I}$, where the class $\mathcal{I}$ of finite groups is defined inductively as follows:

(a) $\{1\} \in \mathcal{I}$,

(b) *if* $\Gamma_1, \Gamma_2 \in \mathcal{I}$ *then* $\Gamma_1 \times \Gamma_2 \in \mathcal{I}$,

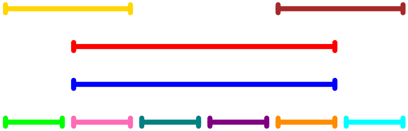
(c) *if* $\Gamma \in \mathcal{I}$ *and* $n \geq 2$ *then* $\Gamma \wr \mathbb{S}_n \in \mathcal{I}$.

(d) *if* $\Gamma_1, \ldots, \Gamma_n \in \mathcal{I}$, $n \geq 3$ *and* G_i *is the graph for which* $\mathrm{Aut}(G_i) = \Gamma_i$, *then* $(\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \mathbb{Z}_2 \in \mathcal{I}$ *if* $G_1 \cong G_n$, $G_2 \cong G_{n-1}$, *and so on.*

Further research

- Circle graphs
- Circular-arc graphs
- Intersection graphs in general

Thank you!



$$\mathrm{Aut}(G) \cong (\mathbb{Z}_2 \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$$