Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

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Kombinatorika matematických struktur

Katedra algebry

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Studijní program: Matematika, matematické struktury

2010

Děkuji svým rodičům za podporu v celém průběhu studia a to nejen za podporu finanční. Dále děkuji svému vedoucímu profesoru Krajíčkovi za shovívavost, trpělivost a cenné rady. Děkuji též slečně Zuzaně Safernové za obětavou kontrolu pravopisu a ostatní připomínky. Děkuji i docentu Mlčkovi za to, že mě seznámil s větou o mnoha modelech a upřesnil mi její znění.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 19. července 2010 Pavel Paták

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Název práce: Kombinatorika matematických struktur
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Abstrakt: Kombinatorika matematické struktury prvního řádu je třída všech formulí, které platí ve všech strukturách v ní definovatelných. Tento pojem poprvé zavedl Krajíček v [6]. V předložené práci se zabýváme charakterizací a srovnáním kombinatorik známých matematických struktur (reálná a komplexní čísla, husté lineární uspořádání, ...). Dále se věnujeme otázce výpočetní složitosti, tj. otázce, jak těžké je zjistit, zda daná formule leží v kombinatorice dané struktury. Dokážeme, že v případě modelů úplných teorií bez vlastnosti striktního uspořádání (SOP) či v případě pseudokonečných struktur je tento problém korekurzivně spočetně úplný a tudíž algoritmicky neřešitelný.
Klíčová slova: definovatelnost, matematické struktury, složitost, vlastnost striktního uspořádání (SOP), pseudokonečné struktury

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Abstract: The combinatorics of a first order mathematical structure is the class of all formulas valid in all in it definable structures. This notion was first introduced by Krajíček in [6]. In the present work we try to characterize and compare the combinatorics of several different prominent structures (reals, complex number, dense linear order, ...). We also study the question of algorithmical complexity, i.e. the question how hard it is to check whether a given formula lies in the combinatorics of a given structure. We prove that this question is corecursively enumeratively complete and therefore algorithmicaly undecidable in the case of models of complete theories without strict order property (SOP) and in the case of pseudofinite structures.
Keywords: definability, mathematical structures, complexity, strict order property (SOP), pseudo-finite structures

Chapter 1

Motivation

There are several interesting statements that do hold in all finite structures. For example, we know that every injective mapping from a finite set into itself must be a bijection, that every linear order on a finite set has the minimal and the maximal element, etc. We call such statements combinatorial principles. Trahtenbrot [7] proved that the set of all combinatorial principles is co-r.e. complete.

But the situation is radically less complicated in the case of infinite sets. From Completeness Theorem follows that only logically valid formulas do hold in all infinite sets.

If we want to measure how much an $\mathcal{L}$-structure $\mathcal{A}$ differs from being finite, we can look at all the formulas that hold in the given structure. But then we get only the set of all formulas valid in $\mathcal{A}$.

We get a much more interesting information if we look at the class of all formulas that hold in all structures that can be defined in $\mathcal{A}$.

This motivated Krajíček [6] to define the combinatorics of a structure $\mathcal{A}$ as the set of all first order formulas in a fixed language valid in all structures definable in $\mathcal{A}$.

The combinatorics tell us some interesting properties of $\mathcal{A}$. For example, if the pigeon hole principle is in the combinatorics, the structures allows us to define ordered weak Euler characteristics on it. It shows up that if the minimum principle[1] is not in the combinatorics, the structure is unstable, etc.

In this thesis we systematically study the combinatorics of various structures and answer some questions from [6].

[1] I.e. the statement that every order has a minimal element.

The first three chapters are introductory. Their larger size is due to the fact that the last chapters need many different theorems and notions from elementary logic and model theory. As the proofs of all the theorems are scattered through several books and papers, we decided to prove some of them. We hope that the reader will spend less time looking through numerous references.

Second chapter introduces the used notion and basic theorems which we will use later.

In the third chapter we list some interesting combinatorial statements and give a rigorous definition of the combinatorics of a structure. We also look at some tools that will help us to characterize combinatorics of various structures.

The last four chapters present the new results we discovered. For the sake of completeness we also list some older results, but we always announce when doing so.

The fourth chapter is about pseudo-finite structures. Although these structures are infinite, their combinatorics is exactly the same as the combinatorics of finite structures. We present several examples of pseudo-finitene structures.

In the fifth chapter we study how much the combinatorics depends on the set of all $\mathcal{L}$-formulas contained in it. We show that the combinatorics is fully determined by the set of all formulas in the language of graphs, i.e. in the language with one binary predicate. We also prove some interesting results about combinatorics in the empty language and languages with unary predicates.

The sixth chapter talks about the relations among combinatorics of several different prominent structures. We study the combinatorics of reals, integers, complex numbers, etc.

The last chapter is about the complexity results. We try to describe how hard it is to decide whether a given formula lies in the combinatorics of a given structure. We show that in many cases it is hard – many such problems are co-r.e. (resp. r.e.) complete.

Chapter 2

Preliminaries

In this chapter we introduce the basic concepts of mathematical logic and model theory. All the theorems and definitions are well known and can be found in any book on mathematical logic, i.e. [9] or [15]. The only exception is Many Models Theorem in Section 2.10, which is usually taught at advanced courses on model theory.

2.1 Used notion

The notions used here are quite standard. The letters k, l, m, n, i, j are always used to denote natural numbers $1, 2, \ldots$, if not stated otherwise. The set of natural numbers with zero is denoted by ω. The set of all integers with addition, minus and zero is denoted by $\mathbb{Z}$. The class of ordinal numbers is denoted by On. Cn stands for cardinals. The symbol ω_1 is used to denote the least uncountable cardinal.

$|S|$ stands for the cardinality of set S. The sign $:=$ is used to define new objects. If s and t are strings, $s\smile t$ means the concatenation of them with comma, i.e. the string s, t. The string $f^n(x)$ stands for $\underbrace{f \circ f \circ f \circ \cdots \circ f}_{n\times}(x)$. The string $f^0(x)$ means x and if f is injective, with inverse mapping g, $f^{-n}(x)$ stands for $g^n(x)$. If X is a set, R a relation on X and $S \subseteq X$, then $R[S] := \{y \colon R(x, y) \text{ for some } x \in S\}$, especially if f is a function, $f[S] := \{f(x) \colon x \in S\}$.

If $f \colon A \to B$ is a function and C is a set, then $f \restriction C$ stands for the restriction of f onto C, i.e. the mapping $g \colon C \to B$ defined by $g(x) = f(x)$. If $A \subseteq B^n$ are sets, π_i means the projection on the i-th coordinate.

2.2 Language

Let us state precisely what the first order language means. In the case that the reader is familiar with this notion, he can skip these definitions and continue directly to Subsection 2.10.

A language consists of symbols and rules how to concatenate these symbols in a reasonable way. We will use calligraphic letter $\mathcal{L}$ with different indices to denote languages.

Let us first define the logical symbols, that are the symbols common for all first order languages.

Definition 2.1 (Logical symbols)**.**

- ***Logical connectives*** $\rightarrow$, $\neg$, $\vee$, $\wedge$, $\leftrightarrow$
- ***Quantifiers*** $\forall$, $\exists$
- ***Variables*** $v_1, v_2, \ldots$ *(each first order language has countable many variables, the set of them is denoted by* $\textsc{Vars}$*)*
- ***Auxiliary symbols*** $), ($ *(right parenthesis, comma, left parenthesis)*
- ***Equality sign*** $=$

We will also use symbols x, y, z, u, v, x_1, x_2, $\ldots$, y_1, y_2, etc. for variables.

First order languages differ in a part called signature:

Definition 2.2 (Signature)**.**
A pair $((\mathcal{R}, \mathcal{F}, \mathcal{C}), n)$ is called ***a signature*** *if*

- *$\mathcal{R}, \mathcal{F}, \mathcal{C}$ are disjoint sets of non-logical symbols,*
- *$n\colon (\mathcal{R} \cup \mathcal{F}) \rightarrow \{1, 2, 3, \ldots\}$ is a mapping (it tells us the arity of relational and functional symbols).*

*A symbol is said to be k-**ary** if its arity is equal to k. Set $\mathcal{R}$ is the set of relational symbols, $\mathcal{F}$ is the set of functional symbols, $\mathcal{C}$ is the set of constants.*
*The **cardinality of a language** $|\mathcal{L}|$ is defined as $|R| + |F| + |C| + \omega$ and is therefore always infinite.*
*Two signatures $\sigma = ((\mathcal{R}, \mathcal{F}, \mathcal{C}), n)$ and $\sigma' = ((\mathcal{R}', \mathcal{F}', \mathcal{C}'), n')$ are called **disjoint** if $(\mathcal{R} \cup \mathcal{F} \cup \mathcal{C}) \cap (\mathcal{R}' \cup \mathcal{F}' \cup \mathcal{C}') = \emptyset$.*

Two signatures $\sigma = ((\mathcal{R}, \mathcal{F}, \mathcal{C}), n)$ *and* $\sigma' = ((\mathcal{R}', \mathcal{F}', \mathcal{C}'), n')$ *are said to be* ***isomorphic*** *if there exist three bijections* $r\colon \mathcal{R} \to \mathcal{R}'$, $f : \mathcal{F} \to \mathcal{F}'$ *and* $c\colon \mathcal{C} \to \mathcal{C}'$ *such that for any relational symbol* R *(resp. functional symbol* F*) in* σ *the following holds true:* $n'(r(R)) = n(R)$ *(resp.* $n'(f(F)) = n(F)$*). Two languages are said to be* ***isomorphic*** *if their signatures are isomorphic.*

We will often identify the language with its signature. There is no danger of confusion in this double usage since the language is determined uniquely by its signature. If no misunderstanding arises, we also use just the sets $\mathcal{R}, \mathcal{F}, \mathcal{C}$ for describing a signature.

Terms are the smallest strings that have some meaning.

Definition 2.3 (Terms)**.** *Let* $\mathcal{L}$ *be a language. The set of all* $\mathcal{L}$*-terms* $\textsc{Term}_{\mathcal{L}}$ *is the smallest set of strings satisfying:*

1. *Any constant symbol is a term,*
2. *any variable is a term,*
3. *if* $t_1, t_2, \dots, t_n$ *are terms and* F *is an* n*-ary functional symbol, then* $F(t_1, t_2, \dots, t_n)$ *is a term.*

We usually denote terms by small letters s, t, t_1, t_2, *etc.*

Intuitively terms correspond to functions that can be built up in our language.

We will now define the atomic formulas, which are the most elementary statements in a given language:

Definition 2.4 (Atomic formulas)**.** *Let* $\mathcal{L}$ *be a language. The set of all atomic* $\mathcal{L}$*-formulas is the smallest set* $\text{A}\textsc{Form}_{\mathcal{L}}$ *of strings satisfying:*

- *If* t *and* s *are* $\mathcal{L}$*-terms,* $t = s$ *is an atomic formula,*
- *if* R *is a* n*-ary relational symbol of* $\mathcal{L}$ *and* $t_1, t_2, t_3, \dots, t_n$ *are* $\mathcal{L}$*-terms,* $R(t_1, t_2, \dots, t_n)$ *is an atomic* $\mathcal{L}$*-formula.*

We can compose atomic formulas in a certain way to get compound formulas.

Definition 2.5 (Formulas). [1] *Let $\mathcal{L}$ be a language. The set of all formulas in the language $\mathcal{L}$ is the smallest set $\textsc{Form}_{\mathcal{L}}$ of strings satisfying the following conditions:*

- *Every atomic formula is a formula,*
- *if φ and ψ are formulas, $(\varphi \to \psi)$ is a formula,*
- *if φ is a formula, $\neg\varphi$ is a formula,*
- *if φ is a formula and x is a variable, $(\forall x)\varphi$ is a formula.*

We will usually denote formulas by small Greek letters θ,φ, χ, ψ.

Now look at the notion of subformulas:

Definition 2.6 (Subformulas).

- *The subformula of an atomic formula φ is only φ itself,*
- *the subformulas of $(\neg\varphi)$ are $(\neg\varphi)$ itself and all subformulas of φ,*
- *the subformulas of $(\varphi \to \psi)$ are $(\varphi \to \psi)$ itself, all subformulas of φ and all subformulas of ψ,*
- *the subformulas of $(\forall x)\varphi$ are $(\forall x)\varphi$ itself and all subformulas of φ.*

We shall now define the notion of free and bound occurrences of a variable. Intuitively an occurrence is bound if it is in the scope of a quantifier. Let's state it precisely. As usually this will be done via induction:

Definition 2.7 (Bound and free occurrences).

- *Any occurrence of a variable in an atomic formula is free,*
- *a free (resp. bound) occurrence of a variable x in φ is also a free (resp. bound) occurrence of x in $(\varphi \to \psi)$,*
- *a free (resp. bound) occurrence of a variable x in ψ is also a free (resp. bound) occurrence of x in $(\varphi \to \psi)$,*

[1] For technical reasons we use only small set of connectives, the induction based on the construction of formulas is then simpler. Other logical connectives are later introduced as abbreviations.

- *a free (resp. bound) occurrence of a variable x in φ is also a free (resp. bound) occurrence of x in $\neg\varphi$,*

- *if x and y are different variables then a free (resp. bound) occurrence of y in φ is also a free (resp. bound) occurrence of y in $(\forall x)\varphi$,*

- *a free or bound occurrence of a variable x in φ is called a bound occurrence of x in $(\forall x)\varphi$.*

A variable is said to be ***free*** *in φ if it has at least one free occurrence in φ, a variable is said to be* ***bound*** *in φ if it has at least one bound occurrence in φ. A formula is called* ***quantifier free*** *(or* ***open****) if it contains no bound occurrence of any variable.*

The variable can have both free and bound occurrences in a formula, as is easily seen on the example $(x = y \vee (\forall x)x = x)$. We will avoid such formulas (they are logically equivalent to formulas in which no variable has bound and free occurrence simultaneously).

A formula with no free occurrence of any variable is called a ***sentence****.*

Let x be a variable and t a term. The formula $\varphi(x/t)$ is obtained from φ by replacing all free occurrences of x by t.

We will write $\varphi(x)$ to denote that φ contains no other free variables than x.

It is convenient to introduce some abbreviations, so that we can express the desired properties in $\mathcal{L}$ more effectively.

Convention 2.8 (Common abbreviations)**.** *Let $\mathcal{L}$ be a language. Then,*

- $(\varphi \vee \psi)$ *means* $(\neg\varphi \to \psi)$,

- $(\varphi \wedge \psi)$ *means* $(\neg(\varphi \to \neg\psi))$,

- $(\varphi \leftrightarrow \psi)$ *means* $((\varphi \to \psi) \wedge (\psi \to \varphi))$,

- $(\exists x)\varphi$ *stands for* $\neg(\forall x)\neg\varphi$,

- $(\exists! x)\varphi$ *means* $\Big((\exists x)\varphi \wedge (\forall x)(\forall y)\big((\varphi \wedge \varphi(x/y)) \to x = y\big)\Big)$,

- $a \neq b$ *is an abbreviation for* $\neg a = b$,

- *if R is a binary relation, we write aRb instead of $R(a, b)$,*

- *if F is a binary function, we write aFb instead of $F(a,b)$,*
- *if R is a unary relation, we write Ra instead of $R(a)$,*
- *if F is a unary function, we write Fa instead of $F(a)$.*

Convention 2.9 (Dropping parenthesis)**.** *To avoid confusion we will add parenthesis if needed. (For example if $\odot$ is a binary function, $a \odot b \odot c$ can be interpreted as $(a \odot b) \odot c$ or $a \odot (b \odot c)$ which are two different terms.) If no misunderstanding arises, we leave out unnecessary parenthesis. We shall follow the commonly accepted usage in dropping parenthesis. Thus functional symbols are considered more binding than equality sign or relational symbols, which in turn are more binding than quantification. Third comes the $\neg$ sign, $\wedge$ and $\vee$ are fourth, $\rightarrow$ and $\leftrightarrow$ are fifth. In the case that the priority is the same, we consider the rightmost written operation as the most binding, e.g. $\varphi \rightarrow \psi \rightarrow \chi$ means $\varphi \rightarrow (\psi \rightarrow \chi)$.*

Sometimes we may also make use of the following handy expressions:

Convention 2.10.

- $\bigwedge_{i=1}^{n} \varphi_i$ *means* $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n$,
- *for technical reasons* $\bigwedge_{i=n}^{k} \varphi_i$, *where* $k < n$ *stands for* $v_1 = v_1$ *or any other logically valid formula,*
- $\bar{x}$ *is an abbreviation for* $x_1, x_2, x_3, \ldots, x_n$, *where* n *(**the length of** $\bar{x}$) can be given, unspecified or even zero,*
- $\varphi(\bar{x})$ *means that all free variables of* φ *are among* $x_1, x_2, \ldots, x_n$,
- $\bar{x} = \bar{y}$ *stands for* $x_1 = y_1 \wedge x_2 = y_2 \wedge x_3 = y_3 \wedge \cdots \wedge x_n = y_n$,
- $(\forall \bar{x})\varphi$ *stands for* $(\forall x_1)(\forall x_2)(\forall x_3)\ldots(\forall x_n)\varphi$,
- *similarly* $(\exists \bar{x})\varphi$ *stands for* $(\exists x_1)(\exists x_2)(\exists x_3)\ldots(\exists x_n)\varphi$,
- $(\exists! \bar{x})\varphi(\bar{x})$ *means* $(\exists! x_1)(\exists! x_2)\ldots(\exists! x_n)\varphi(\bar{x})$,
- $\big(\exists^{\geq n} x\big)\varphi(x)$ *means* $(\exists x_1, x_2, \ldots, x_n) \bigwedge_{i=1}^{n} \left(\varphi(x_i) \wedge \bigwedge_{j=i+1}^{n} (x_i \neq x_j) \right)$,

- $\big(\exists^{\leq n} x\big)\varphi(x)$ *means* $\neg\big(\exists^{\geq(n+1)} x\big)\varphi(x)$,
- $\big(\exists^{=n} x\big)\varphi(x)$ *means* $\big(\exists^{\geq n} x\big)\varphi(x) \wedge \big(\exists^{\leq n} x\big)\varphi(x)$,
- $\big(\exists^{=\infty} x\big)\varphi(x)$ *is the set of formulas* $\big\{\big(\exists^{\geq n} x\big)\varphi(x) \colon n \in \omega\big\}$. [2]

Let us define some interesting languages:

Example 2.11 (Languages)**.**

$\mathcal{L}_G$ *the language of graphs, it contains one binary relational symbol* E,

$\mathcal{L}_{Ab}$ *the language of abelian groups, it contains one binary functional symbol* $+$, *one unary functional* $-$ *and constant* 0,

$\mathcal{L}_R$ *the language of rings, it is the language* $\mathcal{L}_{Ab}$ *with a constant* 1 *and a binary functional symbol* $\cdot$ *added,*

$\mathcal{L}_{Set}$ *the language of set theory*[3] *with one binary relational symbol* $\in$,

$\mathcal{L}_O$ *the language of order with one binary relational symbol* $<$,

$\mathcal{L}_{OR}$ *the language of ordered rings which is the language of rings together with one binary relational symbol* $<$,

$\mathcal{L}_\emptyset$ *the empty language containing neither relational, functional nor constant symbols.*

Now comes an example of an $\mathcal{L}_R$-formula in a full and an abbreviated forms:

Example 2.12.

Full version $(\forall x)(\forall y)(\cdot(x, y) = 0 \rightarrow ((\neg x = 0) \rightarrow y = 0))$

Abbreviated version $\forall x \forall y(x \cdot y = 0 \rightarrow x = 0 \vee y = 0)$

[2] For the sake of simplicity, if we write $(\exists^{=\infty} x)\varphi(x)$ as an element of a set S, we actually mean that S contains all the formulas from the set $(\exists^{=\infty} x)\varphi(x)$.

