Chapter 6

Representability of matroids

6.1 Matroid representations

Recall that a matroid $\mathcal{M}$ is $\mathbb{F}$-*representable* if there exists a matrix A with columns one-to-one corresponding to the elements of $\mathcal{M}$ such that a set of columns if linearly independent over $\mathbb{F}$ if and only if the set of corresponding elements of $\mathcal{M}$ is independent in $\mathcal{M}$. A matroid is *representable* if it is representable over some field $\mathbb{F}$ and it is *regular* if it is representable over all fields $\mathbb{F}$.

Not all matroids are representable. Let $E = \{1, 2, \ldots, 8\}$ and

$$\mathcal{T}_1 = \{\{1,2,3,4\},\{1,4,5,6\},\{1,4,7,8\},\{2,3,5,6\},\{2,3,7,8\}\}$$

. Further, let

$$\mathcal{T} = \mathcal{T}_1 \cup \{T \subseteq E, |T| = 3, T \not\subseteq T_1,\ \forall T_1 \in \mathcal{T}_1\}\ .$$

It is straightforward (but little bit tedious) to verify that there exists a matroid $\mathcal{M}$ on E such that $\mathcal{T}$ is the family of the hyperplanes of $\mathcal{M}$, i.e., inclusion-wise maximal sets of rank $r(\mathcal{M}) - 1 = 3$. This matroid, denoted by V_8 and called the *Vámos matroid*, is depicted in Figure 6.1. We show that the Vámos matroid is not representable over any field.

Proposition 6.1. *The Vámos matroid V_8 is not representable over any field.*

Proof. Assume that the matroid V_8 is representable over a field $\mathbb{F}$. Since the rank of V_8 is four, there exists a mapping $\psi : E(V_8) \to \mathbb{F}^4$ such that $r(X) = \dim \mathcal{L}(\psi(X))$ for any subset $X \subseteq E(V_8)$ where $\mathcal{L}(Z)$ denotes the linear hull of the vectors of Z. For $\{x_1, \ldots, x_k\} \subseteq E(V_8)$, $W(x_1, \ldots, x_k)$ will denote the subspace $\mathcal{L}(\{\psi(x_1), \ldots, \psi(x_k)\})$. It holds that

$$\begin{aligned}\dim\,(W(5,6) \cap W(1,2,3,4)) &= \dim W(5,6) + \dim W(1,2,3,4) \\ &\quad -\dim \mathcal{L}(W(5,6) \cup W(1,2,3,4)) \\ &= 2 + 3 - 4 = 1\,.\end{aligned}$$

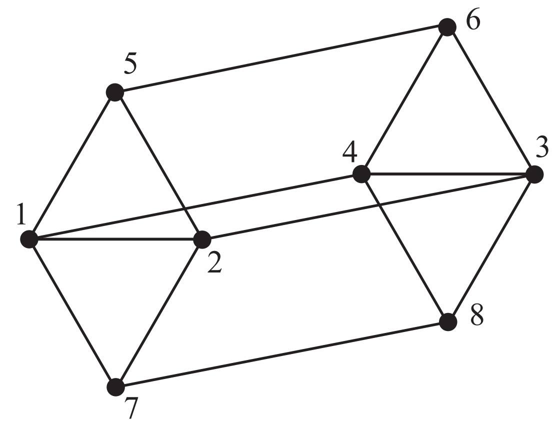

Figure 6.1: The diagram of the Vámos matroid.

Hence, $W(5,6) \cap W(1,2,3,4) = \mathcal{L}(\{v\})$ for some non-zero vector $v \in \mathbb{F}^4$.