[3] also called epsilon-language

2.3 Structures

A language is a useful tool for describing structures mathematicians work with. We will now define the notion of a first order mathematical structure.

Definition 2.13 ($\mathcal{L}$-structure)**.**
Let $\mathcal{L}$ be a first order language of signature σ.
*An **$\mathcal{L}$-structure** $\mathcal{A}$ is a quadruple $\big(A, \{R^A\}_{R\in\mathcal{R}}, \{F^A\}_{F\in\mathcal{F}}, \{C^A\}_{C\in\mathcal{C}}\big)$, where*

- *A is a nonempty set,*
- *each R^A is an $n(R)$-ary relation on A (i.e. subset of $A^{n(R)}$),*
- *each F^A is an $n(F)$-ary operation on A (i.e. mapping of $A^{n(F)}$ into A),*
- *each C^A is an element of A.*

*The set A is called the **universe** of $\mathcal{A}$ or the **underlying set** of $\mathcal{A}$. We will often write $a \in \mathcal{A}$ or $\bar{a} \in \mathcal{A}$ to denote that $a \in A$ or $\bar{a} \in A^n$.*
*We call R^A, F^A and C^A the **realizations** of symbols R, F and C.*
*If the context is clear, we use the original symbols from signature to describe the realization of them, i.e. we write $(\mathbb{R}, +, -, 0, 1)$ instead of the full form $(\mathbb{R}, +^{\mathbb{R}}, -^{\mathbb{R}}, 0^{\mathbb{R}}, 1^{\mathbb{R}})$. A structure with $|A| = 1$ is called **trivial**, otherwise it is **non-trivial**.*

Here we list some basic (and for the whole of mathematics quite important) examples:

Example 2.14 (Structures)**.**

1. $(\mathbb{Q}, <)$*: the dense linear order without endpoints*
2. (V, E)*: directed graphs*
3. $(\mathbb{N}, +, 0, 1)$*: the set of non-negative integers with addition, and two constants 0 and 1*
4. $(\mathbb{Z}, +, 0, 1, \cdot)$*: integers ($\cdot$ is a binary operation)*
5. $(\mathbb{Q}, +, 0, 1, \cdot)$*: rational numbers*
6. $(\mathbb{R}, +, 0, 1, \cdot, <)$*: real numbers*

7. $(\mathbb{C}, +, 0, 1, \cdot)$: *complex numbers*

8. $(\mathbb{H}, +, 0, 1, \cdot)$: *quaternions*

9. $(V, 0, +, -, \cdot_f)$: *vector space over a field F (with one unary operation $\cdot_f$ for each $f \in F$)*

10. (S): *any structure in empty language (only $=$ is allowed) (S being non-empty)*

Other examples of mathematical structures are groups, rings, etc.

One of the basic notion in mathematics is isomorphism, which describes when two structures are the same up to renaming of their elements.

Convention 2.15. *Let us recall that $\bar{a}$ stands for $a_1, a_2, \ldots, a_n$.*
If h is a mapping, $h(\bar{a})$ stands for $h(a_1), h(a_2), \ldots, h(a_n)$.

Definition 2.16 (Embedding and isomorphism)**.**
Let $\mathcal{A} = \big(A, \{R^A\}_{R\in\mathcal{R}}, \{F^A\}_{F\in\mathcal{F}}, \{C^A\}_{C\in\mathcal{C}}\big)$
*and $\mathcal{B} = \big(B, \{R^B\}_{R\in\mathcal{R}}, \{F^B\}_{F\in\mathcal{F}}, \{C^B\}_{C\in\mathcal{C}}\big)$ be two $\mathcal{L}$-structures. A mapping $h\colon A \to B$ is called **an embedding**, if*

- $h\big(F^A(\bar{a})\big) = F^B\big(h(\bar{a})\big)$ *for all functional symbols F and all $\bar{a} \in A$,*
- $(\bar{a}) \in R^A \Leftrightarrow \big(h(\bar{a})\big) \in R^B$.

***Isomorphism** is a bijective embedding. We say that two structures $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic**, if there exists an isomorphism between them. We write $\mathcal{A} \cong \mathcal{B}$ in this case.*

Definition 2.17 (Substructure)**.** *Let $\mathcal{L}$ be a language. An $\mathcal{L}$-structure $\mathcal{B}$ is said to be a **substructure** of $\mathcal{A}$ ($\mathcal{B} \subseteq \mathcal{A}$), if the underlying sets satisfy $B \subseteq A$ and the identity mapping $Id\colon B \to A$ is an embedding, which means $(S^B) = S^A \cap B^n$ for any n-ary relational symbol S, $C^B = C^A$ for any constant symbol C and $F^B = F^A \restriction B^n$ for any n-ary functional symbol F.*

Given a formula we want to decide if it is true in a mathematical structure. Tarski's Truth definition formalizes this concept:

Definition 2.18 (Interpretation of terms)**.** *Let $\mathcal{A}$ be an $\mathcal{L}$-structure.*
*By a **valuation** of variables we mean any assignment $v : \textsc{Vars} \to A$.*
Let us write $t[v]$ for the interpretation of t under v.
*The valuation of variables can be extended to the **interpretation** of all terms $v : \textsc{Term} \to A$ by the following definition:*

- *$t[v] = c^A$ if term t is equal to constant symbol c,*
- *$t[v] = v(x)$ if term t equals variable x,*
- *$F(t_1, t_2, \ldots, t_n)[v] = F^A\big(t_1[v], t_2[v], \ldots, t_n[v]\big)$ for F an n-ary functional symbol and F^A its realization.*

Definition 2.19 (Definition of truth)**.**
$\mathcal{A} \models \varphi[v]$ means φ is true in $\mathcal{A}$ under the valuation v,
$\mathcal{A} \not\models \varphi[v]$ means φ is not true in $\mathcal{A}$ under the valuation v.

- *If t and s are terms, $\mathcal{A} \models (t = s)[v] \Leftrightarrow t[v] = s[v]$,*
- *if R is an n-ary relation and t_1, t_2, $\ldots t_n$ are terms:*
  $\mathcal{A} \models R(t_1, t_2, \ldots, t_n)[v] \Leftrightarrow \big(t_1[v], t_2[v], \ldots, t_n[v]\big) \in R^A$,
- *$\mathcal{A} \models (\neg\varphi)[v] \Leftrightarrow \mathcal{A} \not\models \varphi[v]$,*
- *$\mathcal{A} \models (\varphi \to \psi)[v] \Leftrightarrow (\mathcal{A} \not\models \varphi[v]$ or $\mathcal{A} \models \psi[v])$*
- *$\mathcal{A} \models (\forall x)\varphi[v] \Leftrightarrow \mathcal{A} \models \varphi[v']$ for each valuation v' which agrees with v on all variables different from x.*

*We say that a formula φ is **true (or valid) in** $\mathcal{A}$ ($\mathcal{A} \models \varphi$) if it is true for all valuations on A.*
*We say that an $\mathcal{L}$-formula φ is **valid** ($\models \varphi$) if it is true in all $\mathcal{L}$-structures.*
*We say that φ is **satisfiable in** $\mathcal{A}$ if there exists a valuation v on $\mathcal{A}$ such that $\mathcal{A} \models \varphi[v]$, we say that φ is **satisfiable** if there exists a mathematical structure A and a valuation v on it satisfying $\mathcal{A} \models \varphi[v]$.*

Let us remark that under this definition the equality is absolute, i.e. it is always interpreted as equality on the underlying set.

Now look at examples of some true and false sentences in different structures.

1. $(\mathbb{Q}, <) \models \forall x \forall y \forall z (x < y \wedge y < z \to x < z)$
2. $(\mathbb{C}, +, -, \cdot, 0, 1) \models \forall x \exists y (y \cdot y = x)$
3. $(\mathbb{R}, +, -, \cdot, 0, 1, <) \not\models \forall x \exists y (y \cdot y = x)$

The following example plays an important role in logic.

Example 2.20. *The sentence* $\varphi_n := \big(\exists^{\geq n} x\big)(x = x)$ *can be formulated in any first order language. Moreover, the formula* $\big(\exists^{=n} x\big)(x = x)$ *is valid in* $\mathcal{A}$ *if and only if the underlying set of* $\mathcal{A}$ *has exactly* n *elements.*

Furthermore if we have a language $\mathcal{L}$ *with finite signature and a finite* $\mathcal{L}$*-structure* $\mathcal{A}$*, we can easily find a formula* φ*, such that an* $\mathcal{L}$*-structure* $\mathcal{B}$ *is isomorphic to* $\mathcal{A}$ *if and only if* $\mathcal{B}$ *satisfies* φ*. We can namely take the conjunction of formulas: "The structure has exactly the same number of elements as* $\mathcal{A}$*.", "The relation* R*, function* F *or constant* C *is interpreted in the same way as in* $\mathcal{A}$*." for each relation, function and constant symbols of* $\mathcal{L}$*.*

For example, let us consider the additive group $\mathbb{Z}_2$ *in* $\mathcal{L}_{Ab}$*. This structure is up to isomorphism fully determined by* $\varphi := (\exists x_1)(\exists x_2)(x_1 \neq x_2 \wedge (\forall x_3)(x_3 = x_1 \vee x_3 = x_2) \wedge (x_1 = 0 \wedge -x_1 = x_1 \wedge -x_2 = x_2 \wedge x_1 + x_1 = x_1 \wedge x_1 + x_2 = x_2 \wedge x_2 + x_1 = x_2 \wedge x_2 + x_2 = x_1)$.

For other structures the defining formula can be longer.

Definition 2.21 (Formulas with parameters)**.**
Let $\varphi(\bar{x}, \bar{y})$ *be an* $\mathcal{L}$*-formula with* $\bar{y} = (y_1, y_2, \ldots, y_n)$*.*
Let $\mathcal{A}$ *be an* $\mathcal{L}$*-structure and* $\bar{a} = (a_1, a_2, \ldots, a_n) \in A^n$*. (A is the underlying set of* $\mathcal{A}$*.)*
We say that $\mathcal{A} \models \varphi(\bar{x}, \bar{a})$ *if and only if* $\mathcal{A} \models \varphi(\bar{x}, \bar{y})[v]$ *for each valuation satisfying* $v(\bar{y}) = \bar{a}$*. The elements* $a_1, a_2, \ldots, a_n$ *are called* ***parameters*** *of* $\varphi(\bar{x}, \bar{a})$*. If* $\mathcal{A}$ *is an* $\mathcal{L}$*-structure with universe* A *and* $X \subseteq A$ *is a set of parameters,* $\mathrm{FORM}_{\mathcal{L}}^{X}$ *means the set of all* $\mathcal{L}$*-formulas with parameters from* X*. The symbol* $\mathrm{FORMP}_{\mathcal{L}}$ *is used to denote the class of all* $\mathcal{L}$*-formulas with parameters.*

Definition 2.22 (Elementarily equivalence)**.**
Two $\mathcal{L}$*-structures* $\mathcal{A}$ *and* $\mathcal{B}$ *are* ***elementarily equivalent*** *(*$\mathcal{A} \equiv \mathcal{B}$*), if for every* $\mathcal{L}$*-sentence* φ *the following equivalence holds true:*

$$\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi.$$

A substructure $\mathcal{B} \subseteq \mathcal{A}$ *of an* $\mathcal{L}$*-structure is called* ***elementary****, if for every* $\mathcal{L}$*-formula* $\varphi(\bar{x})$ *and parameters* $\bar{b} \in \mathcal{B}$ *the following equivalence holds true:*

$$\mathcal{A} \models \varphi(\bar{b}) \Leftrightarrow \mathcal{B} \models \varphi(\bar{b}).$$

Let us remark that according to Example 2.20 no finite structure is elementarily equivalent to an infinite one and two finite $\mathcal{L}$-structures are elementarily equivalent if and only if they are isomorphic.

Definition 2.23 (Elementary embedding)**.** *An $\mathcal{L}$-structure $\mathcal{B}$ is **elementarily embeddable** into an $\mathcal{L}$-structure $\mathcal{A}$, if it is isomorphic to some elementary substructure of $\mathcal{A}$.*

2.4 Complexity classes

In the following text we will often study whether a set of natural numbers is given effectively. To distinguish between effectively and non-effectively given theories we will use the notation of recursive sets. This section is rather informal, formal definitions can be found in [15].

Definition 2.24 (Recursive sets)**.**

1. *An n-ary relation R is said to be **recursive** if there exists an algorithm which for given $\bar{y}$ decides whether $R(\bar{y})$.*

2. *A relation S is said to be **recursively enumerable (r.e.)** if there exists a recursive relation $R(\bar{x}, \bar{y})$ such that*

$$S(\bar{x}) \Leftrightarrow (\exists \bar{y}) R(\bar{x}, \bar{y}).$$

3. *A relation S is said to be **co-r.e.** if there exists a recursive relation R, such that*
$$S(\bar{x}) \Leftrightarrow \neg(\exists \bar{y}) R(\bar{x}, \bar{y}).$$

4. *A set S is said to be **recursive or r.e., co-r.e.**, if the membership relation $\bar{x} \in S$ is recursive or r.e., co-r.e., respectively.*

More sharp criterion of effectivity is the criterion of polynomially solvable(**P**) problems. For the sake of the following definition $|x|$ means the number of bites in the standard[4] binary encoding of x.

Definition 2.25 (**NP** problems)**.**
Polynomial algorithm is an algorithm, that solves the given problem in time polynomial in the length of input.

1. *A relation R is in **P** if there exists a polynomial algorithm that given $\bar{y}$ decides whether $R(\bar{y})$. Such relations are called **polynomial**.*

[4] I.e. reproducable on any computer.

2. *A relation S is in* ***NP*** *if there exists a polynomial relation $R(\bar{x}, \bar{y})$ and $c \geq 1$ such that*

$$S(\bar{x}) \Leftrightarrow (\exists \bar{y})\big(R(\bar{x}, \bar{y}) \wedge |\bar{y}| \leq |\bar{x}|^c\big).$$

3. *A relation S is in* ***co-NP*** *if there exists a polynomial relation $R(\bar{x}, \bar{y})$ and $c \geq 1$, such that*

$$S(\bar{x}) \Leftrightarrow \neg(\exists \bar{y})\big(R(\bar{x}, \bar{y}) \wedge |\bar{y}| \leq |\bar{x}|^c\big).$$

We give here an example of an NP problem.

Example 2.26. *Consider the set $\mathcal{G}$ of all simple graphs with finite set of vertices $V \subset \omega$, which have property that there exists a coloring of their vertices with four colors, such that no vertices connected by an edge have the same color.*

The set $\mathcal{G}$ of four-colorable graphs is in ***NP****. Namely, if we have a coloring of G, we can verify in time polynomial in the size of G that no two connected vertices have the same color.*

Definition 2.27 (Reducibility)**.** *Let $R(\bar{x})$, $S(\bar{y})$ be two relations on natural numbers. We say that R is* ***polynomially-time*** *reducible to S, if there exists a polynomially-time computable function f such that*

$$R(\bar{x}) \Leftrightarrow S\big(f(\bar{x})\big).$$

Definition 2.28 (Complete problems)**.** *Let $\mathcal{T}$ be a set of relations on natural numbers.*

1. *We say that a relation Q is* ***$\mathcal{T}$-hard****, if any relation R from $\mathcal{T}$ is polynomially-time reducible to Q.*

2. *We say that a relation Q is* ***$\mathcal{T}$-complete****, if Q is in $\mathcal{T}$ and is $\mathcal{T}$-hard.*

2.5 Theories

Theories describe classes of interesting mathematical structures:

Definition 2.29 (Theory).
An ***$\mathcal{L}$-theory*** *T is any set of $\mathcal{L}$-sentences. The sentences in T are called* ***non-logical axioms*** *of T.*
An $\mathcal{L}$-structure $\mathcal{A}$ is called a ***model of*** *T (or a T*-***model****) if $\mathcal{A} \models \varphi$ for each φ in T. We write $\mathcal{A} \models T$ in this case.*
A theory T is said to be ***satisfiable*** *if there exists a model of T.*

We will consider only theories that have at least one model, i.e. are satisfiable.

An interesting question is what sentences are consequences of a given theory.

Definition 2.30 (Consequences).
We say that φ ***is a consequence of a theory*** *T if $\mathcal{A} \models \varphi$ for each model $\mathcal{A}$ of T. We denote this situation by $T \models \varphi$.*
A satisfiable $\mathcal{L}$-theory is said to be ***complete*** *if for all $\mathcal{L}$-sentences φ either $T \models \varphi$ or $T \models \neg\varphi$. Two $\mathcal{L}$-theories T_1 and T_2 are said to be* ***equivalent*** *if for each formula φ*

$$T_1 \models \varphi \Leftrightarrow T_2 \models \varphi.$$

Definition 2.31 (Equivalence of formulas).
Let T be a theory. Two formulas φ and ψ are ***equivalent in*** *T ($\varphi \sim_T \psi$) if $T \models \varphi \leftrightarrow \psi$. If the context is clear, we just write that φ and ψ are equivalent.*
Two formulas are ***logically equivalent*** *($\varphi \sim \psi$) if $\emptyset \models \varphi \leftrightarrow \psi$.*

Definition 2.32 (Theory of a structure).
The $\mathcal{L}$-theory of an $\mathcal{L}$-structure $\mathcal{A}$ is the set of all $\mathcal{L}$-sentences valid in $\mathcal{A}$, we denote this theory by $Th_{\mathcal{L}}(\mathcal{A})$.

The theory of any structure is complete, because either φ or $\neg\varphi$ is valid in $\mathcal{A}$ according to the definition of truth.

Definition 2.33 (Spectrum of categoricity).
Let T be a theory. The ***spectrum of categoricity*** *$I(T, \kappa)$ is defined as the number*[5] *of mutually non-isomorphic T-models of cardinality κ.*
A theory T is said to be ***κ-categorical*** *if there exists (up to isomorphism) only one T-model of cardinality κ.*

[5] $I(T, \kappa)$ is often an infinite cardinal number.

Observation 2.34. *All models of a complete theory are elementarily equivalent.*

Proof. It follows from the definition. ☐

If we work with a countable language $\mathcal{L}$, we have some encoding of all its symbols into natural numbers, which can be extended into an encoding of all finite strings created by these symbols. So we may assume that all strings (especially formulas) from $\mathcal{L}$ are natural numbers. We use this in the following definition.

Definition 2.35. *Let $\mathcal{L}$ be a countable language. An $\mathcal{L}$-theory T is called* ***recursive*** *if T is recursive as a subset of natural numbers.*

At the end of this section we give an example of a first order theory.

Example 2.36 (DeLO)**.** *The theory* ***DeLO*** *(dense linear order without endpoints) is the theory in $\mathcal{L}_O$ having these non-logical axioms:*

1. $(\forall x)(\forall y)(\forall z)\big(x < y \wedge y < z \rightarrow x < z\big)$ *[transitivity]*
2. $(\forall x)(\forall y)\big(x < y \rightarrow (\neg y < x)\big)$ *[antisymmetry]*
3. $(\forall x)(\forall z)\Big(x < z \rightarrow \big((\exists y)(x < y \wedge y < z)\big)\Big)$ *[density]*
4. $(\forall x)(\exists y)(y < x)$ *[non-existence of the minimal element]*
5. $(\forall x)(\exists y)(x < y)$ *[non-existence of the maximal element]*

All models of DeLO are clearly infinite. As we show later DeLO is a complete theory with $I(T, \kappa) = 2^\kappa$ for all uncountable cardinals κ and $I(T, \omega) = 1$. It is equivalent to $\mathrm{Th}_{\mathcal{L}_O}(\mathbb{Q}, <)$. DeLO has only finitely many axioms and it is therefore recursive.

2.6 Proof system

Whether a formula is a consequence of a given theory can depend on a proper class of structures. It is surprising that checking whether the formula is a consequence of T is equivalent to finding some finite object, namely a proof of the formula. Let us define precisely what a proof is[6]. First we need to define one technicality:

[6] We will use Hilbert's calculus, but there are many others.

Definition 2.37 (Substitution)**.**
Let φ be a formula and x be a variable. We say that a term t ***is free to be substituted into*** φ ***for*** *x, if all occurrences of all variables of term t stay free in the formula $\varphi(x/t)$.*
We usually write $\varphi(x/t)$ if and only if t is free to be substituted into φ for x.

Definition 2.38 (Formal proof)**.** ***A formal proof of a formula*** φ ***in a theory*** *T is a finite sequence of formulas, in which φ is the last one and each formula is either an axiom or is obtained from previous formulas by a rule of inferences. We have just two rules of inferences:*

- ***Modus ponens*** *From φ and $(\varphi \to \psi)$ derive ψ.*
- ***Generalization*** *If x is a variable, from the formula φ derive $(\forall x)\varphi$.*

But there are many ***axioms****:*

- ***Non-logical axioms*** *These are just all the formulas in T.*
- ***Logical axioms*** *Let φ, ψ and χ be any formulas. Then:*
  1. $\big(\varphi \to (\psi \to \varphi)\big)$ *is an axiom,*
  2. $\Big(\big(\varphi \to (\psi \to \chi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \chi)\big)\Big)$ *is an axiom,*
  3. $\big((\neg\varphi \to \neg\psi) \to (\psi \to \varphi)\big)$ *is an axiom,*
  4. *if term t can be substituted into φ for x, then*
  $$\big((\forall x)\varphi \to \varphi(x/t)\big)$$
  is an axiom,
  5. *if the variable x is not free in the formula φ, then*
  $$\Big((\forall x)(\varphi \to \psi) \to \big(\varphi \to (\forall x)\psi\big)\Big)$$
  is an axiom.
- ***Axioms of equality***[7]
  7. *If x is a variable, $x = x$ is an axiom,*

[7] We use Convention 2.9 about dropping parenthesis.

8. *if x_1, x_2, y_1, y_2 are variables,*

$$x_1 = y_1 \to x_2 = y_2 \to x_1 = x_2 \to y_1 = y_2$$

is an axiom,

9. *if R is an n-ary relational symbol of $\mathcal{L}$*

$$x_1 = y_1 \to \ldots \to x_n = y_n \to R(x_1, \ldots, x_n) \to R(y_1, \ldots, y_n)$$

is an axiom,

10. *if F is an n-ary functional symbol of $\mathcal{L}$*

$$x_1 = y_1 \to \ldots \to x_n = y_n \to F(x_1, \ldots, x_n) = F(y_1, \ldots, y_n)$$

is an axiom.

We say that φ is ***provable in*** *T if there exists a formal proof of φ in T. We will write $T \vdash \varphi$ if φ is provable in T and $T \nvdash \varphi$ otherwise.*
If $T = \emptyset$ we say that φ is logically valid and write $\vdash \varphi$ instead of $\emptyset \vdash \varphi$. An $\mathcal{L}$-theory is called ***consistent*** *if there exists an $\mathcal{L}$-sentence such that either $T \nvdash \varphi$ or $T \nvdash \neg\varphi$.*

2.7 Completeness and compactness

We will now state the three most important results in mathematical logic:

Theorem 2.39 (Soundness). *Let T be an $\mathcal{L}$-theory and φ an $\mathcal{L}$-sentence, then $T \vdash \varphi \Rightarrow T \models \varphi$.*

Proof. It is enough to check that all the axioms are true in any structure $\mathcal{A}$ with $T \models \mathcal{A}$, which is easily done. We can then proceed by induction on the length of the proof and use the fact that modus ponens and generalization allows us to derive only valid formulas (if we start with valid formulas). □

Theorem 2.40 (Completeness). *Let T be an $\mathcal{L}$-theory and φ an $\mathcal{L}$-sentence, then $T \models \varphi \Rightarrow T \vdash \varphi$. Moreover[8] if $|\mathcal{L}| = \kappa$, we can find a model of T with cardinality less or equal κ.*

[8] Let us remark that according to Definition 2.2, $|\mathcal{L}|$ is always infinite.

Proof. We will omit the proof, which is rather long and technical. It can be found in [15]. □

This is the key theorem of mathematical logic. It tells us that if we want to check whether the sentence is true in a theory T, it is sufficient to look at finite objects, namely the proofs in T.

Corollary 2.41. *The set of all formulas φ valid in a recursive theory T is recursively enumerable (r.e.).*

Proof. If there exists an algorithm that decides whether a formula[9] is an axiom of T, then given a finite string y, there exists an algorithm that checks whether y is a proof of φ in T. □

Corollary 2.42. *A theory T is satisfiable if and only if it is consistent.*

Proof. It T is not satisfiable, every sentence φ is true in all models of T, thus every sentence φ is provable in T according to Completeness Theorem. Hence φ and $\neg\varphi$ are provable in T for any sentence φ. The other implication is clear. □

Theorem 2.43 (Compactness)**.** *Let T be an $\mathcal{L}$-theory.*

1. *If $T \models \varphi$, then there exists finite $T' \subseteq T$ such that $T' \models \varphi$.*
2. *A theory T has a model if and only if each its finite subset has a model.*

Proof. The proof that $T \models \varphi$ is a finite object, therefore it uses finitely many axioms of T, so we can set $T' = \{\text{axioms used in the proof of } \varphi\}$. If T has no model, it satisfies $T \models x \neq x$, according to the previous part some finite subset T' of T satisfies $T' \models x \neq x$. Using Completeness Theorem we get $T' \vdash x \neq x$, hence T' has no model, which is a contradiction. The other implication is clear. □

One of the interesting consequences of Compactness Theorem is the following proposition:

Theorem 2.44 (Löwenheim-Skolem)**.**
If a theory T has an infinite model $\mathcal{A}$, then there exist models of T with cardinality λ for each $\lambda \geq |\mathcal{L}|$.

[9] More precisely its encoding

Proof. Let us consider the extended language $\mathcal{L}$ together with $\lambda \geq |\mathcal{L}|$ new constant symbols $\{c_\alpha\}_{\alpha<\lambda}$. Let $T' = T \cup \{c_\alpha \neq c_\beta \colon \alpha < \beta < \lambda\}$. Then T' is a theory in the language $\mathcal{L}' = \mathcal{L} \cup \{c_\alpha\}_{\alpha<\lambda}$, which has the cardinality λ. Every finite subset of T' has a model – $\mathcal{A}$ with the constants interpreted arbitrarily (but differently). According to Compactness Theorem, T' has a model with cardinality less or equal to $|\mathcal{L} \cup \{c_\alpha\}_{\alpha<\lambda}| = \lambda$. But the cardinality cannot be less than λ because each constant c_α must be interpreted differently. □

Theorem 2.45 (Łoś-Vaught test)**.** *Let T be an $\mathcal{L}$-theory which has only infinite models. If $I(T,\kappa) = 1$ for some $\kappa \geq |\mathcal{L}|$ then T is complete.*

Proof. Let us assume that there exists a sentence φ with neither φ nor $\neg\varphi$ consequences of T. It means that we can find models $\mathcal{A} \models \varphi$ and $\mathcal{B} \models \neg\varphi$. These models are infinite according to the assumption. Let S be the set of all sentences true in $\mathcal{A}$ and U denote the set of all sentences true in $\mathcal{B}$. $T \cup S$ and $T \cup U$ are $\mathcal{L}$-theories with infinite models, from Löwenheim-Skolem Theorem there exist a model of $T \cup S$ and a model of $T \cup U$ of cardinality κ. But these models are also models of T and there exists only one such model up to isomorphism. It follows that for this model both φ and $\neg\varphi$ are true, which is a contradiction. □

We will most often use the previous corollary in the following form:

Corollary 2.46 (Test of completeness)**.** *Let T be an $\mathcal{L}$-theory.*

1. *If the signature of $\mathcal{L}$ is finite, $I(T,n) = 1$ for some $n < \omega$ and T has no models of cardinalities $\kappa \neq n$, then T is complete.*

2. *If there exists $\kappa \geq |\mathcal{L}|$, such that $I(T,\kappa) = 1$ and for all $n < \omega$ $I(T,n) = 0$, T is complete.*

3. *If there exists $n \in \omega$, such that $I(T,n) > 1$, then T is not complete.*

4. *If there exist $n \in \omega$ and $\kappa \in Cn$, such that $\kappa \neq n$ and $I(T,n) > 0$, $I(T,\kappa) > 0$, then T is not complete.*

Proof. It follows easily from the previous corollary and Example 2.20. □

2.8 Quantifier elimination

Definition 2.47 (Elimination of quantifiers).
An $\mathcal{L}$-theory T ***allows the elimination of quantifiers*** *if for each formula $\varphi(\bar{x})$ there exists a quantifier-free $\mathcal{L}$-formula $\psi(\bar{x})$ such that $\varphi(\bar{x})$ and $\psi(\bar{x})$ are equivalent in T.*

Definition 2.48. *A formula ψ is said to be a* ***Boolean combination*** *of formulas φ_1,φ_2,. . . ,φ_n if it is obtained from φ_1, φ_2,. . . , φ_n only by connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$ (i.e. no quantification).*

Definition 2.49. *A* ***literal*** *is an atomic formula or its negation. A formula ψ is said to be in* ***disjunctive normal form****, if it is a disjunction of conjunctions of literals.*

Proposition 2.50 (DNF). *Let ψ be a boolean combination of atomic formulas φ_1, . . . φ_n. Then ψ is logically equivalent to some formula in disjunctive normal form.*

Proof. The result can be obtained by induction, here we present a semantical proof.

The two-element boolean algebra $\mathcal{B}$ is a structure containing only two elements 0 and 1. The operation are defined as follows $x \wedge y := \min(x, y)$, $x \vee y := \max(x, y)$, $\neg x = 1 - x$, $x \rightarrow y := (\neg x) \vee y$. Furthermore we set $\vee S = \max S$ for any finite set S. If $t(x)$ is a boolean term, we define $t^1(x) := t(x)$ and $t^0(x) := \neg t(x)$. For $\bar{x} = x_1, \ldots, x_n$ and an arbitrary function $\sigma : \{1, 2, \ldots, n\} \rightarrow \{0, 1\}$, we define $\bar{x}^\sigma$ to be $x_1^{\sigma(1)} \wedge x_2^{\sigma(2)} \wedge \ldots \wedge x_n^{\sigma(n)}$.

Let $t(x)$ be a boolean term, we can easily verify the following equality $t(x) = \bigvee\big\{\wedge\bar{x}^\sigma : \big(\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2\}\big), \mathcal{B} \models t(\bar{x})[\sigma]\big\}$.

From the definition of truth, it follows that this identity is sufficient to prove Proposition 2.50 in the general context of quantifier free formulas, we can namely interpret the value 1 as that the corresponding atomic formula is true and the value 0 as that the corresponding atomic formula is false.

□

Definition 2.51. *A formula is said to be in the* ***prenex form*** *if it has the form $Q_1x_1Q_2x_2 \ldots Q_nx_n\varphi$, where each Q_i is a quantifier (i.e. $\forall$ or $\exists$) and φ is a quantifier-free formula.*

Proposition 2.52. *For every formula ψ there exists a logically equivalent formula φ which is in prenex form.*

Proof. Let Q be a quantifier (i.e. $\forall$, $\exists$) and let us denote Q' the opposite quantifier (i.e. $\exists$, $\forall$). We can easily verify that

1. $(\forall x)\psi \sim (\forall y)\psi(x/y)$, for any two variables x, y,
2. $\neg(Qx)\varphi \sim (Q'x)\neg\varphi$,
3. $(Qx)\psi \to \chi \sim (Q'x)(\psi \to \chi)$, if x is not free in χ
4. $\psi \to (Qx)\chi \sim (Qx)(\psi \to \psi)$, if x is not free in ψ

Let us assume that we have a formula φ. First let us note that if we replace an occurrence of a subformula θ in φ by a logically equivalent subformula θ', the resulting formula φ' will be logically equivalent to φ.

If the formula φ is not in the prenex form, there exists a subformula θ of φ, which has one of the forms written above on the left side. We may assume that the variable x does not occur in both subformulas ψ, χ of θ. We can namely replace $(Qx)\psi(x)$ (or $(Qx)\chi(x)$) by $(Qy)\psi(x/y)$ (or $(Qy)\chi(x/y)$, respectively) for a suitable variable y. The subformula θ satisfies the conditions and can be transformed into a subformula θ' which has more subformulas in the scope of quantifiers (the right side). Because this transformation does not change the total number of subformulas, it follows that the process will eventually terminate. But then the resulting formula φ' is in a prenex form. □

Theorem 2.53. *Let T be a theory. If for every conjunction of literals φ there exists a quantifier-free formula ψ such that $(\exists x)\varphi$ and ψ are T-equivalent, then T allows the elimination of quantifiers.*

Proof. Let χ be a formula. We prove the result by induction on the creation of χ.

If χ is atomic, it is quantifier-free. Suppose that χ has the form $\neg\varphi_0$ or $\varphi_1 \to \varphi_2$, according to the induction hypothesis, ψ_0, ψ_1, ψ_2 are logically equivalent to quantifier-free formulas φ_0, φ_1, φ_2. Consequently, χ is equivalent to quantifier-free formula $\neg\varphi_0$ or $\varphi_1 \to \varphi_2$, respectively.

If χ has the form $\forall x\varphi$, where x does not occur in φ, then χ is logically equivalent to φ, which is logically equivalent to some quantifier free-formula according to the assumption.

Finally, if χ has the form $(\forall x)\varphi$, where x does actually occur in φ, then $(\forall x)\varphi$ is logically equivalent to $\neg(\exists x)\neg\varphi$. But $\neg\varphi$ is logically equivalent to the quantifier-free formula ψ. According to the assumption $(\exists x)\psi$ is logically equivalent to some quantifier-free formula θ. Therefore χ is logically equivalent to quantifier-free formula $\neg\theta$. ☐

Definition 2.54. *An $\mathcal{L}$-theory T is said to be model complete, if $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ for every two T-models satisfying $\mathcal{A} \subseteq \mathcal{B}$.*

Note that if $\mathcal{A} \subseteq \mathcal{B}$ are two models of a model complete theory T, then they are elementarily equivalent.

Proposition 2.55. *If a theory T allows the elimination of quantifiers then it is model complete.*

Proof. Let us suppose that we have two T-models $\mathcal{A} \subseteq \mathcal{B}$. We want to show that for every $\varphi(\bar{x})$ and every parameters $\bar{a} \in \mathcal{A}$ the following equivalence holds true:

$$\mathcal{A} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{B} \models \varphi(\bar{a}).$$

Every atomic formula with parameters from $\mathcal{A}$ does hold in $\mathcal{A}$ if and only if it holds in $\mathcal{B}$ ($\mathcal{A}$ is a substructure), by the induction follows that every quantifier free formula with parameters from $\mathcal{A}$ holds in $\mathcal{A}$ if and only if it holds in $\mathcal{B}$.

Because the theory T allows the elimination of quantifier and both $\mathcal{A}$ and $\mathcal{B}$ are its models, there exists a quantifier free formula $\psi(\bar{x})$ such that $\mathcal{A} \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$ and $\mathcal{B} \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$. Consequently

$$\mathcal{A} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{A} \models \psi(\bar{a}) \Leftrightarrow \mathcal{B} \models \psi(\bar{a}) \Leftrightarrow \mathcal{B} \models \varphi(\bar{a}).$$

☐

2.9 Definability

Definition 2.56. *Let $\mathcal{A}$ be an $\mathcal{L}$-structure with universe A. Any set $B \subseteq A^n$ of the form*

$$B = \{\bar{b} \in A^n \colon \varphi(\bar{b}, \bar{a})\},$$

where φ is any $\mathcal{L}$-formula and $\bar{a} \in A^\ell$ are any parameters from $\mathcal{A}$, is called ***definable in*** *$\mathcal{A}$.*

A k-ary relation on some definable subset $B \subseteq A^n$ is just a subset of $A^{n \cdot k}$, hence we know when a k-ary relation on B is definable in $\mathcal{A}$.

We say that an ℓ-ary function f on B is definable, if its graph[10] is definable.

[10]The graph of the function f is the set $\{(\bar{b}, f(\bar{b})) \colon \bar{b} \in B^\ell\}$.

Definition 2.57. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages with disjoint signatures. Let further the signature of $\mathcal{L}'$ be finite. We denote its set of relational symbols by $\mathcal{R}$, its set of functional symbols by $\mathcal{F}$ and its set of constant symbols by $\mathcal{C}$.*

An $\mathcal{L}'$-structure $\mathcal{B}$ is ***definable in*** *$\mathcal{A}$ if it is isomorphic to some structure*

$$\{D \subseteq A^n, \{R^D\}_{R\in\mathcal{R}}, \{F^D\}_{F\in\mathcal{F}}, \{C^D\}_{C\in\mathcal{C}}\},$$

where D is a definable set in $\mathcal{A}$ and each relation R^D and function F^D is definable[11] in $\mathcal{A}$.

Definition 2.58. *Let a structure $\mathcal{B}$ with universe B be definable in a structure $\mathcal{A}$ with universe A. Without loss of generality we may assume that B is a subset of A^n.*

From the definition of a definable structure follows that we can choose formulas $\varphi_B(\bar{x}, \bar{z})$, $\varphi_R(\bar{x}_1, \ldots, \bar{x}_k, \bar{z}_R)$ and $\varphi_F(\bar{x}_1, \ldots, \bar{x}_\ell, \bar{y}, \bar{z}_F)$ and parameters $\bar{c}_B$, $\bar{c}_R$, $\bar{c}_F$, $\bar{c}_C$ such that:

- *$\bar{x}$, $\bar{y}$ and all $\bar{x}_i$'s have length n,*
- *$\bar{a} \in B \Leftrightarrow \mathcal{A} \models \varphi_B(\bar{a}, \bar{c}_B)$,*
- *$R^B(\bar{a}_1, \ldots, \bar{a}_k) \Leftrightarrow \mathcal{A} \models \varphi_R(\bar{a}_1, \ldots, \bar{a}_k, \bar{c}_R)$ for any k-ary relational symbol $R \in \mathcal{R}$,*
- *$F^B(\bar{a}_1, \ldots, \bar{a}_\ell) = \bar{b} \Leftrightarrow \mathcal{A} \models \varphi_F(\bar{a}_1, \ldots, \bar{a}_n, \bar{b}, \bar{c}_F)$ for any ℓ-ary functional symbol $F \in \mathcal{F}$,*
- *$C^B = \bar{c}_C$ for any constant symbol C.*

Definition 2.59. *Let $X \subseteq B \subseteq A^n$ be a finite set of parameters and Y be the set of parameters used to define $\mathcal{B}$. Let us set $X' := \bigcup_{i=1}^n \pi_i X$. Clearly, X' is a subset of A. Moreover, the parameters X can be defined in $\mathcal{A}$ using the parameters X'.*

It is possible to define[12] the ***interpretation*** *$\mathcal{I}^{\mathcal{B}} \colon \mathrm{F{\scriptsize ORM}}^X_{\mathcal{L}'} \to \mathrm{F{\scriptsize ORM}}^{Y\cup X'}_{\mathcal{L}}$ of an $\mathcal{L}'$-structure in an $\mathcal{L}$-structure such that for every $\mathcal{L}'$-formula φ the following equivalence holds true:*

$$\mathcal{A} \models \mathcal{I}^{\mathcal{B}}(\varphi) \Leftrightarrow \mathcal{B} \models \varphi.$$

[11] Under our assumptions, all constants can also be defined, because there are only finitely many of them.

[12] We skip the details, the basic idea is induction on creation of formulas and using Definition 2.58. The only problem is how to get rid of complicated terms.