Again

$$\begin{aligned} \dim\,(W(1,4,5,6) \cap W(1,2,3,4)) &= \dim W(1,4,5,6) + \dim W(1,2,3,4) \\ &\quad - \dim \mathcal{L}(W(1,4,5,6) \cup W(1,2,3,4)) \\ &= 3 + 3 - 4 = 2\,. \end{aligned}$$

Since $W(1,4)$ is 2-dimensional subspace of $W(1,4,5,6) \cap W(1,2,3,4)$, we obtain that

$$\mathcal{L}(\{v\}) = W(5,6) \cap W(1,2,3,4) \subseteq W(1,4,5,6) \cap W(1,2,3,4) = W(1,4)\,.$$

A symmetric argument (see Figure 6.1 for visualization) yields that $\mathcal{L}(\{v\}) \subseteq W(2,3)$.

The dimension of the intersection of $W(1,4)$ and $W(2,3)$ which is

$$\dim\,(W(1,4) \cap W(2,3)) = 2 + 2 - 3 = 1$$

implies that $W(1,4) \cap W(2,3) = \mathcal{L}(\{v\})$.

By symmetry, we obtain that $\mathcal{L}(\{v\}) = W(5,6) \cap W(7,8)$ which implies that

$$\begin{aligned} \dim W(5,6,7,8) &= \dim \mathcal{L}(W(5,6) \cup W(7,8)) \\ &= \dim W(5,6) + \dim W(7,8) - \dim\,(W(5,6) \cap W(7,8)) \\ &\leq 2 + 2 - 1 = 3\,. \end{aligned}$$

However, $\dim W(5,6,7,8)$ cannot be equal to three since the set $\{5,6,7,8\}$ is independent in V_8. $\square$

To present another example of a matroid that is not representable over any field, we will need an operation of relaxing a circuit-hyperplane in a matroid.

Proposition 6.2. *Let $\mathcal{M}$ be a matroid that contains a subset X of its elements that is both a circuit and a hyperplane. Let $\mathcal{B}' = \mathcal{B}(\mathcal{M}) \cup \{X\}$. The family $\mathcal{B}'$ is a family of bases of a matroid. Moreover, the family of circuits of this matroid is*

$$(\mathcal{C}(\mathcal{M}) \setminus \{X\}) \cup \{X + e : e \in E(\mathcal{M}) \setminus X\} .$$

Proof. Let $\mathcal{I}'$ be the family of all subsets of $\mathcal{B}'$. We verify that $\mathcal{I}'$ has the properties (I1), (I2) and (I3). Since (I1) and (I2) are trivial to verify, we focus on (I3). Let I_1 and I_2 be two members of $\mathcal{I}'$ with $|I_1| < |I_2|$. Clearly, we can assume that $|I_1| = |I_2| - 1$. If $I_2 \neq X$, the claim follows from the fact that the family of independent sets of $\mathcal{M}$ has the property (I3). If $I_1 \subseteq I_2 = X$, the claim also holds. Otherwise, $I_1 + x$ is dependent in $\mathcal{M}$ for every $x \in X$ which implies that $r(I_1 \cup I_2) = r(I_1) \leq r(\mathcal{M}) - 1$ by Lemma 1.10. In other words, $I_1 \cup I_2 \subseteq X$ since X is a hyperplane which violates our assumption that I_1 is not a subset of $I_2 = X$.

Let $\mathcal{M}'$ be the matroid whose bases are those subsets contained in $\mathcal{B}'$. Clearly, any circuit of $\mathcal{M}$ distinct from X is a circuit of $\mathcal{M}'$. So, we have to investigate which supersets of X are circuits in $\mathcal{M}'$. Consider a set $X + e$ for $e \notin X$. This set is dependent and removing any element e' of it results in an independent set; this follows from the fact that X is a hyperplane for $e' \neq e$ and is trivial for $e' = e$. Hence, the family of circuits of $\mathcal{M}'$ is the family described in the statement of the proposition. □

The operation described in Proposition 6.2 is called *relaxing* of a circuit-hyperplane in a matroid.

Another example of a matroid that is not representable is the non-Pappus matroid. The construction is based on relaxing one 3-element set in the Pappus matroid. Both the *Pappus* and the *non-Pappus matroids* are depicted in Figure 6.2. We omit the proof of the non-representability of the non-Pappus matroid.