*Let $\mathcal{L}$ and $\mathcal{L}'$ be two disjoint languages. Let further the signature of $\mathcal{L}'$ be finite. Let $\mathcal{I}\colon \text{FORMP}_{\mathcal{L}'} \to \text{FORMP}_{\mathcal{L}}$ be a mapping. Let us denote the image of φ under $\mathcal{I}$ by $\varphi^{\mathcal{I}}(\bar{x},\bar{y})$. If for every $\mathcal{L}$-structure $\mathcal{A}$ and every choice of parameters $\bar{a} \in \mathcal{A}$ of length t the mapping $\mathcal{I}(\mathcal{A},\bar{a})\colon \varphi \mapsto \varphi^{\mathcal{I}}(\bar{x},\bar{a})$ can be obtained as an interpretation resulting from a definition of some structure in $\mathcal{A}$, we say that $\mathcal{I}$ is **an interpretation of $\mathcal{L}'$ in $\mathcal{L}$**. In this case we also use $\mathcal{I}(\mathcal{A},\bar{a})$ to denote the corresponding structure that is defined in $\mathcal{A}$.*

Let φ be a sentence. We use the notion $\varphi^{\mathcal{B}}(\bar{c})$ for the image $\mathcal{I}^{\mathcal{B}}(\varphi)$ to emphasize all the parameters used in the resulting formula $\mathcal{I}^{\mathcal{B}}(\varphi)$, similarly $\varphi^{\mathcal{I}}(\bar{c})$ means the image $\mathcal{I}(\mathcal{A},\bar{a})(\varphi)$ while emphasizing all the used parameters.

If $\mathcal{B}$ is defined in $\mathcal{A}$ using parameters $\bar{a}$, we have $\mathcal{B} \models \varphi \Leftrightarrow \mathcal{I}^{\mathcal{B}}(\mathcal{A},\bar{a}) \models \varphi$.

If we use parameters $\bar{a}$ from $\mathcal{A}$ to define $\mathcal{B}$, we write $\mathcal{I}^{\mathcal{B}}(\mathcal{A},\bar{a})$ to denote the definable structure $\mathcal{B}$.

For interpretations of languages, we use the calligraphic letter $\mathcal{I}$.

Proposition 2.60. *Let $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$ be three different first order languages and A be an $\mathcal{L}$-structure, B be an $\mathcal{L}'$-structure, C be an $\mathcal{L}''$-structure. If $\mathcal{C}$ is definable in $\mathcal{B}$ which is definable in $\mathcal{A}$, then $\mathcal{C}$ is definable in $\mathcal{A}$.*

Proof. If a relational symbol R is defined via φ_R in $\mathcal{B}$, the formula $\mathcal{I}^{\mathcal{B}}(\varphi_R)$ defines R in $\mathcal{A}$. The same reasoning can be used for other symbols. □

2.10 Many models theorem

Definition 2.61 (SOP and INDP)**.** *Let $\varphi(\bar{x},\bar{y})$ be an $\mathcal{L}$-formula. Let T be a complete $\mathcal{L}$-theory.*

- *We say that φ has the **independence property (INDP)** for T if in every model $\mathcal{A}$ of T and every $0 < n < \omega$ there is a family $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n \in \mathcal{A}$, such that for every subset $X \subseteq \{1,\ldots,n\}$ there is a tuple $\bar{a}$ in $\mathcal{A}$ such that $\mathcal{A} \models \varphi(\bar{a},\bar{b}_i) \Leftrightarrow i \in X$.*
- *We say that φ has the **strict order property (SOP)** for T if for every model $\mathcal{A}$ of T and every $0 < n < \omega$ there are $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n$ such that for any $0 < k, l \leq n$*

$$\mathcal{A} \models (\exists \bar{x})\big(\neg\varphi(\bar{x},\bar{b}_k) \wedge \varphi(\bar{x},\bar{b}_l)\big) \Leftrightarrow k < l.$$

- *We say that T has **SOP (or INDP)**, if there exists a formula φ which has SOP (or INDP, respectively) for T.*

Note that we can even allow φ to contain parameters, as we can easily "hide" these parameters into b_i's.

Definition 2.62. *If a complete theory T has SOP or INDP, it is **unstable**, otherwise T is **stable*** [13].

Theorem 2.63 (Many models theorem)**.** *Let T be an unstable theory in a first-order language L. Then for every cardinal $\kappa > |L|$, there is a family $\{\mathcal{A}_i : i < 2^\kappa\}$ of L-structures which are models of T of cardinality κ, such that for all $i \neq j$, $\mathcal{A}_i$ is not elementarily embeddable in $\mathcal{A}_j$.*

Proof. The proof is rather long, complicated and uses a lot of set theory, Hence we will skip the proof, which can be found in [13]. □

Lemma 2.64. *Let $\mathcal{A}$ be a model of a complete $\mathcal{L}$-theory T with universe A. If there exists an infinite definable set $X \subseteq A$, which is linearly ordered by some formula $\theta(\bar{u}, \bar{v}, \bar{c})$ with parameters $\bar{c}$ and $\bar{u}, \bar{v}$ tuples of variables of the same length, then T has SOP.*

Proof. We can take an infinite linear order $\bar{a}_1 < \bar{a}_2 < \ldots$. We set $\bar{b}_i = \bar{a}_{2a}$. Then the formula $\theta(\bar{x}; \bar{y}, \bar{c})$ corresponding to "$\bar{x}$ lies in X and it is less than $\bar{y}$ in the defined order on X" has SOP (with variables $\bar{x}$ and parameters $\bar{b}_i$ in the place of $\bar{y}$). □

Corollary 2.65 (Undefinability of infinite order)**.**
Let T be a complete $\mathcal{L}$-theory. If T does not have SOP, every definably linearly ordered definable set in any model of T is finite.

In particular, if $I(T, \kappa) < 2^\kappa$ for some $\kappa > |\mathcal{L}|$, every definably linearly ordered definable set in any model of T is finite.

Proof. If a complete theory T has less than 2^κ mutually non-isomorphic models, it clearly has less than 2^κ mutually elementarily non-embeddable models, hence the theory T cannot be unstable. By definition stable theory does not have SOP, hence no infinite order is definable in any model of T by the previous lemma. □

[13] Stability was originally defined differently, Shelah [14] proved that this definitions are equivalent.

Chapter 3

Combinatorics of a structure

3.1 The language of graphs

The language of directed graphs $\mathcal{L}_G$ allows to define many interesting properties of graphs. Let us first describe some auxiliary $\mathcal{L}_G$-sentences.

Definition 3.1 (Auxiliary formulas)**.**

1. *E is function:*
$$\varphi_{func} := (\forall x)(\exists! y)\ xEy$$

2. *E is an injective function:*
$$\varphi_{inj} := \varphi_{func} \wedge (\forall y)\big(\exists^{\leq 1} x\big)\ xEy$$

3. *E is a surjective function:*
$$\varphi_{surj} := \varphi_{func} \wedge (\forall y)(\exists x)\ xEy$$

4. *E avoids at least two elements*
$$\varphi_{two} := \big(\exists^{\geq 2} y\big)\big(\neg \exists x (xEy)\big)$$

5. *E is a strict partial order*
$$\varphi_{ord} := (\forall x\ \neg xEx) \wedge (\forall x \forall y \forall z\ xEy \wedge yEz \rightarrow xEz)$$

6. *E is a strict linear order*

$$\varphi_{lord} := \varphi_{ord} \wedge (\forall x)(\forall y)(x < y \vee x = y \vee y < x)$$

7. *E has maximum*

$$\varphi_{max} := (\exists y \forall x\ \neg yEx)$$

8. *E has minimum*

$$\varphi_{min} := (\exists y \forall x\ \neg xEy)$$

We will be mainly interested in how much structures differ from being finite, hence we list some statements in $\mathcal{L}_G$ that are true in all finite structures.

The main statements are the minimum and the maximum principle and the so called pigeon hole principles. We call these principles combinatorial.

Definition 3.2 (Combinatorial principles)**.**

1. *Every linear order has a minimal element*

$$MIN := \varphi_{lord} \rightarrow \varphi_{min}$$

2. *Every linear order has a maximal element*

$$MAX := \varphi_{lord} \rightarrow \varphi_{max}$$

3. *Every injective mapping is onto*

$$PHP_1 := \varphi_{func} \wedge \varphi_{inj} \rightarrow \varphi_{surj}$$

4. *Every surjective mapping is injective*

$$PHP_2 := \varphi_{func} \wedge \varphi_{surj} \rightarrow \varphi_{inj}$$

5. *There is no injective mapping of A onto A without one element*

$$PHP_3 := (\varphi_{func} \wedge \varphi_{inj} \wedge \neg\varphi_{surj}) \rightarrow \varphi_{two}$$

If the language $\mathcal{L}$ contains at least one functional symbol of arity one, we can formulate analogues of principles PHP_1, PHP_2 and PHP_3. As no misunderstanding arises, we will call these analogues also by PHP_1, PHP_2 and PHP_3.

3.2 Definition of combinatorics of a structure

Definition 3.3 (Combinatorics of a structure)**.** *Let $\mathcal{L}$ and $\mathcal{L}'$ be two first order languages with disjoint signatures. Let $\mathcal{A}$ be an $\mathcal{L}$-structure. We define the* ***Combinatorics of*** *$\mathcal{A}$ as $Comb_{\mathcal{L}'}(\mathcal{A}) :=$*

$$\{\varphi\colon \varphi \text{ is an } \mathcal{L}'\text{-formula valid in all } \mathcal{L}'\text{-structures definable in } \mathcal{A}\}.$$

Let us remark that the combinatorics of a structure is non-empty. It always contains all formulas provable in first order logic (e.g. $x = x$). On the other hand, it is a proper subset of all formulas. It can never contain logically false formulas (e.g. $x \neq x$).

We can even define the combinatorics for a first order theory.

Definition 3.4 (Combinatorics of a theory)**.** *Let $\mathcal{L}$ and $\mathcal{L}'$ be two first order languages with disjoint signatures. Let T be a consistent theory in $\mathcal{L}$. We define the* ***Combinatorics of*** *T by*

$$Comb_{\mathcal{L}'}(T) := \bigcap_{\mathcal{A}:\mathcal{A}\models T} Comb_{\mathcal{L}'}(\mathcal{A}).$$

The assumption that $\mathcal{L}$ and $\mathcal{L}'$ have disjoint signatures is just a minor technicality, as is easily seen in the following definition.

Definition 3.5. *Let X be an $\mathcal{L}$-structure or an $\mathcal{L}$-theory. If in the previous definitions $\mathcal{L}$ and $\mathcal{L}'$ do not have disjoint signatures, we can take a language $\mathcal{K}$ isomorphic to $\mathcal{L}$ such that $\mathcal{K}$ and $\mathcal{L}$ have disjoint signatures and consider X as a $\mathcal{K}$-structure $X_{\mathcal{K}}$ or $\mathcal{K}$-theory $X_{\mathcal{K}}$. Then we define*

$$Comb_{\mathcal{L}}(X) := Comb_{\mathcal{L}}(X_{\mathcal{K}}).$$

Definition 3.6. *Let X be a first-order structure or a theory. Let $\mathbb{L}$ denote the class of all first-order-languages. We define the class*

$$Comb(X) := \bigcup_{\mathcal{L}\in\mathbb{L}} Comb_{\mathcal{L}}(X).$$

Let us look at some notable combinatorics:

Definition 3.7.

1. *$Triv_{\mathcal{L}'} = Comb_{\mathcal{L}'}(\mathcal{A})$, where $\mathcal{A}$ is the trivial structure in $\mathcal{L}_\emptyset$*

2. *$\mathit{Finite}_{\mathcal{L}'} = \mathit{Comb}_{\mathcal{L}'}(\mathcal{A})$, where $\mathcal{A}$ is the two-element structure in $\mathcal{L}_\emptyset$*

3. *$\mathit{Sets}_{\mathcal{L}'} = \{$The set of all $\mathcal{L}'$-formulas valid in all structures$\}$*

Again, if no index $\mathcal{L}'$ is given, Triv means $\bigcup_{\mathcal{L}\in\mathbb{L}} \mathit{Triv}_{\mathcal{L}}$. Finite and Sets are defined analogously.

3.3 Definability of structures

We need some tools to describe combinatorics.

Observation 3.8. *Let $\mathcal{L}$ and $\mathcal{L}'$ be any languages with disjoint signatures. Let $\mathcal{B}$ be an $\mathcal{L}'$-structure definable in an $\mathcal{L}$-structure $\mathcal{A}$. Then*

$$\mathit{Comb}(\mathcal{A}) \subseteq \mathit{Comb}(\mathcal{B}).$$

Proof. The statement is a corollary of Proposition 2.60. □

Definition 3.9. *Two structures $\mathcal{A}$ and $\mathcal{B}$ are said to be **mutually definable** if $\mathcal{A}$ is definable in $\mathcal{B}$ and $\mathcal{B}$ is definable in $\mathcal{A}$.*

Corollary 3.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be two mutually definable structures. Then*

$$\mathit{Comb}(\mathcal{A}) = \mathit{Comb}(\mathcal{B}).$$

Proof. It follows directly from Corollary 3.8. □

Proposition 3.11. *Let us denote the trivial structure in $\mathcal{L}_\emptyset$ by $\mathcal{E}$ and the two-element structure in $\mathcal{L}_\emptyset$ by $\mathcal{D}$.*

1. *$\mathcal{E}$ is definable in any structure,*

2. *any trivial structure is definable in $\mathcal{E}$,*

3. *any trivial structure is definable in any structure,*

4. *$\mathcal{D}$ is definable in any nontrivial structure $\mathcal{A}$,*

5. *any finite structure is definable in $\mathcal{D}$,*

6. *any finite structure is definable in any nontrivial structure $\mathcal{A}$.*

Proof.

1. It is easy to see that the formula $\varphi_{\mathcal{E}}(x,a) \equiv x = a$ with parameter $a \in A$ defines $\mathcal{E}$ in any structure $\mathcal{A}$.

2. Let $\mathcal{A}$ be a trivial structure with universe $\{a\}$. There are no nontrivial functions nor constants in $\mathcal{A}$. Furthermore, there are only two types of relations: the relations of the form $R = \{(a, \ldots, a)\}$ and the empty relations $S = \emptyset$. The first can be defined via $\varphi_R(\bar{x}) \equiv x_1 = x_1$, the second via $\varphi_S(\bar{x}) \equiv x_1 \neq x_1$.

3. It is a consequence of Proposition 2.60.

4. The defining formula is $\varphi_{\mathcal{D}}(x,a,b) \equiv (x = a \vee x = b)$ with parameters a and b, $a \neq b$.

5. Let $\mathcal{A}$ be a finite structure. We take n sufficiently large such that the universe D^n has at least $|A|$ elements and define $\mathcal{A}$ in D^n using $|A|$ different tuples of parameters $\bar{d}_1, \ldots \bar{d}_{|A|}$. We can easily list all elements that are in any relation R via a finite formula (R is finite). The same holds for functions and constants.

6. Again, this is a consequence of Proposition 2.60.

□

3.4 Tools

The following easy propositions can also be useful:

Proposition 3.12.

1. *If $\mathcal{A}$ is a trivial structure, $Comb(\mathcal{A}) = Triv$,*

2. *if $\mathcal{A}$ is non-trivial finite structure, $Comb(\mathcal{A}) = Finite$,*

3. *$Sets_{\mathcal{L}} \subseteq Finite_{\mathcal{L}} \subsetneq Triv_{\mathcal{L}}$, with the first inclusion being strict in the case that $\mathcal{L}$ contains at least one relational symbol of arity at least two or at least one functional symbol.*

4. *For a nontrivial structure $\mathcal{A}$: $Sets_{\mathcal{L}} \subseteq Comb_{\mathcal{L}}(\mathcal{A}) \subseteq Finite_{\mathcal{L}}$, with at least one of these inclusions being strict in the case that $\mathcal{L}$ contains at least one relational symbol of arity at least two or at least one functional symbol.*

Proof.

1. Follows easily from Proposition 3.11.

2. Follows easily from Proposition 3.11.

3. The first inclusion follows from the fact that if something is true in all structures, it must be true in finite structures as well. Without loss of generality we may assume that $\mathcal{L}$ contains a binary relational symbol, or an unary functional symbol. If the relational symbol has arity n greater than two, we can set $S(x, y) := R(\underbrace{x, \ldots, x}_{n\text{-times}})$. A similar trick can be used to modify n-ary operation into a unary one. If $\mathcal{L}$ contains a binary relational symbol, the sentence MAX is clearly in $\mathrm{Finite}_{\mathcal{L}}$ but not in $\mathrm{Sets}_{\mathcal{L}}$. If $\mathcal{L}$ contains a unary functional symbol, the sentence PHP_1 is in $\mathrm{Finite}_{\mathcal{L}}$ but not in $\mathrm{Sets}_{\mathcal{L}}$. The second inclusion is due to Proposition 3.11. The sentence $(\exists^{=1}x)(x = x)$ is clearly in $\mathrm{Triv}_{\mathcal{L}}$ but does not belong to $\mathrm{Finite}_{\mathcal{L}}$.

4. The second inclusion follows from Proposition 3.11, the first is clear – if something is true in all structures, it is also true in all structures definable in $\mathcal{A}$. The fact that at least one of these inclusions is strict follows from the previous point.

□

Theorem 3.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be two elementarily equivalent $\mathcal{L}$-structures. Then*

$$\mathit{Comb}(\mathcal{A}) = \mathit{Comb}(\mathcal{B}).$$

Proof. Let us suppose that an $\mathcal{L}'$-sentence φ is not in $\mathrm{Comb}(\mathcal{A})$. Thus there exists a definable $\mathcal{L}'$-structure $\mathcal{C}$ in $\mathcal{A}$ such that $\mathcal{C} \models \neg\varphi$. Let us denote $\neg\varphi$ by ψ. According to the definition of definability, we know that $\mathcal{A} \models \psi^C(\bar{a})$, where $\bar{a}$ are some parameters.

It means that the sentence $(\exists \bar{x})\psi^C(\bar{x})$ without parameters is true in $\mathcal{A}$. Therefore $\mathcal{B} \models (\exists \bar{x})\psi^C(\bar{x})$, because $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent. It follows that $B \models \psi^C(\bar{b})$, for some parameters $\bar{b}$.

But this means that the structure $\mathcal{C}'$ defined in $\mathcal{B}$ with the same formulas as $\mathcal{C}$ in $\mathcal{A}$ with parameters $\bar{b}$ instead of $\bar{a}$ is a structure definable in $\mathcal{B}$ such that $\mathcal{C}' \models \psi$ and $\mathcal{C}' \not\models \varphi$.

Hence $\mathrm{Comb}(B) \subseteq \mathrm{Comb}(A)$. The opposite inclusion follows from the symmetry. □

Corollary 3.14. *If T is a complete theory, then $Comb(T) = Comb(\mathcal{A})$ for any model $\mathcal{A}$ of T.*

Proof. All models of a complete theory are elementarily equivalent. ☐

Theorem 3.15 (Fundamental Theorem of Ultraproducts)**.** *Let $\mathcal{L}$ be a first order language. Let I be a set and $(\mathcal{A}_i)_{i\in I}$ be $\mathcal{L}$-structures. Let $\mathcal{U}$ be an ultrafilter on I. Then the ultraproduct*

$$U := \prod_{i\in I} \mathcal{A}_i/\mathcal{U}$$

satisfies an $\mathcal{L}$-sentence φ if and only if the set of indices $A = \{i : A_i \models \varphi\}$ lies in $\mathcal{U}$.