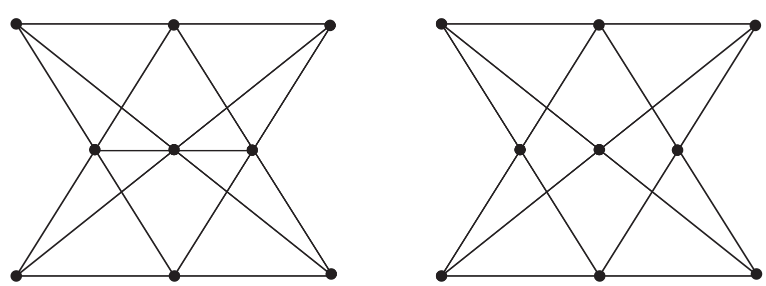

Figure 6.2: The diagrams of the Pappus and the non-Pappus matroids.

Proposition 6.3. *The non-Pappus matroid is not representable over any field.*

We will now study representability of the Fano matroid F_7, which was introduced in Section 1.3, and the matroid F_7^-, called the *non-Fano matroid* , that is obtained from F_7 by relaxing a circuit-hyperplane (the circuit-hyperplane $\{2, 4, 6\}$ in Figure 6.3).

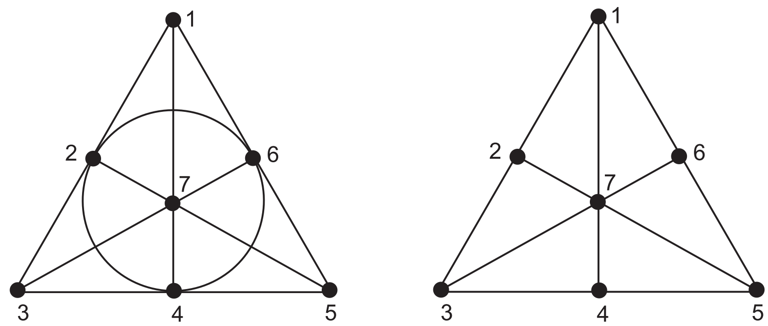

Figure 6.3: The diagrams of the Fano and the non-Fano matroids.

Let $\mathcal{M}$ be a matroid with a base B. For every element e of $\mathcal{M}$ not contained in B, the set $B + e$ contains a unique circuit by the property (C3) from Lemma 1.1. This circuit is called the *fundamental circuit* of e with respect to the base B.

Consider a standard representation A of a matroid $\mathcal{M}$ over a field $\mathbb{F}$ such that $A = [I_r|D]$ where r is the rank of $\mathcal{M}$ and the first r columns correspond to the elements of the base B. The elements of B correspond in a natural ways to the rows of A, too. Observe that the fundamental circuit of e with respect to B is formed by those elements of B that have in the column corresponding to e non-zero entries in the corresponding rows. Let $D^{\#}$ be the matrix obtained from D by replacing each non-zero entry by a 1. The columns of $D^{\#}$ are now the incidence vectors of the fundamental circuits with respect to B restricted to B. This matrix $D^{\#}$ is called the *B-fundamental-circuit incidence matrix* of $\mathcal{M}$. The fundamental-circuit incidence matrix for the matroids F_7 and F_7^- is the same and can be found in Figure 6.4. The matrix $[I_r|D^{\#}]$ is called a *partial representation* of $\mathcal{M}$. Note that a partial representation and the B-fundamental-circuit incidence matrix $D^{\#}$ is also well-defined for non-representable matroids and is unique because of the uniqueness of fundamental circuits with respect to a chosen base. Let us state this fact as a separate proposition.

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Figure 6.4: The fundamental-circuit incidence matrix for F_7 and F_7^-.

Proposition 6.4. *Let $\mathcal{M}$ be a matroid of rank r. If $[I_r|D_1]$ and $[I_r|D_2]$ are two representations of $\mathcal{M}$ over a field $\mathbb{F}$ such that their columns correspond to the elements of $\mathcal{M}$ in the same way, then $D_1^{\#} = D_2^{\#}$.*

Theorem 2.14 and Proposition 6.4 combine to the following.

Proposition 6.5. *Let $\mathcal{M}$ be a representable matroid with a ground set E and B a base of $\mathcal{M}$. If $D^{\#}$ is the B-fundamental-circuit incidence matrix of $\mathcal{M}$, then $(D^{\#})^T$ is the $(E \setminus B)$-fundamental-incidence matrix of $\mathcal{M}^*$.*

Let us remark that the assumption that $\mathcal{M}$ is representable can be omitted in Proposition 6.5.

We now study representability of the matroids F_7 and F_7^- over different fields. Let us start with the following proposition whose proof lies a routine check of linear dependencies of vectors over a field and is left to a reader.

Proposition 6.6. *Let $\mathbb{F}$ be a field and X the matrix given in Figure 6.4. If the characteristic of $\mathbb{F}$ is two, then $[I_3|X]$ is a representation of the matroid F_7 and if the characteristic of $\mathbb{F}$ is not two, then it is a representation of the matroid F_7^-.*

Complementing the previous proposition, we prove the following.

Proposition 6.7. *Let $\mathcal{M}$ be one of the matroids F_7 and F_7^-. If $[I_3|X]$ is a representation of $\mathcal{M}$ over a field $\mathbb{F}$, then X is the matrix in Figure 6.4.*

Proof. We can choose a base B of $\mathcal{M}$ such that the matrix X is the B-fundamental-circuit base. In particular, since $[I_3|D]$ is an $\mathbb{F}$-representation of $\mathcal{M}$, $D^{\#}$ is the matrix X. By multiplying rows and columns with non-zero elements of $\mathbb{F}$, we can assume D to be of the following form

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c & 1 \\ a & b & 0 & 1 \end{pmatrix}.$$

Since the third, last but one and last column of $[I_3|D]$ correspond to a circuit of $\mathcal{M}$, we obtain $c = 1$. Similarly, we get that $b = 1$ and $a = 1$. In particular, D must be equal to X. □

We infer from Propositions 6.6 and 6.7 the following.

Proposition 6.8. *(i) The matroid F_7 is $\mathbb{F}$-representable for a field $\mathbb{F}$ if and only if the characteristics of $\mathbb{F}$ is two.*

(ii) The matroid F_7^- is $\mathbb{F}$-representable for a field $\mathbb{F}$ if and only if the characteristics of $\mathbb{F}$ is different from two.

Corollary 6.9. *The matroid $F_7 \oplus F_7^-$ is not representable over any field.*

6.2 Representability over finite fields

A classical approach to finding necessary and sufficient conditions for a matroid to be $\mathbb{F}$-representable is to determine minimal obstructions to $\mathbb{F}$-representability. Since the class of $\mathbb{F}$-representable matroids is closed under taking minors one way to characterize such class is by listing all minor-minimal matroids that do not belong to the class. These matroids are called *excluded minors* for $\mathbb{F}$-representability. Finding the complete set of excluded minors for representability over a particular field is a notorious difficult problem and has, in fact, only been solved for 2-element and 3-element fields. GF(2)-representable matroids are called *binary* and GF(3)-representable matroids are called *ternary*. Nevertheless, it is still possible to find some properties of excluded minors for other fields, too. For example, since the class of $\mathbb{F}$-representable matroids is closed under duality, we have the following.

Proposition 6.10. *If a matroid $\mathcal{M}$ is an excluded minor for $\mathbb{F}$-representability, then so is its dual $\mathcal{M}^*$.*

Excluded minors for $\mathbb{F}$-representability falls loosely into two categories: those that are excluded because the field is too small, and those that are excluded for structural reasons. A class of matroids of the first type is the class of rank-two uniform matroids.