Proof. We will omit the proof. It can be found in [1]. ☐

Theorem 3.16. *Let $\mathcal{L}$ and $\mathcal{L}'$ be first order languages with disjoint signatures. Let Γ be a set of $\mathcal{L}'$-sentences and let I be a set. Let $(A_\alpha)_{i\in I}$ be a collection of $\mathcal{L}$-structures with $Comb_{\mathcal{L}'}(A_i) = \Gamma$ for all $i \in I$. Let $\mathcal{U}$ be an ultrafilter on I. Let $U := \prod_{i\in I} A_i/\mathcal{U}$ be the ultraproduct of (A_α). Then $\Gamma \subseteq Comb_{\mathcal{L}'}(U)$.*

Proof. Suppose that φ is a sentence in Γ, which is not in $\mathrm{Comb}_{\mathcal{L}'}(U)$. There exists a witness of that, some $\mathcal{L}'$-structure $\mathcal{B}$ definable in U with $\mathcal{B} \models \neg\varphi$. Let us denote $\neg\varphi$ by ψ and the universe of $\mathcal{B}$ by B. $U \models \psi^B(\bar{a})$ for some parameters $\bar{a} \in U$ according to Definition 2.59. So $U \models (\exists\bar{x})\psi^B(x)$. From the Fundamental Theorem of Ultraproducts follows that the set of indices $A = \{\alpha : \mathcal{A}_\alpha \models (\exists\bar{x})\psi^B(\bar{x})\}$ lies in $\mathcal{U}$ and it is thus nonempty. Hence at least one $\mathcal{A}_\alpha$ satisfies $\mathcal{A}_\alpha \models (\exists x)\psi^B(\bar{x})$. Denote one such $\mathcal{A}_\alpha$ by $\mathcal{C}$ and choose some parameters $\bar{c}$ such that $\mathcal{C} \models \psi^B(\bar{c})$. But this means that the structure $\mathcal{D}$ defined in the $\mathcal{C}$ with the same formulas as B in U (with the parameters $\bar{a}$ replaced by $\bar{c}$) satisfies ψ. Thus $\mathcal{D}$ does not satisfy φ, which is a contradiction with $\varphi \in \Gamma$. ☐

Corollary 3.17. *If an $\mathcal{L}$-structure $\mathcal{A}$ can be obtained as an ultraproduct of non-trivial finite $\mathcal{L}$-structures, then $Comb(\mathcal{A}) = Finite$.*

Proof. This corollary follows from the previous theorem and the fact that $\mathrm{Comb}(\mathcal{A}) \subseteq$ Finite, which was stated in Corollary 3.12. ☐

From now on, the word ultraproduct means only ultraproducts with respect to a non-trivial ultrafilter $\mathcal{U}$.

Definition 3.18 (Euler characteristic)**.** *Let $\mathcal{A}$ be an $\mathcal{L}$-structure with universe A. Denote the family of all sets definable in $\mathcal{A}$ by Def($\mathcal{A}$). By an* ***Euler characteristic*** *on $\mathcal{A}$ we mean any mapping $E\colon \mathit{Def}(\mathcal{A}) \to \mathbb{Z}$ satisfying:*

- $E(C \cup D) = E(C) + E(D)$ *for every two disjoint sets* $C, D \in \mathit{Def}(\mathcal{A})$[1]*,*
- $E(\{a\}) = 1$ *for every* $a \in A$*,*
- $E(f[X]) = E(X)$ *for every definable injective mapping f and every definable set $X \subseteq dom(f)$,*
- $E(C \times D) = E(C) \cdot E(D)$*, for every sets* $C, D \in \mathit{Def}(\mathcal{A})$*.*

Observation 3.19. *If an $\mathcal{L}$-structure $\mathcal{A}$ allows us to define an Euler characteristic on it, then $PHP_3 \in Comb_{\mathcal{L}'}(\mathcal{A})$ for any language $\mathcal{L}'$ in which PHP_3 can be formulated.*

Proof. The universe of $\mathcal{A}$ is denoted by A. Let E be an Euler characteristic on $\mathcal{A}$.

Let us suppose that PHP_3 does not lie in Comb($\mathcal{A}$), in other words, there exist a definable mapping f and a definable set X, such that $f([X]) = X \setminus \{\bar{a}\}$ for some $\bar{a} \in A$.

From the definition of Euler characteristic follow the equalities:

$$E(X) = E(f[X]) = E(X \setminus \{\bar{a}\}). \tag{3.1}$$

On the other hand, $\bar{a} = g(b)$ for the definable mapping $g(b) = \bar{a}$, hence

$$E(\bar{a}) = 1. \tag{3.2}$$

The definable sets $X \setminus \{\bar{a}\}$ and $\{\bar{a}\}$ are disjoint, hence using the definition of Euler characteristic and 3.1 and 3.2 we obtain

$$E(X) = E(X \setminus \{\bar{a}\} \cup \{\bar{a}\}) = E(X \setminus \{\bar{a}\}) + E(\{\bar{a}\}) = E(X) + 1,$$

which yields a contradiction. □

[1] In other words, E is a finitely additive measure.

Chapter 4

Pseudo-finite structures

By Theorem 3.16 it is easy to describe the combinatorics of structures which can be obtained as ultraproducts of finite sets. This result motivated the systematic study of such structures in this chapter.

4.1 Definition of pseudo-finiteness

Definition 4.1. *An infinite structure $\mathcal{A}$ is called* ***pseudo-finite*** *if it is elementarily equivalent to an ultraproduct of finite structures.*

Corollary 4.2. *Let $\mathcal{A}$ be a pseudo-finite structure. Then*

$$Comb(A) = Finite.$$

Moreover, if T is a complete theory which has an infinite pseudo-finite model, then $Comb(T) = Finite$.

Proof. It is a rephrasing of Corollary 3.14 and Corollary 3.17. □

4.2 Structures in $\mathcal{L}_\emptyset$

Let us first study the theories in $\mathcal{L}_\emptyset$.

Definition 4.3 ($\mathcal{L}_\emptyset$-theories)**.** *If $n \geq 1$ is an integer or ∞, we set*

- $PE_n := \{(\exists^{=n} x) x = x\}$.

All models of PE_n with the same cardinality are isomorphic, because any bijection is an isomorphism. From Corollary 2.46 follows that PE_n are the only satisfiable complete theories in $\mathcal{L}_\emptyset$ up to equivalence.

Theorem 4.4 (Combinatorics of $\mathcal{L}_\emptyset$-theories)**.**

1. *$Comb(PE_1) = Triv$,*
2. *$Comb(PE_n) = Finite$, for each $n = 2, 3, 4, \dots$ or ∞.*

Furthermore, if $\mathcal{A}$ is an $\mathcal{L}_\emptyset$-structure:

4. *$Comb(\mathcal{A}) = Triv$, if the structure is trivial,*
5. *$Comb(\mathcal{A}) = Finite$, otherwise.*

Proof. It suffices to prove that there exists a pseudo-finite model of PE_ω. But such a model U can be constructed as an ultraproduct of $(\mathcal{A}_i)_{2<i<\omega}$, where the cardinality of each A_i is i. (For a nontrivial ultrafilter and arbitrary $n \in \omega$ we have that $(\exists^{\geq n} x) x = x$ holds true in all $\mathcal{A}_i$'s except finitely many. Thus $(\exists^{\geq n} x) x = x$ holds true in U for every n and U is infinite.) The rest is clear. □

4.3 Unary predicates

The combinatorics of $\mathcal{L}_\emptyset$-structures is easy to describe, let us now look what happens if the language $\mathcal{L}$ contains only unary relational symbols.

Definition 4.5 (Theories with unary predicates)**.** *Let the language $\mathcal{L}$ contain only finitely many unary predicates $R_1, R_2, \dots, R_k$. For a function $f : \mathcal{P}(\{1, 2, \dots, k\}) \to \omega \cup \{\infty\}$ which is not identically zero, we define the theory UP_f as follows:*

$$UP_f := \left\{ \left(\exists^{=f(S)} x\right) \left(\bigwedge_{i \in S} R_i(x) \wedge \bigwedge_{i \notin S} \neg R_i(x) \right) : S \subseteq \mathcal{P}(1, 2, \dots, k) \right\}.$$

We claim that these are the only satisfiable complete theories in $\mathcal{L}$. Choose a function f which is not identically zero. If there are only finite models of UP_f, we can easily find an isomorphism between any such two models. If there is an infinite model of UP_f, there exists a countable model of UP_f. Moreover, all the countable models of UP_f are isomorphic (again, the isomorphism is easy to construct).

So UP_f is complete according to Corollary 2.46. On the other hand, all $\mathcal{L}$-structures $\mathcal{A}$ have to satisfy some UP_f. It follows that UP_f are the only complete $\mathcal{L}$-theories up to equivalence.

Proposition 4.6 (Combinatorics of structures in unary predicates)**.** *Let $\mathcal{L}$ be a language containing only unary relational symbols and only finitely many of them. Let $\mathcal{L}'$ be a first order language with the signature disjoint with $\mathcal{L}$, then*

1. *$Comb(UP_f) = Triv$, if* $\left(\sum_{S\in\mathcal{P}(1,2\ldots k)} f(S)\right) = 1$
2. *$Comb(UP_f) = Finite$, otherwise.*

Furthermore, if $\mathcal{A}$ is an $\mathcal{L}$-structure:

4. *$Comb(\mathcal{A}) = Triv$, if the structure is trivial,*
5. *$Comb(\mathcal{A}) = Finite$, otherwise.*

Proof. The first two statements are clear. Corollary 4.2 says that it is sufficient to find a pseudo-finite model of UP_f to finish the proof. Consider functions $g_n : \omega \cup \{\infty\} \to \omega$, defined by

$$g_n(\ell) := \begin{cases} \ell & \text{if } \ell \in \omega, \\ n+1 & \text{if } \ell = \infty. \end{cases}$$

Let us denote the models of $\mathrm{UP}_{g_n \circ f}$ by M_n. All M_n are non-trivial finite structures according to their definitions. The ultraproduct of $(M_n)_{n\in\omega}$ is a model of UP_f, which can be proved similarly as in Theorem 4.4. ☐

Theorem 4.7. *Let $\mathcal{L}$ be a language containing only unary relational symbols. If $\mathcal{A}$ is an $\mathcal{L}$-structure, then:*

1. *$Comb(\mathcal{A}) = Triv$, if the structure is trivial,*
2. *$Comb(\mathcal{A}) = Finite$, otherwise.*

Proof. According to the previous theorem it is sufficient to investigate the situation when $\mathcal{L}$ contains an infinite number of unary relational symbols. If $\mathcal{A}$ is trivial, the statement is obvious. Let us therefore further suppose that $\mathcal{A}$ is a non-trivial structure. From Proposition 3.12 we already have $\mathrm{Comb}(\mathcal{A}) \subseteq$ Finite. Suppose that an $\mathcal{L}'$-sentence $\varphi \in$ Finite is not in $\mathrm{Comb}(\mathcal{A})$. It means that there exists a structure $\mathcal{B}$ definable in $\mathcal{A}$ such that $\mathcal{B} \models \neg\varphi$. Hence $\mathcal{A} \models \varphi^{\mathcal{B}}$. But $\varphi^{\mathcal{B}}$ used only finitely many relations of $\mathcal{A}$. Let us denote set of the used relations by $\mathcal{R}$. The language $\mathcal{L}_{\mathcal{R}}$ with signature $\mathcal{R}$ has only finitely many unary relational symbols. If we look at $\mathcal{A}$ as on an $\mathcal{L}_{\mathcal{R}}$-structure (denote this structure by $\mathcal{A}_{\mathcal{R}}$), then it satisfies $\varphi^{\mathcal{B}}$. Therefore there exists a structure $\mathcal{B}'$ definable in $\mathcal{A}_{\mathcal{R}}$, which satisfies $\neg\varphi$.

Moreover, $\mathrm{Th}_{\mathcal{L}_{\mathcal{R}}}(\mathcal{A}_{\mathcal{R}})$ is a complete theory in $\mathcal{L}_{\mathcal{R}}$ and we have described all such theories above.

Hence $\varphi \notin \mathrm{Comb}(\mathcal{A}_R) =$ Finite which yields a contradiction. ☐

4.4 Vector spaces over finite fields

The characterization of vector spaces follows the same lines as the previous two sections. First we look at all complete theories extending the theory VF of vector spaces over F. We show that $VF \cup \mathrm{PE}_\omega$ is complete and find its pseudo-finite model.

Definition 4.8. *Let F be a finite field. We will denote $\mathcal{L}_F$ the language of vector spaces over F. Thus $\mathcal{L}_F$ contains binary functional symbol $+$, unary functional symbol $-$ and constant 0 and it contains unary functional symbol $f\cdot$ for each $f \in F$. The vector space will be usually denoted by V. The theory of vector spaces over F will be denoted by VF.*

F is finite therefore the formula

$$\varphi_n := \left(\exists^{\geq |F|^n}\right) x = x$$

is well defined for any $n \in \omega$ and states that the dimension of V is at least n.

Let us now define some theories in $\mathcal{L}_F$.

Definition 4.9 (Complete $\mathcal{L}_F$-theories of vector spaces)**.**

1. *$VF_n := \{\varphi_n \wedge \neg\varphi_{n+1}\} \cup VF$*
2. *$VF_\omega := \{\varphi_n : n < \omega\} \cup VF$.*

Two vector spaces V and W over F are isomorphic if and only if they have the same dimension. From this and Corollary 2.45 follows that the above given theories are complete. Moreover these are the only complete theories (up to equivalence) which extend VF.

Theorem 4.10 (Combinatorics of VF_k)**.**

1. *$Comb(VF_0) = Triv$,*

2. *$Comb(VF_n) = Finite$ for each $n \geq 1$ or $n = \infty$.*

Moreover, if V is a vector space over a finite field F, then

4. *$Comb(V) = Triv$ if V is trivial, i.e. zero-dimensional*

5. *$Comb(V) = Finite$, otherwise.*

Proof. The first statement is obvious. The second statement is clear, since every nontrivial finite dimensional vector space over a finite field is finite and nontrivial. The third follows from the fact that there exists an infinite dimensional vector space which is pseudo-finite. We can take the ultraproduct of $(V_i)_{i<\omega}$, where each V_i has dimension i, the result is a vector space according to Fundamental Theorem of Ultraproducts and according to the same theorem its dimension is infinite (because for each n the formula φ_n is satisfied in all but finitely many V_i's). $\square$

4.5 Bijective mappings without cycles

In this section we will study the theory FL of one functional symbol without cycles. Again we show FL is complete and find its pseudo-finite model.

Definition 4.11. *Let $\mathcal{L}$ be a language with one functional symbol f. Let us recall that $f^n(x)$ stands for $\underbrace{f \circ f \circ f \circ \cdots \circ f}_{n\times}(x)$. The string $f^0(x)$ means x and if f is injective with inverse mapping g, $f^{-n}(x)$ stands for $g^n(x)$. By the theory FL we mean the theory:*

$$FL := \Big\{ (\forall x)\big(x \neq f^n(x)\big) : n \geq 1 \Big\} \cup \Big\{ \big(\forall x \exists! y\big) f(y) = x \Big\}.$$

Theorem 4.12. *Let $\mathcal{L}$ be a language only containing one functional symbol f.*

1. *FL is a complete theory in $\mathcal{L}$,*

2. *Comb(FL) = Finite.*

Proof. First we note that every model of FL is infinite. Let us prove that all models of FL with cardinality $\kappa > \omega$ are isomorphic.

$(\mathbb{Z}, S)$ stands for the set of integers with successor function S, i.e. the unary mapping $S(x) = x+1$. The idea of the proof is to show that any model of FL is isomorphic to $(\mathbb{Z}, S)^\lambda$ for some cardinal λ. Although two countable models need not to be isomorphic, as in the case of $(\mathbb{Z}, S)^1$ and $(\mathbb{Z}, S)^2$, the situation is different in uncountable cardinalities. Let us formulate this idea precisely:

If we have two models $\mathcal{A}$, $\mathcal{B}$ with universes A, B and with cardinalities $|A| = |B| = \kappa \geq \omega_1$, we can well-order both models $\mathcal{A}$ and $\mathcal{B}$. Let say that $A = (a_i)_{i<\kappa}$, $B = (a_i)_{i<\kappa}$. We set $h_0 := \emptyset$. For $\gamma+1 < \kappa$ the mapping h_γ has both image and pre-image of cardinality less than κ, therefore there exist minimal indices α_γ and β_γ such that a_{α_γ} is not in the range of h_γ and b_{β_γ} is not in the image of h_γ. We construct $h'_{\gamma+1}$ by setting $h'_{\gamma+1}(f^z(a_{\alpha_\gamma})) = f^z(b_{\beta_\gamma})$ for each $z \in \mathbb{Z}$. We set $h_{\gamma+1} = h_\gamma \cup h'_{\gamma+1}$. We notice that $h_{\gamma+1}$ is well defined due to the axioms of FL and that $h_{\gamma+1}$ has both image and pre-image of cardinality less than κ. For a limit ordinal δ we set $h_\delta = \bigcup_{\gamma<\delta} h_\gamma$. Again, this is a well defined mapping and if δ is less than κ, so are the image and pre-image of h_δ. From the construction follows that h_κ is an isomorphism. From Corollary 2.46 follows that FL is a complete theory.

To prove the second part it is sufficient to find a pseudo-finite model of FL. Such a model U can be constructed as an ultraproduct of $(M_i)_{i\in\omega\setminus\{0,1\}}$, where each M_i is an i-element set with f being one cycle of length i. The mappings f are bijections in all M_i's, so f is a bijection in U. Moreover, for every $n \in \omega$ the formula $\big(\forall x\big)\big(x \neq f^n(x)\big)$ is true in all but finitely many M_i's. Therefore $\big(\forall x\big)\big(x \neq f^n(x)\big)$ holds true in U for every n and U is a pseudo-finite model of FL. □

4.6 Discrete linear order

Now comes the most difficult example of a pseudo-finite structure in this text. In particular, the corresponding theory is not κ-categorical for any cardinality $\kappa \geq \omega$. Our proof that this theory is complete is taken from [8].

Definition 4.13. *The theory of discrete linear order with maximal and minimal element* $DiLO_{-\infty}^{\infty}$ *is the theory in* $\mathcal{L}_O$ *with the following set of non-logical axioms:*

1. $(\forall x)(\neg x < x)$ *(ireflexivity)*
2. $(\forall x)(\forall y)(\forall z)(x < y \wedge y < z \rightarrow x < z)$ *(transitivity)*
3. $(\forall x)(\forall y)(x < y \vee x = y \vee y < x)$ *(linearity)*
4. $(\exists x)(\forall y)(x < y \vee x = y)$ *(existence of minimal element)*
5. $(\exists y)(\forall x)(x < y \vee x = y)$ *(existence of maximal element)*
6. $(\forall x)\Big((\exists y)x < y \rightarrow \big(\exists z\big)\big(x < z \wedge \neg(\exists v)(x < v \wedge v < z)\big)\Big)$
   (existence of successor for non-maximal elements)
7. $(\forall x)\Big((\exists y)y < x \rightarrow \big(\exists z\big)\big(z < x \wedge \neg(\exists v)(z < v \wedge v < x)\big)\Big)$
   (existence of predecessor for non-minimal elements)

Let us recall that if we have two orders $(A, <)$, $(B, <')$, we define their sum $(A, <) + (B, <')$ by putting B after A. Formally, we have a structure $(A \times \{0\} \cup B \times \{1\}, <'')$, where $<''$ stands for the order defined by the relations $(0, a) <'' (1, b)$ for all $a \in A, b \in B$; $(0, a) <'' (0, a')$ if and only if $a < a'$; $(0, b) <'' (0, b')$ if and only if $b <' b'$.

Lemma 4.14. *Let* ω *be the ordinal number with its natural ordering. Let* ω^* *stand for the ordinal* ω *with the inverse ordering. Then* $\omega + \omega^*$ *can be embedded into any model of* $DiLO_{-\infty}^{\infty} \cup PE_{\infty}$.