Proposition 6.11. *Let $\mathbb{F}$ be a finite field and $k \geq 2$. The matroid $U_{2,k}$ is $\mathbb{F}$-representable if and only if $|\mathbb{F}| \geq k-1$.*

Proof. Let $[I_2|D]$ be an $\mathbb{F}$ representation of $U_{2,k}$. Without lost of generality, all the entries of the first row of D are equal to one. The entries of the second row of D have to be mutually distinct non-zero elements of $\mathbb{F}$ and thus $k-2 \leq |\mathbb{F}|-1$. In the other direction, if D is a matrix with all the entries of the first row equal to one and the entries of the second row equal to mutually distinct non-zero elements of $\mathbb{F}$, then $[I_2|D]$ is an $\mathbb{F}$-representation of D. □

Propositions 6.10 and 6.11 combine to the following. Recall that all minors of uniform matroids are uniform matroids.

Corollary 6.12. *If q is a prime number, then the matroids $U_{2,q+2}$ and $U_{q,q+2}$ are excluded minors for* GF(q)*-representability.*

We can now characterize excluded minors for binary matroids.

Theorem 6.13. *A matroid is binary if and only if it does not contain $U_{2,4}$ as a minor.*

Proof. By Corollary 6.12, $U_{2,4}$ is not binary. Hence, it is enough to prove that any matroid that is not binary contains $U_{2,4}$ as a minor. Let $\mathcal{M}$ be an arbitrary minor-minimal non-binary matroid. Since $\mathcal{M}^*$ is also a minor-minimal non-binary matroid, thus we can assume that $2r(\mathcal{M}) \le |E(\mathcal{M})|$. Moreover, the choice of $\mathcal{M}$ implies that $\mathcal{M}$ has neither loops nor parallel edges.

Let B be a base of $\mathcal{M}$ and D the B-fundamental-circuit incidence matrix of $\mathcal{M}$. If $\mathcal{M}$ were binary, then $[I_r|D]$ would be its representation. Since $\mathcal{M}$ is not binary, the matroid $\mathcal{M}_b$ represented by $[I_r|D]$ differs from $\mathcal{M}$. In particular, there exists a base B' of $\mathcal{M}$ such that the B'-fundamental-circuits are not properly represented in $[I_r|D]$. By pivoting operations which preserve the represented matroid, we can assume that the bases B and B' differ in one element, say $B' = (B - x) + y$.

Assume that x corresponds to the first column of I_r and y to the first column of D. Clearly, the first entry in the first row of D is non-zero (otherwise, B' would not be a base of $\mathcal{M}$). Add the first row of $[I_r|D]$ to every row having a non-zero entry in the first column of D. After switching the first columns of I_r and D, we obtain a matrix $[I_r|D_b]$ still representing the matroid $\mathcal{M}_b$ where D_b differs from the partial representation D' of $\mathcal{M}$ with respect to B'. Without loss of generality, we can assume that the matrices D_b and D' differ in the second entry of the second row (otherwise, permute the rows and columns). By the choice of $\mathcal{M}$ as a minor-minimal matroid that is not binary, the matrices D_b and D' have only two rows and two columns. Moreover, since $\mathcal{M}$ has no loops or parallel elements, $\mathcal{M}$ must be isomorphic to $U_{2,4}$. □

Now, we show two interesting properties of binary matroids.

Lemma 6.14. *Let $\mathcal{M}$ be a binary matroid. The symmetric difference of any two circuits of $\mathcal{M}$ is a disjoint union of circuits.*

Proof. Fix a representation A of $\mathcal{M}$ over GF(2). Let C_1 and C_2 be two circuits of $\mathcal{M}$. Since C_i, $i = 1, 2$, is a circuit, the columns of A corresponding to the elements of C_i sum to the zero vector. Let C be the symmetric difference of C_1 and C_2. Since the columns corresponding to the elements of $C_1 \cap C_2$ are counted in the sums twice, we obtain that the sum of the columns corresponding to the elements of C is the zero vector.

Let $C^1, \ldots, C^k$ be inclusion-wise minimal subsets of C such that the columns corresponding to the elements of C^i, $i = 1, \ldots, k$, sum to the zero vector. Observe that all C^i, $i = 1, \ldots, k$, are disjoint and their union is equal to C. Clearly, each C^i is a circuit. The lemma now follows. □

We now study intersections of circuits and cocircuits of binary matroids.