Proof. Let us fix some model $\mathcal{A}$ of $\mathrm{DiLO}_{-\infty}^{\infty} \cup \mathrm{PE}_{\infty}$. We want to embed $\omega + \omega^*$ into $\mathcal{A}$. We can define the successor operation S and the predecessor operation P on $\omega + \omega^*$ and analogously we can define the successor and predecessor operations S' and P' on $\mathcal{A}$. (For the sake of correctness we set $S(m) = m$ and $S'(m') = m'$ for the maximal elements m or m', respectively. Snalogously we set $P(\ell) = \ell$ snf $P'(\ell') = \ell'$ for the minimal elements ℓ or ℓ', respectively.) The order $\omega + \omega^*$ has a minimal element ℓ and $\mathcal{A}$ has a minimal element ℓ'. First we set $h_1\big(S^n(\ell)\big) = S^n(\ell')$ for each $n \in \omega$. Analogously, the order $\omega + \omega^*$ has a maximal element m and $\mathcal{A}$ has a maximal element m'. We set $h_2\big(P^n(m)\big) = P^n(m')$ for each $n \in \omega$. The order $\mathcal{A}$ is infinite, hence the mappings h_1 and h_2 are both injective with disjoint images. We have constructed both of them to preserve order, i.e. $(h_i(x) < h_i(y) \Leftrightarrow x < y)$. Moreover, the range of h_1 is $\omega \times 0$ and the range of h_2 is $\omega \times 1$. From the definition of $\omega + \omega^*$ follows that $h := h_1 \cup h_2$ is an embedding of $\omega + \omega^*$ into $\mathcal{A}$. $\square$

Definition 4.15. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages. Let T be an $\mathcal{L}$-theory. We say that S extends T by definitions of symbols in $\mathcal{L}'$, if*

- *there for each functional symbol f from $\mathcal{L}' \setminus \mathcal{L}$ exists an $\mathcal{L}$-formula $\varphi_f(\bar{x}, y)$ such that the axiom $(\forall \bar{x})(\forall y)\big(f(\bar{x}) = y \leftrightarrow \varphi_f(\bar{x}, y)\big)$ lies in S,*
- *there for each constant symbol c from $\mathcal{L}' \setminus \mathcal{L}$ exists an $\mathcal{L}$-formula φ_c such that the axiom $(\exists! x)\varphi_c(x) \wedge \varphi_c(c)$ lies in S,*
- *there for each relational symbol R from $\mathcal{L}' \setminus \mathcal{L}$ exists an $\mathcal{L}$-formula φ_R such that the axiom $(\forall \bar{x})\big(R(\bar{x}) \leftrightarrow \varphi_R(\bar{x})\big)$ lies in S,*
- *S contains no other axioms.*

Let us now consider the language $\mathcal{L} = (<, S, \{m_i\}_{i\in\omega}, \{M_i\}_{i\in\omega})$. The relational symbol $<$ is binary and stands for the order. The functional symbol S is unary and stands for the successor function[1]. The constant symbol m_0 is used for the minimal element. The constant symbols m_i represent $S^i m_0$. The constant symbol M_0 represents the maximal element. For $i \geq 1$ the constant symbol M_i stands for an element t which satisfies $S^i t = M_0 \wedge S^{i-1} t \neq M_0$.

Lemma 4.16. *Let $\mathcal{L}$ be a language extending $\mathcal{L}_O$ by some functional symbols and constants. Let T be an $\mathcal{L}_O$-theory containing the following axioms:*

1. $(\forall x)(\neg x < x)$ *[ireflexivity],*
2. $(\forall x)(\forall y)(\forall z)(x < y \wedge y < z \rightarrow x < z)$ *[transitivity],*
3. $(\forall x)(\forall y)(x < y \vee x = y \vee y < x)$ *[linearity].*

Let further S be a theory extending T by definitions of functional symbols and constants from $\mathcal{L}$. Let every $\mathcal{L}$-formula φ of the form $\big(\exists x\big) \bigwedge_{i=1}^{n} \alpha_i$, where α_i's are atomic formulas be S-equivalent to an open $\mathcal{L}$-formula. Then S allows the elimination of quantifiers in $\mathcal{L}$ and T is model complete.

[1] If M is the maximal element, we set $SM = M$.

Proof. To prove the first part it is sufficient (according to Theorem 2.53) to prove that for every boolean combination φ of atomic formulas in the disjunctive normal form, the formula $(\exists x)\varphi$ is T-equivalent to a quantifier-free formula ψ.

Let t and s be two terms. The axiom of linearity gives us that $\neg t < s$ is equivalent to $t = s \vee s < t$, similarly $t \neq s$ is equivalent to $t < s \vee s < t$. We may therefore assume that φ does not contain the symbol $\neg$.

Because $\theta \wedge (\psi \vee \chi)$ is logically equivalent to $(\theta \vee \psi) \wedge (\theta \vee \chi)$ and $\theta \vee (\psi \wedge \chi)$ is logically equivalent to $(\theta \wedge \psi) \vee (\theta \wedge \chi)$, we may assume that φ is a disjunction of conjunctions of atomic formulas.

Because $(\exists x)(\psi \wedge \chi)$ is logically equivalent to $\big((\exists x)\psi \wedge (\exists x)\chi\big)$, it is sufficient to prove our claim for conjunctions of atomic formulas.

If some of these formulas α_j does not contain the variable x, φ is logically equivalent to $\alpha_j \wedge \big(\exists x\big) \bigwedge_{i=1, i\neq j}^{n} \alpha_i$. This ends the first part of the proof.

If $\mathcal{L}$ can be T-defined in $\mathcal{L}_O$, then every $\mathcal{L}_O$-model of T can be viewed as a model of S if we add the realizations of $\mathcal{L}$-symbols, which are uniquely determined by S. Therefore we can use Proposition 2.55 and conclude that T is model complete. □

Lemma 4.17. *$DiLo_{-\infty}^{\infty}$ allows the elimination of quantifiers in $\mathcal{L}$ and is therefore model complete.*

Proof. The assumptions of the previous lemma do apply, therefore it is sufficient to prove that $(\exists x)\varphi$ is $\mathrm{DiLO}_{-\infty}^{\infty}$-equivalent to an open $\mathcal{L}$-formula for every conjunction of atomic formulas φ.

Let us suppose that φ contains an atomic formula containing the term $S^n x$. We can list all possible values of x for which S^n is not a bijective mapping. For each of these possibilities, we can replace all occurrences of x by some m_i or M_i. Hence $\varphi \equiv (\varphi_1 \vee \varphi_2 \dots \varphi_n \vee \theta)$, where φ_i are formulas which do not contain the variable x and S^n is a bijection in θ. Therefore

$$(\exists x)\varphi \sim_{\mathrm{DiLo}_{-\infty}^{\infty}} \big(\varphi_1 \vee \varphi_2 \vee \dots \vee \varphi_n \vee (\exists x)\theta\big),$$

where θ has the form $\bigwedge_{i\in I} x \neq m_i \wedge \bigwedge_{j\in J} x \neq M_j \wedge \theta'$ for some suitable sets I and J.

By induction we may assume that all S^n for which $S^n x$ occurs in θ' are bijections. If an atomic formula in θ' has the form $S^n x = S^m x$, where

$n \neq m$, then $(\exists x)\theta$ is equivalent to $v_1 < v_1$. Therefore we can replace[2] all occurrences of $S^n x$ in θ by x and assume that no term $S^n x$ occurs in θ. If an atomic formula in θ has the form $x = S^n v_i$, we can replace all occurrences of x by $S^n v_i$ and hence assume that φ does not contain x. Consequently, $(\exists x)\varphi$ is equivalent to φ.

So we may assume that θ' has the form

$$(x > t_1 \wedge x > t_2 \dots x > t_n) \wedge (x < r_1 \wedge x < r_2 \wedge x < r_m).$$

Now we list all possible orderings of $t_1, \dots, t_n$ and $r_1, \dots, r_m$ and for each combination let us denote the greatest element among $t_1, \dots, t_n$ by t and the least element among $r_1, \dots, r_m$ by r. Then θ' is equivalent to the disjunction of formulas $Sr < t$ over all possible combinations of orderings. (We have to leave some space to fit x in.) □

Theorem 4.18. *Let $\mathcal{L}'$ be a language with signature disjoint with $\mathcal{L}_O$.*

1. *The theory $DiLO^{\infty}_{-\infty} \cup PE_{\infty}$ is a complete theory in $\mathcal{L}_O$.*

2. *$Comb_{\mathcal{L}'}(DiLO^{\infty}_{-\infty} \cup PE_{\infty}) = Finite_{\mathcal{L}'}$.*

Proof. The theory $\mathrm{DiLO}^{\infty}_{-\infty}$ is model complete. Let us take a model $\mathcal{A}$ of $\mathrm{DiLO}^{\infty}_{-\infty}$. According to Lemma 4.14 $\omega+\omega^*$ is a substructure of $\mathcal{A}$. Moreover, it is a model of $\mathrm{DiLO}^{\infty}_{-\infty}$, therefore $\mathcal{A}$ is elementarily equivalent to $\omega+\omega^*$, due to the model completeness and any theory with all its models elementarily equivalent is complete.

To prove the second part it is sufficient to find a pseudo-finite model of $\mathrm{DiLO}^{\infty}_{-\infty}$. Such a model can be easily constructed as an ultraproduct of $(M_n)_{n\in\omega}$, where each M_n is an arbitrary linearly ordered set of size $n+1$. It can be easily check that all the axioms are true in all but finitely many structures M_n. □

[2] $S^n x < S^m v$ by $x < S^{m-n}$, $S^n x = S^n x$ by $v_1 = v_1$, $S^m x = S^n y$ by $x = S^{n-m} y$, $S^n y < S^m x$ by $S^{n-m} y < x$

Chapter 5

Combinatorics as a function of the language

In this chapter we will explain what does it mean that some languages are universal with respect to the combinatorics. First look at languages that are obviously not universal.

5.1 Combinatorially equivalent languages

Definition 5.1. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two first order languages. We say that $\mathcal{L}'$ is **combinatorially subvalent to** $\mathcal{L}$, if for every structure $\mathcal{A}$ and every structure $\mathcal{B}$,*

$$Comb_{\mathcal{L}}(\mathcal{A}) = Comb_{\mathcal{L}}(\mathcal{B}) \quad \Rightarrow \quad Comb_{\mathcal{L}'}(\mathcal{A}) = Comb_{\mathcal{L}'}(\mathcal{B}).$$

*We write $\mathcal{L}' \preccurlyeq \mathcal{L}$ in this case. Two languages $\mathcal{L}'$ and $\mathcal{L}$ are **combinatorially equivalent** if $\mathcal{L} \preccurlyeq \mathcal{L}'$ and $\mathcal{L}' \preccurlyeq \mathcal{L}$. In this case we write $\mathcal{L} \approx \mathcal{L}'$.*

Proposition 5.2. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages.*

1. *If $\mathcal{L}$ is isomorphic to $\mathcal{L}'$, then $\mathcal{L} \approx \mathcal{L}'$.*
2. *If $\mathcal{L}' \subseteq \mathcal{L}$, then $\mathcal{L}' \preccurlyeq \mathcal{L}$.*
3. *If $\mathcal{L}'' \preccurlyeq \mathcal{L}'$ and $\mathcal{L}' \preccurlyeq \mathcal{L}$, then $\mathcal{L}'' \preccurlyeq \mathcal{L}$.*
4. *If $\mathcal{L}'' \preccurlyeq \mathcal{L}'$ and $\mathcal{L}' \approx \mathcal{L}$, then $\mathcal{L}'' \preccurlyeq \mathcal{L}$.*
5. *If $\mathcal{L}'' \approx \mathcal{L}'$ and $\mathcal{L}' \preccurlyeq \mathcal{L}$, then $\mathcal{L}'' \preccurlyeq \mathcal{L}$.*

6. *The relation $\approx$ is an equivalence on the class of all languages.*

7. *Let $\mathcal{I}$ be an $\mathcal{L}$-interpretation*[1] *of $\mathcal{L}'$ with parameters $\bar{z}$, such that for every infinite $\mathcal{L}'$-structure $\mathcal{C}$ there is an $\mathcal{L}$-structure ${}^{\mathcal{I}}\mathcal{C}$ definable in $\mathcal{C}$ and parameters $\bar{c}_{\mathcal{C}} \in {}^{\mathcal{I}}\mathcal{C}$ such that $\mathcal{I}({}^{\mathcal{I}}\mathcal{C}, \bar{c}_{\mathcal{C}}) \equiv \mathcal{C}$. Then $\mathcal{L}' \preccurlyeq \mathcal{L}$.*

Proof. Because each formula contains only finitely many symbols we may assume that $\mathcal{L}'$ and $\mathcal{L}$ are both finite.

The first two points are obvious. Third point follows easily from the definition of combinatorial subvalence. Points 4, 5 and 6 are its consequences. The proof of point 7 uses the formalism introduced in Definition 2.59. Let us suppose that we have an $\mathcal{L}''$-structure $\mathcal{A}$, with $\mathrm{Comb}_{\mathcal{L}}(\mathcal{A}) = C$ and $\mathrm{Comb}_{\mathcal{L}'}(\mathcal{A}) = D$.

If $\mathcal{A}$ is a trivial structure, then $(\forall x)(\forall y)(x = y)$ lies in both C and D, consequently $C = \mathrm{Triv}_{\mathcal{L}}$, $D = \mathrm{Triv}_{\mathcal{L}'}$. Hence any structure $\mathcal{B}$ with $\mathrm{Comb}_{\mathcal{L}}(\mathcal{B}) = C$ is trivial and $\mathrm{Comb}_{\mathcal{L}'}(\mathcal{B}) = \mathrm{Triv}_{\mathcal{L}'} = D$.

So we may assume that $\mathcal{A}$ is a non-trivial $\mathcal{L}''$-structure and $C \subseteq \mathrm{Finite}_{\mathcal{L}}$, $D \subseteq \mathrm{Finite}_{\mathcal{L}'}$.

Therefore it suffices to prove that for any non-trivial structure $\mathcal{A}$ we can decide whether an $\mathcal{L}'$-formula φ from $\mathrm{Finite}_{\mathcal{L}'}$ lies in $\mathrm{Comb}_{\mathcal{L}'}(\mathcal{A})$ fully based on the set $\mathrm{Comb}_{\mathcal{L}}(\mathcal{A})$ only.

Let us look at an $\mathcal{L}'$-formula $\varphi \in \mathrm{Finite}_{\mathcal{L}'}$.

For every finite $\mathcal{L}'$-structure $\mathcal{C}$ we have $\mathcal{C} \models \varphi$. On the other hand, for an infinite $\mathcal{L}'$-structure $\mathcal{C}$ definable in $\mathcal{A}$ the following equivalence holds true due to Definition 2.59:

$${}^{\mathcal{I}}\mathcal{C} \models \varphi^{\mathcal{I}}(\bar{c}_{\mathcal{C}}) \Leftrightarrow \mathcal{I}\big({}^{\mathcal{I}}\mathcal{C}, \bar{c}_{\mathcal{C}}\big) \models \varphi.$$

From the assumption then follows ${}^{\mathcal{I}}\mathcal{C} \models \varphi^{\mathcal{I}}(\bar{c}_{\mathcal{C}}) \Leftrightarrow \mathcal{C} \models \varphi$.

If $\varphi \notin \mathrm{Comb}_{\mathcal{L}'}(\mathcal{A})$, there exists an infinite $\mathcal{L}'$-structure $\mathcal{C}$ definable in $\mathcal{A}$ with $\mathcal{C} \models \neg\varphi$. So we have

$$\mathcal{C} \models \neg\varphi \Rightarrow {}^{\mathcal{I}}\mathcal{C} \models \neg\varphi^{\mathcal{I}}(\bar{c}_{\mathcal{C}}) \Rightarrow {}^{\mathcal{I}}\mathcal{C} \not\models (\forall \bar{z})\varphi^{\mathcal{I}}(\bar{z}) \Rightarrow$$

$$\Rightarrow \big((\forall \bar{z})\varphi^{\mathcal{I}}(\bar{z})\big) \notin \mathrm{Comb}_{\mathcal{L}}(\mathcal{A}),$$

where the last implication follows from the fact that ${}^{\mathcal{I}}\mathcal{C}$ is an $\mathcal{L}$-structure definable in $\mathcal{C}$ and therefore in $\mathcal{A}$.

[1] See Definition 2.59.

If $\big((\forall z)\varphi^{\mathcal{I}}(\bar{z})\big) \notin \mathrm{Comb}_{\mathcal{L}}(\mathcal{A})$, there exists an $\mathcal{L}$-structure $\mathcal{D}$ definable in $\mathcal{A}$ and parameters $\bar{d} \in \mathcal{D}$ such that $\mathcal{D} \models \neg\varphi^{\mathcal{I}}(\bar{d})$ and consequently the definable $\mathcal{L}'$-structure $\mathcal{I}(\mathcal{D}, \bar{d})$ satisfies $\neg\varphi$. Because $\mathcal{I}(\mathcal{D}, \bar{d})$ is definable in $\mathcal{A}$ due to Proposition 2.60, $\varphi \notin \mathrm{Comb}_{\mathcal{L}'}(\mathcal{A})$.

We conclude that for any $\varphi \in \mathrm{Finite}_{\mathcal{L}'}$:

$$\varphi \in \mathrm{Comb}_{\mathcal{L}'}(\mathcal{A}) \Leftrightarrow \big((\forall \bar{z})\varphi^{\mathcal{I}}(\bar{z})\big) \in \mathrm{Comb}_{\mathcal{L}}(\mathcal{A}).$$

Hence $\mathrm{Comb}_{\mathcal{L}'}(\mathcal{A})$ is fully determined by $\mathrm{Comb}_{\mathcal{L}}(A)$.

□

5.2 Trivial languages

Proposition 5.3. *Let $\mathcal{L}'$ be a language such that all relation symbols in $\mathcal{L}'$ have arity one and $\mathcal{L}'$ contains no functional symbols. Then $\mathcal{L}' \approx \mathcal{L}_{\emptyset}$. Furthermore, let $\mathcal{L}$ be a language with signature disjoint with $\mathcal{L}'$ and let $\mathcal{A}$ be a non-trivial $\mathcal{L}$-structure. Then $Sets_{\mathcal{L}'} = Comb_{\mathcal{L}'}(A) = Finite_{\mathcal{L}'} \subsetneq Triv_{\mathcal{L}'}$.*

Proof. It is obviously sufficient to prove the last part. If $\mathcal{A}$ is nontrivial, $\mathrm{Comb}(\mathcal{A}) \subseteq \mathrm{Finite}$ according to Proposition 3.12. For contradiction let us assume that there exists an $\mathcal{L}'$-sentence φ in $\mathrm{Finite}_{\mathcal{L}'}$, which is not in $\mathrm{Comb}_{\mathcal{L}'}(\mathcal{A})$. The sentence φ contains only finitely many $\mathcal{L}'$-symbols, therefore we may assume that $\mathcal{L}'$ is finite. Let $\mathcal{B}$ be an $\mathcal{L}'$-structure definable in $\mathcal{A}$ such that $\mathcal{B} \models \neg\varphi$. $\mathrm{Th}_{\mathcal{L}'}(\mathcal{B})$ is a complete $\mathcal{L}'$-theory, hence it is equivalent to one of the complete theories described in Sections 4.3 or 4.2 (if $\mathcal{L}' = \mathcal{L}_{\emptyset}$). From Theorems 4.4 and 4.7 we know that $\mathrm{Comb}\big(\mathrm{Th}_{\mathcal{L}'}(\mathcal{B})\big) = \mathrm{Finite}$. Thus $\mathcal{B} \models \varphi$ (φ is valid in all structures definable in $\mathcal{B}$, especially in $\mathcal{B}$ itself) and we have a contradiction. □

For the language $\mathcal{L}'$ with properties described above, the Combinatorics tell us only whether the structure is trivial or not, hence $\mathrm{Comb}_{\mathcal{L}'}$ is not interesting.

On the other hand, this is the only uninteresting case.

Observation 5.4. *Let $\mathcal{L}'$ be a language that contains a functional symbol or a relational symbol of arity at least two, then*

$$Sets_{\mathcal{L}'} \subsetneq Finite_{\mathcal{L}'} \subsetneq Triv_{\mathcal{L}'}.$$

Consequently, $\mathcal{L}' \not\approx \mathcal{L}_{\emptyset}$.

Proof. Let us first look at the case where f is a functional symbol of arity n. Let us write $g(x) := f(\underbrace{x, x, \ldots, x}_{n\times})$. Then the statement "If g is surjective, then g is injective." is clearly in $\text{Finite}_{\mathcal{L}'}$ but not in $\text{Sets}_{\mathcal{L}'}$.

Consider now the case of one relational symbol R of arity $n > 1$. Let us write $E(x, y) := R(\underbrace{x, x, \ldots, x}_{(n-1)\times}, y)$. Then the statement "If $E(x, y)$ is an ordering, then $E(x, y)$ has minimal element." is clearly in $\text{Finite}_{\mathcal{L}'}$ but not in $\text{Sets}_{\mathcal{L}'}$. The consequence is clear. □

5.3 Universal language

Observation 5.5. *If $\mathcal{L}$ contains at least one relational or functional symbol of arity at least 2, then $\mathcal{L}_G \preccurlyeq \mathcal{L}$.*

Proof. If $\mathcal{L}$ contains a relation symbol S of arity $k \geq 2$, we can set

$$E(x, y) := S(\underbrace{x, x, \ldots, x}_{(k-1)\times}, y)$$

and use Proposition 5.2(7) to get $\mathcal{L}_G \preccurlyeq \mathcal{L}$.

Let us now suppose that $\mathcal{L}$ contains a functional symbols F or arity $k \geq 2$. We denote the language with one binary functional symbol G by $\mathcal{L}_B$. We can set

$$G(x, y) := F(\underbrace{x, x, \ldots, x}_{(k-1)\times}, y)$$

and use Proposition 5.2(7) to get $\mathcal{L}_B \preccurlyeq \mathcal{L}$.

Now it is sufficient to prove that $\mathcal{L}_G \preccurlyeq \mathcal{L}_B$. If $E(x, y)$ is a binary relation, we can choose two different constants a and b and set

$$f(x, y) := \begin{cases} a & \text{if } E(x, y) \text{ holds,} \\ b & \text{otherwise.} \end{cases}$$

This definition satisfies the conditions of Proposition 5.2(7), thus $\mathcal{L}_G \preccurlyeq \mathcal{L}_B$ holds.

Consequently, $\mathcal{L}_G \preccurlyeq \mathcal{L}$ due to Proposition 5.2(3). □

Theorem 5.6. *Let $\mathcal{L}$ be a finite language. Then $\mathcal{L} \preccurlyeq \mathcal{L}_G$. Consequently if $\mathcal{L}$ contains at least one relational or functional symbol of arity at least two, then $\mathcal{L} \approx \mathcal{L}_G$ and if X is a structure or a first-order-theory, the class $Comb(X)$ is fully determined by the set*

$$Comb_{\mathcal{L}_G}(X) = Comb(X) \cap \textsc{Form}_{\mathcal{L}_G}.$$

Proof. Krajíček proved in [5] that any finite language $\mathcal{L}$ can be interpreted using (sufficiently many) different parameters in $\mathcal{L}_G$, even more his proof went in such a way that the assumptions of the Proposition 5.2(7) do hold. This finishes the proof. The consequences are clear. □

Problem 1.

- *We have seen $\approx$ is an equivalence relation on the class of all first-order languages. How many equivalence classes does it have?*
- *In particular, is it true that $\mathcal{L} \approx \mathcal{L}' \Rightarrow \mathcal{L} \approx (\mathcal{L} \cup \mathcal{L}')$? This would imply that there are indeed only three equivalence classes.*

Chapter 6

Relations among different structures

In this chapter we will explore the relations between combinatorics of several different prominent structures. We will look which principles PHP_1, PHP_2, PHP_3, MAX and MIN defined in Section 3.1 do hold in studied structures. Our main tool will be Corollary 3.8, which states that if a structure $\mathcal{B}$ is definable in $\mathcal{A}$, then $\text{Comb}(\mathcal{A}) \subseteq \text{Comb}(\mathcal{B})$.

6.1 $\mathbb{N}$,$\mathbb{Z}$,$\mathbb{Q}$, Sets

Let us now look at the combinatorics of numbers.

Convention 6.1.

1. $\mathbb{N}$ *means the set of non-negative integers with* $+, \cdot, 0, 1$*,*
2. $\mathbb{Z}$ *means the set of integers with the same operations,*
3. $\mathbb{Q}$ *means the set of rationals with the same operations.*

Observation 6.2.

1. $\mathbb{Z}$ *can be defined in* $\mathbb{N}$*,*
2. $\mathbb{N}$ *can be defined in* $\mathbb{Z}$*,*
3. $\mathbb{Q}$ *can be defined in* $\mathbb{Z}$*,*

4. *$\mathbb{Z}$ can be defined in $\mathbb{Q}$.*

Proof. First and third points are trivial. From Lagrange's Four-square Theorem (see e.g. [4]) follows that non-negative integers are precisely the numbers which can be written as a sum of four squares. This proves the second point. The last point is due to Julia Robinson [12]; the defining formula was recently simplified, see e.g. [11]. □

Corollary 6.3. *Let $\mathcal{L}$ be a language, then*

$$Comb(\mathbb{N}) = Comb(\mathbb{Z}) = Comb(\mathbb{Q}) = Sets.$$

Moreover, $MAX, MIN, PHP_1, PHP_2, PHP_3 \notin Sets$.

Proof. The first two equalities are clear, the last follows from the fact that a consistent recursive theory in a countable language has a model definable in $\mathbb{N}$. This is due to the fact, that if we observe $\mathrm{Sets}_{\mathcal{L}}$, we may assume that $\mathcal{L}$ is finite [1] and the proof of Completeness Theorem 2.40 for $\mathcal{L}$ can be formalized in $I\Sigma_1$ fragment of Peano arithmetic, see [3]. □

6.2 $\mathbb{R}$,$\mathbb{H}$,$\mathbb{O}$,$\mathbb{S}$

Let us now look on other rings and fields of numbers. What can be said about their combinatorics?

Convention 6.4. *$\mathbb{R}$ stands for the set of reals [2] with $+, -, \cdot, 0, 1$.*

We now define some finite dimensional algebras over $\mathbb{R}$, namely the quaternions $\mathbb{H}$, octonions $\mathbb{O}$ and sedenions $\mathbb{S}$.

Definition 6.5.

1. *$\mathbb{H}$ is a four-dimensional algebra over $\mathbb{R}$ with generators $1, i, j, k$.*

2. *$\mathbb{O}$ is an eight-dimensional algebra over $\mathbb{R}$. We denote its generators by $1, i, j, k, l, il, jl, kl$.*

[1] Every formula $\varphi \in \mathrm{Sets}_{\mathcal{L}}$ contains only finitely many symbols from $\mathcal{L}$.

[2] There is no reasonable way how to define a field order on the other studied structures, therefore we do not include the symbol $<$ into the signature for $\mathbb{R}$. On the other hand, as $<$ is definable in $\mathbb{R}$ by $x < y \leftrightarrow (\exists z)\big(x + (y \cdot y) = z\big)$, it does not matter whether $<$ is included in the signature or not, the definable sets, which we study, will be the same.

3. $\mathbb{S}$ *is a sixteen-dimensional algebra over* $\mathbb{R}$*. We denote its generators by* $1, e_1, e_2, \ldots, e_{15}$.

In each of these algebras we define the multiplication of its generators, the multiplication of other elements is uniquely determined via linearity. Multiplication tables

1. In $\mathbb{H}$*:*

$\cdot_{\mathbb{H}}$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	$-i$
k	k	j	$-i$	-1

2. In $\mathbb{O}$*:*

$\cdot_{\mathbb{O}}$	1	i	j	k	l	il	jl	kl
1	1	i	j	k	l	il	jl	kl
i	i	-1	k	$-j$	il	$-l$	$-kl$	jl
j	j	$-k$	-1	i	jl	kl	$-l$	$-il$
k	k	j	$-i$	-1	kl	$-jl$	il	$-l$
l	l	$-il$	$-jl$	$-kl$	-1	i	j	k
il	il	l	$-kl$	jl	$-i$	-1	$-k$	j
jl	jl	kl	l	$-il$	$-j$	k	-1	$-i$
kl	kl	$-jl$	il	l	$-k$	j	i	-1

3. Multiplication in $\mathbb{S}$*: Although it is possible to define the multiplication on sedenions using a similar table, we use another approach*[3]*. If* a, b, c, d, e, f, g, h *are real numbers, we define the conjugation* * *in octonions by* $(a + bi + cj + dk + el + fil + gij + hil)^* :=$

$$a - bi - cj - dk - el - fil - gil - hil.$$

Sedenions can be viewed as a two-dimensional vector space over $\mathbb{O}$*. For two sedenions* (p, r) *and* (s, t) *with* $x, y, u, v \in \mathbb{O}$*, we define*

$$(p, q) \cdot_{\mathbb{S}} (r, s) := (p \cdot_{\mathbb{O}} r - s^* \cdot_{\mathbb{O}} q, s \cdot_{\mathbb{O}} p + q \cdot_{\mathbb{O}} r^*).$$

Observation 6.6.

1. $\mathbb{H}$*,* $\mathbb{O}$ *and* $\mathbb{S}$ *can be defined in* $\mathbb{R}$*.*

[3]Called Cayley–Dickson construction

2. $\mathbb{R}$ *can be defined in* $\mathbb{H}$, $\mathbb{O}$ *and* $\mathbb{S}$.

Proof. The first point easily follows from the constructions above.

To prove the second one it is sufficient to check that the center[4] of any of these algebras is isomorphic to $\mathbb{R}$. Let us first look at quaternions. The center $C := \{x \in \mathbb{H} \colon (\forall a \in \mathbb{H}) x \cdot_{\mathbb{H}} a = a \cdot_{\mathbb{H}} x\}$ is isomorphic to $\mathbb{R}$. Let x be an element of C, we can write x as $a + bi + cj + dk$. This number should commute with everything, in particular $x \cdot_{\mathbb{H}} i = i \cdot_{\mathbb{H}} x$, hence $ai - b - ck + dj = ai - b + ck - dj$ and $c = d = 0$. Similarly we obtain $b = d = 0$ from $x \cdot_{\mathbb{H}} j = j \cdot_{\mathbb{H}} x$. Therefore $x = a \cdot 1$. Furthermore,

$$(a \cdot 1 + 0i + 0j + 0k) \cdot_{\mathbb{H}} (e \cdot 1 + 0i + 0j + 0k) = (ae \cdot 1 + 0i + 0j + 0k).$$

So we can identify $(\mathbb{R}, 0, 1, +, -, \cdot)$ with $(C, 0_{\mathbb{H}}, 1_{\mathbb{H}}, +_{\mathbb{H}}, -_{\mathbb{H}}, \cdot_{\mathbb{H}})$. The proofs for $\mathbb{O}$ and $\mathbb{S}$ are analogous, but longer. □

Corollary 6.7. *Let $\mathcal{L}'$ be a language which contains one unary functional symbol. Let $\mathcal{L}$ be a language with $\mathcal{L}' \preccurlyeq \mathcal{L}$. Then*

$$Sets_{\mathcal{L}} \subsetneq Comb_{\mathcal{L}}(\mathbb{R}) = Comb_{\mathcal{L}}(\mathbb{H}) = Comb_{\mathcal{L}}(\mathbb{O}) = Comb_{\mathcal{L}}(\mathbb{S}) \subsetneq Finite_{\mathcal{L}}.$$

Consequently

$$Sets \subsetneq Comb(\mathbb{R}) = Comb(\mathbb{H}) = Comb(\mathbb{O}) = Comb(\mathbb{S}) \subsetneq Finite.$$

Proof. The equalities are direct consequences of the previous observation. The rest of the proof is taken from [6].

The first inclusion is strict, because we can define an Euler characteristic on $\mathbb{R}$, see [2]. In particular, PHP_3 is in $\mathrm{Comb}_{\mathcal{L}}$ due to Corollary 3.19. On the other hand, PHP_2 and PHP_1 do not belong to $\mathrm{Comb}_{\mathcal{L}}(\mathbb{R})$, which can be easily seen on the two following examples $f(x) = (x \cdot x) \cdot (x + 1)$ and $g(x) = \begin{cases} \frac{1}{x-1} & \text{for } x < 0, \\ \frac{1}{x+1} & \text{for } x \geq 0. \end{cases}$ That the last inequality is strict is clear. □

Computation shows that:

1. $\mathbb{R}$ is commutative $(x \cdot y = y \cdot x)$, associative $((x \cdot y) \cdot z = x \cdot (y \cdot z))$ and without zero-divisors $(x \cdot y = 0 \rightarrow x = 0 \vee y = 0)$.

[4]I.e. the set of elements that commute with everything.

2. $\mathbb{H}$ is not commutative ($ij = k \neq -k = ji$), but it is associative and without zero-divisors.

3. $\mathbb{O}$ is neither commutative, nor associative

$$(i \cdot j) \cdot l = kl \neq -kl = i \cdot (j \cdot l),$$

but has no zero-divisors.

4. $\mathbb{S}$ is neither commutative, nor associative and it even contain zero-divisors ($(e_3 + e_{10}) \cdot (e_6 - e_{15}) = 0$).

So all of these structures are mutually elementarily non-equivalent. This affirmatively answers the question whether there exist two elementarily non-equivalent structures with the same combinatorics which is different from both Sets and Finite.

Problem 2. *Do there exist two different structures $\mathcal{A}$, $\mathcal{B}$, with*

$$Sets \subsetneq Comb(\mathcal{A}) = Comb(\mathcal{B}) \subsetneq Finite,$$

which are not mutually definable?

6.3 $\mathbb{C}$

The complex numbers are to some extent exceptional.

Convention 6.8. $\mathbb{C}$ *stands for complex numbers with* $+, \cdot, -, 0, 1$.

Theorem 6.9. *Let $T = Th_{\mathcal{L}_R}(\mathbb{C})$. Then T is κ-categorical for every uncountable cardinal κ.*

Proof. We omit the proof. It can be found in [10]. □

Observation 6.10. $\mathbb{C}$ *is definable in* $\mathbb{R}$. *Consequently*

$$Comb(\mathbb{R}) \subseteq Comb(\mathbb{C}).$$

The first two points of the following proposition are taken from [6].

Proposition 6.11.

1. $PHP_2 \notin Comb(\mathbb{C})$.

2. $PHP_1, PHP_3 \in Comb(\mathbb{C})$.

3. $MAX, MIN \in Comb(\mathbb{C})$.

Proof.

1. Consider the mapping $f(x) = x^2$.

2. The first part follows from the fact that $\mathbb{C}$ allows definition of Euler characteristics, the second is basically Ax' Theorem [10].

3. From Theorem 6.9 and Corollary 2.65 follows that only finite definable set of $\mathbb{C}^n$ can be linearly ordered and every linearly ordered set has the maximal resp. minimal element.

□

Corollary 6.12. *Let $\mathcal{L}$ be a language that contains at least one functional symbol or at least one at least binary relational symbol. Then*

$$Sets_{\mathcal{L}} \subsetneq Comb_{\mathcal{L}}(\mathbb{R}) \subsetneq Comb_{\mathcal{L}}(\mathbb{C}) \subsetneq Finite_{\mathcal{L}}$$

and consequently $Sets \subsetneq Comb(\mathbb{R}) \subsetneq Comb(\mathbb{C}) \subsetneq Finite$.

6.4 DeLO

In this section we will study the theory of dense linear order without endpoints.

Lemma 6.13. *DeLO allows the elimination of quantifiers in $\mathcal{L}_O$ and is therefore model complete.*

Proof. This is a well known fact, the proof is taken from [8]. According to Lemma 4.16 it is sufficient to show that every formula φ in the form $(\exists w)\left(\bigwedge_{i=1}^{n} \alpha_i\right)$ where each α_i has one of the following forms $w < w$, $w = w$, $w < x_i$, $x_j < w$, is equivalent to a quantifier free formula. In the first case φ is equivalent to $x_1 < x_1$. In the second case, we may delete the atomic formula $w = w$ and obtain a logically equivalent formula φ'. Hence we may assume that only third and fourth case occurs.

Let us write I for the set of indices i such that the atomic formula $x_i < w$ occurs in φ. Let us use J for the set of indices j such that the atomic formula $w < x_j$ occurs in φ.

Then φ is equivalent to $\bigwedge_{i \in I, j \in J} x_i < x_j$. □

Lemma 6.14. $(\mathbb{Q}, <)$ *can be embedded into any model of DeLO, consequently DeLO is a complete theory.*

Proof. $(\mathbb{Q}, <)$ is a countable structure, hence there exists an ordering of its elements $(q_1, q_2, q_3, \ldots)$ of type ω. Any model $\mathcal{A}$ of DeLO is infinite. Let us suppose that it is well ordered by $\prec$. We define $f(q_i)$ by induction.

$$f(q_i) := \min_{\prec}\Big\{a\colon (\forall j < i)\Big(\big(q_j < q_i \to f(q_j) < a\big) \wedge \big(q_i < q_j \to a < f(q_j)\big)\Big)\Big\}.$$

The mapping f is clearly an embedding.

As DeLO is model complete, $\mathcal{A}$ is elementarily equivalent to $(\mathbb{Q}, <)$, consequently all models of DeLO are elementarily equivalent and DeLO is a complete theory. □

Definition 6.15. *Let $\mathcal{A}$ be an $\mathcal{L}$-structure with underlying set A and $X \subseteq A$ be a set of parameters. A definable set $D \subseteq A^n$ is said to be* ***minimal definable*** *over X, if it is definable over X and for every other set $C \subseteq A^n$ definable over X either $C \cap D = \emptyset$ or $C \cap D = D$.*

Observation 6.16. *Let $\mathcal{A}$ be a model of DeLO with universe A and X be a finite subset of A. Then there exist only finitely many mutually non-equivalent $\mathcal{L}_O$-formulas with n-free variables and parameters from X. Consequently there exist only finitely many subsets of A^n that are minimal definable over X (we will denote them by p_1, p_2,..., p_N). Every set $S \subseteq A^n$ definable using parameters from X is a finite union of some p_i's.*

Proof. DeLO allows the elimination of quantifiers (Lemma 6.13), hence every formula φ can be written as a Boolean combination of atomic formulas in disjunctive normal form (Theorem 2.50). Since there exist only two kinds of atomic formulas ($x < y$ and $x = y$, where x, y can be variables or parameters) and since $\neg x = y$ is equivalent to $x < y \vee y < x$ and $\neg x < y$ is equivalent to $y < x \vee x = y$, we may assume that φ is equivalent to a disjunction of conjunctions of atomic formulas.

If we have finite number of variables and finite number of parameters, there exists only finitely many atomic formulas. Consequently, there are only finitely many conjunctions of them, and only finitely many disjunctions of these conjunctions, let us call them $\psi_1, \dots \psi_M$. Hence φ is equivalent to some ψ_i.

Let us observe that $A^n = \bigcup p_i$, hence if some set S definable with parameters from X is not a finite union of $p_i's$, then $X \cap p_i$ is a definable proper subset of p_i for some i, which is a contradiction. □

We want to describe all the minimal definable sets.

Proposition 6.17. *Let $\mathcal{A}$ be a model of DeLO with universe A, let further $X = \{c_1, \dots, c_\ell\} \subseteq A$ be a finite set of parameters. Let $p \subseteq A^n$ be a minimal definable set over X. Then $p = \{\bar{a} \colon \varphi(\bar{a}, \bar{c})\}$, where φ is a conjunction of atomic formulas such that for every $t, s \in \{x_1, \dots, x_n, c_1, \dots, c_\ell\}$ the formula φ contains exactly one of the formulas $t < s$, $t = s$, $t > s$ as subformula.*

Proof. It follows easily from the proof of the previous observation. □

So we can say that p is fully determined by the order of variables and parameters. In other words for every minimal definable set p there exists a function $f \colon \{x_1, x_2, \dots, x_n, c_1, \dots, c_m\} \to \mathcal{A}$, such that $f(c_i) = c_i$ and the structure $\mathcal{A}$ satisfies $\varphi(f(x_1), f(x_2), \dots, f(x_n), \bar{c})$. If two such functions f, g are isomorphic (i.e. $f(x) < f(y) \Leftrightarrow g(x) < g(y)$), then

$$\mathcal{A} \models \varphi\big(f(x_1), f(x_2), \dots, f(x_n), \bar{c}\big) \Leftrightarrow \mathcal{A} \models \varphi\big(g(x_1), g(x_2), \dots, g(x_n), \bar{c}\big).$$

This motivates the following definition:

Definition 6.18. *Let $p_i \subseteq A^n$ be a minimal definable set over X defined by formula φ satisfying the conditions of the previous proposition.* ***The dimension of*** p_i ***(denoted by dim***(p_i)***)*** *is defined as the size $\big|\{a_1, a_2, \dots, a_n\} \backslash X\big|$ for any $\bar{a} \in p_i$, i.e. as the number of mutually different elements among a_1, a_2, ..., a_n that are different from all parameters.*

The following theorem is special case of a corollary taken from van den Dries [2] (Chapter IV, §1, Corollary 1.6(ii)).

Theorem 6.19. *Let $\mathcal{A}$ be a model of DeLO with universe A. Let $X \subseteq A$ be a finite set of parameters. Let $B \subseteq A^n$ be a definable set over X and $f \colon B \to A^m$ be a definable map over X. Then for each $d \in \{0, \dots, n\}$ the set $S_f(d) := \{a \in A^m \colon \dim\big(f^{-1}(a) = d\big)\}$ is definable and*

$$\dim\Big(f^{-1}\big(S_f(d)\big)\Big) = \dim\big(S_f(d)\big) + d.$$

Moreover, $\dim\big(f[B]\big) \leq \dim\big(B\big)$.

Note that in this situation we have $\dim(B) = dim\big(f[B]\big)$, if each fiber $f^{-1}\big[\{a\}\big]$ is finite.

Proof. The notion of dimension introduced here coincides with the definition from [2]. □

Proposition 6.20. *Let $\mathcal{L}'$ be a language with one unary functional symbol f. Let $\mathcal{A}$ be a model of DeLO. Let $\mathcal{B}$ be an $\mathcal{L}'$-structure definable in $\mathcal{A}$. The definition of $\mathcal{B}$ uses only a finite sets of parameters X, hence we can use the notion from Observation 6.16. Moreover, we may without loss of generality assume that $\mathcal{B} = \bigcup_{i=1}^{n} p_i$.*

The mapping $\tilde{f}\colon \{p_1, \dots, p_n\} \to \{p_1, \dots, p_n\}$ defined by $\tilde{f}(p_i) = f[p_i]$ is well-defined and satisfies:

- *If f is injective or onto, $\tilde{f}$ is a bijection and $\dim\big(f[p_i]\big) = \dim(p_i)$.*

Proof.

- It is enough to show that $f[p_i] = p_j$ for some j. Since $f[p_i]$ is a definable set, it can be written as union of p_j's.

$$f[p_i] = \bigcup_{j\in J} p_j,$$

  for some set $J \subseteq 1, \dots, n$. Let us suppose that $|J| > 1$. Let k be the minimal element of J and let us set

$$S := \bigcup_{j\in J\setminus\{k\}} p_j.$$

  Then $f[p_i] = p_k \cup S$ and $p_i = f^{-1}[p_k] \cup f^{-1}[S]$, with $f^{-1}[p_k]$ and $f^{-1}[S]$ non-empty disjoint and definable from X. But this is a contradiction with p_i having no proper non-empty definable subsets.

- If f is injective or onto, so is $\tilde{f}$, but $\tilde{f}$ is a mapping from a finite set to the same finite set and hence has to be a bijection.

  If $\dim\big(f[p_i]\big) < \dim(p_i)$, then because $\tilde{f}$ is a bijection, we have for some j the inequality $\dim\big(f[p_j]\big) > \dim(p_j)$. This is a contradiction with Theorem 6.19, therefore the equality $\dim(p_i) = \dim\big(f[p_i]\big)$ holds.

□

Theorem 6.21.

1. *MAX, MIN* $\notin$ *Comb(DeLO),*

2. $PHP_1, PHP_2, PHP_3 \in$ *Comb(DeLO).*

Proof. The first point is clear, the relation $<$ clearly satisfies neither MAX nor MIN.

Let us assume that there exist a definable mapping f on some definable set B. We will use the notion from Proposition 6.20. If f is injective, $\tilde{f}$ is a bijection, hence

$$f[B] = f\left[\bigcup_{i=1}^{n} p_i\right] = \bigcup_{i=1}^{n} f[p_i] = \bigcup_{i=1}^{n} p_i = B$$

and f is onto, consequently PHP_1, PHP_3 lie in Comb($DeLO$).

If f is onto, $\tilde{f}$ is a bijection. Let us denote the set of all parameters used to define f by X. It remains to show that $f|p_i$ is injective for each p_i.

Let B be subset of A^n and $f\colon B \to B$ be onto. We choose an arbitrary set $p_i \subseteq B$ and define the set $S := \{(\bar{x}, f(\bar{x}))\colon \bar{x} \in p_i\}$. Let $\pi_1\colon A^{2n} \to A^n$ or $\pi_2\colon A^{2n} \to A^n$ denote the projections on the first n or the last n coordinates, respectively.

The set S is clearly definable. Moreover, if $S = U \cup V$, where U, V are definable non-empty proper subsets of S, then $\pi_1 U$ is a proper non-empty subset of p_i – a contradiction with p_i being minimal definable. We conclude that S is a minimal definable set in A^{2n}.

For every $(\bar{a}, \bar{b}) \in S$, the set $\{\bar{x}\colon (\bar{x}, \bar{b}) \in S\}$ has exactly one element. Therefore $\dim(S) = \dim\big(\tilde{f}(p_i)\big) = \dim(p_i)$ due to Theorem 6.19.

For any $\bar{a} \in \pi_1 S$ and $\bar{b} \in \pi_2 S$,

$$\dim(p_i) = \big|\{a_1, \ldots, a_n\} \setminus X\big| = \big|\{b_1, \ldots, b_n\} \setminus X\big| = \dim\big(f(p_i)\big).$$

Let us set $Y := \{a_1, \ldots, a_n\} \setminus X$ and $Z := \{b_1, \ldots, b_n\} \setminus X$. $\tilde{Y}$ will be the set of coordinates corresponding to a_i's in Y and $\tilde{Z}$ will be the set of coordinates corresponding to b_i's in Z. S is minimal definable, therefore the sets $\tilde{Y}$ and $\tilde{Z}$ do not depend on the choice of $\bar{a}$, $\bar{b}$.

As $\dim(S) = |\tilde{X} \cup \tilde{Z}| = |\tilde{X}| = \dim(p_i) = |\tilde{Z}|$ and $|\tilde{Y}| = |\tilde{Z}|$, S is a bijection of $\tilde{Y}$ onto $\tilde{Z}$. Therefore $f \restriction p_i$ is just a permutation of variables and is injective. The mapping f is injective due to Proposition 6.20. Finally, $\mathrm{PHP}_2 \in$ Comb($DeLO$). ∎

Remark 6.22. *Using Theorem 6.19, we can even show that for any definable mapping f its restriction $f_{\restriction p_i}$ is just a permutation and omitting of variables, moreover every definable set is just a union of minimal definable sets. Therefore, if we have a definable $\mathcal{L}$-structure with $\mathcal{L}_G \not\preccurlyeq \mathcal{L}$, we can find an isomorphism to some structure definable in some $\mathcal{L}_\emptyset$-structure. Consequently $Comb_{\mathcal{L}}(DeLO) = Finite_{\mathcal{L}}$ for any language $\mathcal{L}$ which contains only unary relational and unary functional symbols.*

6.5 $(\mathbb{Z}, +, 0)$

Let us now consider integers as an Abelian group $(\mathbb{Z}, +, 0)$.

Proposition 6.23.

1. $PHP_2, PHP_1 \notin Comb(\mathbb{Z}, +, 0)$,
2. $MAX, MIN \in Comb(\mathbb{Z}, +, 0)$.

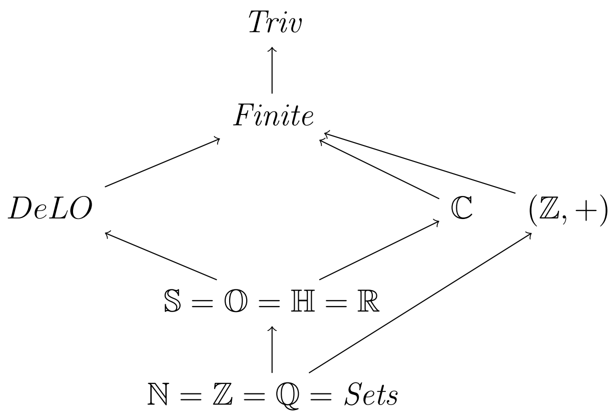
Proof. The first part is clear. The mapping $f(z) = z + z$ is clearly injective, but not onto. The definable mapping $f(z) = \begin{cases} x & \text{if } z = x + x, \\ y & \text{if } z = y + y + 1 \end{cases}$ is onto but not injective.

For the second part: The theory of integers viewed as an Abelian group has $\min(2^{2^\omega}, 2^\kappa)$ models of cardinality κ, which was shown in [14]. From Corollary 2.65 of the Many Models Theorem 2.63 follows that no linear order on an infinite set is definable in $(\mathbb{Z}, +, 0)$. Every finite linear order has the smallest and the largest element. □

Problem 3. *Is PHP_3 in $Comb(\mathbb{Z}, +, 0)$?*

We will conclude the result of this chapter with a figure depicting the relations between combinatorics of various structures.

Theorem 6.24. *The combinatorics Comb of studied structures is ordered as in the picture. All arrows correspond to a proper inclusion, = means that* *the combinatorics of the structures is the same.*



Chapter 7

Complexity results

In this section we will discuss complexity results about the combinatorics of structures.

7.1 Trivial structure

We will begin with a trivial observation.

Theorem 7.1. *Let $\mathcal{A}$ be a trivial structure. Then the problem of deciding whether a given formula φ lies in $Comb_{\mathcal{A}}$ is co-NP-complete.*

Proof. We will denote the problem of deciding whether $\varphi \in \mathrm{Triv}$ by TrivDec. TrivDec is clearly in co-NP, as we can delete all quantifiers and suppose that φ contains neither functional symbols nor relational symbols or arity greater than one. Then φ does not lie in Triv if and only if there exists a realization of unary relations $S \in \mathcal{S}$ from φ such that $\big(\{a\}, \{S_s\}_{s \in \mathcal{S}}\big) \nvDash \varphi$.

On the other hand, it is easy to check that TrivDec is co-NP-hard (easy reduction from TAUT[1]). ☐

We can easily see that the triviality of the structure $\mathcal{A}$ was used only to show that TrivDec lies in co-NP, so we can conclude that deciding whether a given sentence is in $\mathrm{Comb}(\mathcal{A})$ is at least co-NP-hard.

Theorem 7.2. *Let $\mathcal{L}$ be a language containing at least one functional symbol or at least one relational symbol of arity at least two. Let $\mathcal{A}$ be an $\mathcal{L}$-structure, then deciding whether a given $\mathcal{L}$-formula φ lies in $Comb(\mathcal{A})$ is co-NP-hard.*

[1]For the definition of TAUT, see [15].

Proof. It is easy to see that

$$\big((\forall x)(\forall y)(x = y) \to \varphi\big) \in \mathrm{Comb}(\mathcal{A}) \Leftrightarrow \varphi \in \mathrm{Triv}.$$

□

7.2 Trakhtenbrot's theorem

Theorem 7.3 (Trakhtenbrot's theorem)**.** *$\mathit{Finite}_{\mathcal{L}_G}$ is co-r.e. complete.*

Proof. Given a finite structure $\mathcal{A}$ witnessing that $\varphi \notin \mathrm{Finite}_{\mathcal{L}_G}$, we can recursively check that $\mathcal{A} \models \neg\varphi$, hence deciding whether $\varphi \in \mathrm{Finite}_{\mathcal{L}_G}$ is co-r.e. The second part is due to Trakhtenbrot's theorem, see [7]. □

On the other hand, we have the following theorem.

Theorem 7.4. *Let $\mathcal{L}$ be a language, then $\mathit{Sets}_{\mathcal{L}}$ is r.e. complete.*

Proof. To prove that a given formula is in Sets it is sufficient to find its proof. Every formula contains only finitely many symbols, hence we may assume that $\mathcal{L}$ is finite and we have an encoding of all sequences of symbols from $\mathcal{L}$ into natural numbers. Define a relation $\mathrm{Proof}(x, y)$ deciding whether the string encoded by number x is a proof of sentence encoded by a number y. It can be shown that $\mathrm{Proof}(x, y)$ is recursive (see [15]). Consequently $\mathrm{Thm}(y) = (\exists x)\mathrm{Proof}(x, y)$ is r.e. Hence the problem is r.e.

On the other hand, we can easily reduce any r.e. problem to finding a proof of the statement in Peano arithmetic. Recursive enumerable problem can be written as the problem of deciding whether $(\exists y)R(x, y)$ for some recursive relation R, but this problem is equivalent to finding a proof that there exists y satisfying $(\exists y)R(x, y)$. □

7.3 Definability of the finiteness

We want to describe the complexity of other structures.

Definition 7.5. *We say that a non-trivial structure $\mathcal{A}$ has strongly definable finiteness if:*

- *there exist a first order language $\mathcal{L}$ and an $\mathcal{L}$-formula Fin, such that if some $\mathcal{L}'$-structure $\mathcal{B}$ definable in $\mathcal{A}$ (with $\mathcal{L}' \supseteq \mathcal{L}$) satisfies Fin, then $\mathcal{B}$ is finite and*

- *if $\mathcal{L}''$ and $\mathcal{L}$ are disjoint and $\mathcal{C}$ is a finite $\mathcal{L}''$-structure, we can expand*[2] *$\mathcal{C}$ into an $\mathcal{L} \cup \mathcal{L}''$-structure satisfying Fin.*

Theorem 7.6. *Let $\mathcal{A}$ be an $\mathcal{L}$-structure. Let us suppose that $Th_{\mathcal{L}}(\mathcal{A})$ is recursive. Let further $\mathcal{A}$ have strongly definable finiteness. Then $Comb_{\mathcal{L}_G}(\mathcal{A})$ is co-r.e. complete. Furthermore, if a recursive theory T has less than 2^κ elementarily non-embeddable models for some $\kappa > \omega$, the set $Comb_{\mathcal{L}_G}(T)$ is co-r.e. complete.*

Proof. We will first show that the problem is in co-r.e. To do so, it is sufficient to observe that if the formula φ does not lie in the combinatorics, we have some definable counterexample and hence we have the proof that this example actually proves $\neg\varphi$.

It is now sufficient to show

$$\varphi \in \text{Finite}_{\mathcal{L}_G} \Leftrightarrow (\text{Fin} \rightarrow \varphi) \in \text{Comb}_{(\mathcal{L}_G \cup \mathcal{L})}(\mathcal{A}).$$

The rest follows from Trakhtenbrot's Theorem 7.3. If $\varphi \in \text{Finite}_{\mathcal{L}_G}$, then clearly $(\text{Fin} \rightarrow \varphi) \in \text{Comb}_{\mathcal{L}_G \cup \mathcal{L}}$, as if the definable structure $\mathcal{B}$ is infinite, $\mathcal{B}$ does not satisfy Fin, hence it satisfies the implication, and if $\mathcal{B}$ is finite, it satisfies φ. On the other hand, if $\varphi \notin \text{Finite}_{\mathcal{L}_G}$, there exists a finite definable structure $\mathcal{C} \models \neg\varphi$. This structure is finite and hence is definable in $\mathcal{A}$. According to the assumptions, we can define realizations of $\mathcal{L}$ symbols on $\mathcal{C}$ such that $\mathcal{C} \models \text{Fin}$. But then $\mathcal{C}$ is a definable subset of $\mathcal{A}$ satisfying $\mathcal{C} \models \neg(\text{Fin} \rightarrow \varphi)$, hence $(\text{Fin} \rightarrow \varphi) \notin \text{Comb}_{(\mathcal{L} \cup \mathcal{L}_G)}$.

Let us now suppose that T has less than 2^κ mutually non-embeddable models, according to Corollary 2.65, the $\mathcal{L}_G$-formula φ_{lord} strongly defines finiteness. The only technical detail is that we have to change the language $\mathcal{L}_G$ to some other isomorphic language with one binary relation, in order to avoid confusion.

The rest follows from the fact that $\mathcal{L}_G$ is a universal language. ☐

Example 7.7.

1. *$Comb_{\mathcal{L}_G}(\mathbb{C}, +, \cdot, -, 0, 1)$ is co-r.e. complete,*
2. *$Comb_{\mathcal{L}_G}(\mathbb{Z}, +, 0)$ is co-r.e. complete,*
3. *$Comb_{\mathcal{L}_G}(V)$ is co-r.e. complete, if V is a non-trivial vector space over any field.*

[2] I.e. add realization of new symbols.

Proof.

1. For every $\kappa > \omega$ there exists only one model of $\mathbb{C}$ with cardinality κ. For the proof see [10].

2. For every $\kappa > 2^{\omega}$ there exist only $2^{2^{\omega}}$ non-isomorphic models of cardinality κ. The proof can be found in [14].

3. The vector space is uniquely determined by its dimension $\dim(V)$. If $\mathrm{card}(V) > \omega$, then $\dim(V) = \mathrm{card}(V)$ and consequently there exists only one model of V with cardinality $\mathrm{card}(V)$.

□

This negatively answers the question whether the combinatorics of complex numbers is recursively enumerable, which was asked in [6].

Problem 4. *Is $Comb_{\mathcal{L}_G}(\mathbb{R})$ resp. $Comb_{\mathcal{L}_G}(DeLO)$ recursively enumerable?*

Conclusion

The combinatorics of many structures is really interesting. Here we list our new results and open problems.

In [6] Krajíček give a sketch of proof that the combinatorics of pseudo-finite structures equals to the combinatorics of finite sets. We have written the proof completely and found some interesting examples of pseudo-finite structures.

Furthermore, we found examples of elementarily non-equivalent structures with the same combinatorics which is not extremal. The remaining question is whether there exist two such mutually non-definable structures.

One of our main results is the proof that the variants PHP_3, PHP_2, PHP_1 of pigeon-hole principle are all in Comb(DeLO). We conjecture that $\mathrm{Comb}_{\mathcal{L}}(\mathrm{DeLO}) = \mathrm{Finite}_{\mathcal{L}}$ for any language $\mathcal{L}$ that contains no symbols of arity greater than two.

The second main result that we proved is that the combinatorics of every complete theory without SOP is co-r.e. complete. In particular we solved the problem from [6] how complex is the combinatorics of complex numbers – it is co-r.e. complete. The remaining question is what can be said about the complexity of $\mathrm{Comb}(\mathbb{R})$.

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