Lemma 6.15. *Let $\mathcal{M}$ be a binary matroid. If C is a circuit of $\mathcal{M}$ and C^* is a cocircuit, then the size of $C \cap C^*$ is even.*

Proof. If the intersection of C and C^* is empty, the lemma holds. Hence, we can assume that there exists an element x contained in both C and C^*. Consequently, there exists a base B^* of $\mathcal{M}^*$ such that $C^* - x \subseteq B^*$. Let $B = E \setminus B^*$ be the complementary base of $\mathcal{M}$, and let $[I_r|D]$ be the representation of $\mathcal{M}$ with the first r columns corresponding to B and we can assume that $x \in B$ corresponds to the first column of I_r. Hence, $[D^T|I_{n-r}]$ is a representation of $\mathcal{M}^*$. Since C is a circuit of $\mathcal{M}$, the corresponding columns of $[I_r|D]$ sum to the zero-vector. Since $x \in C$, there is an odd number of columns of D having non-zero entry in the first row. However, these columns are precisely the columns of corresponding to the elements of C^* since such non-zero entries correspond to the non-zero entries of the first column of D^T. Hence, $C - x$ and $C^* - x$ have an odd number of common elements. Consequently, the intersection of C and C^* has an even number of elements. □

The properties given in Lemmas 6.14 and 6.15 actually give different characterizations of binary matroids whose proof we omit.

Theorem 6.16. *Let $\mathcal{M}$ be a matroid. The following statements are equivalent:*

(i) The matroid $\mathcal{M}$ is binary.

(ii) Every circuit C and every cocircuit C^* of $\mathcal{M}$ have intersection of even size.

(iii) The symmetric difference of any two circuits of $\mathcal{M}$ contains a circuit.

(iv) The symmetric difference of any two circuits is a disjoint union of circuits.

(v) The symmetric difference of any set of circuits of $\mathcal{M}$ is either empty or contains a circuit.

(vi) The symmetric difference of any set of circuits is $\mathcal{M}$ a disjoint union of circuits (which includes the case that it is empty).

(vii) Let B be an arbitrary base of $\mathcal{M}$. Every circuit C of $\mathcal{M}$ is the symmetric difference of e-fundamental circuits of $\mathcal{M}$ with respect to B where e runs over all the elements of C.

(viii) There exists a base B such that every circuit C of $\mathcal{M}$ is the symmetric difference of e-fundamental circuits of $\mathcal{M}$ with respect to B where e runs over all the elements of C.

Let us turn our attention to matroids representable over fields with characteristic different from two.

Proposition 6.17. *The matroids F_7 and F_7^* are excluded minors for $\mathbb{F}$-representability for any field $\mathbb{F}$ with characteristic different from two.*

Proof. By Proposition 6.8, F_7 is not $\mathbb{F}$-representable. Observe that for an arbitrary element e, the matroids $F_7 \setminus \{e\}$ and $F_7^* \setminus \{e\}$ are representable over any field. Hence, both $F_7 \setminus \{e\}$ and $F_7/\{e\}$ are representable over $\mathbb{F}$ for any element and F_7 is an excluded minor for $\mathbb{F}$-representability. By Proposition 6.10, the matroid F_7^* is also an excluded minor for $\mathbb{F}$-representability. □

Without proof, we give a list of excluded minors for ternary matroids.

Theorem 6.18. *A matroid is ternary if and only if it has no minor isomorphic to any of the matroids $U_{2,5}$, $U_{3,5}$, F_7, and F_7^*.*

Though the concept of excluded minors for matroids is similar to that for graphs, there are substantial differences. One of the most important theorems in the theory of graph minors is the following deep theorem of Robertson and Seymour [20].

Theorem 6.19. *For every proper class $\mathcal{G}$ of graphs closed under taking minors, there exists a finite set of graphs $excl(\mathcal{G})$ such that $G \in \mathcal{G}$ if and only if G has no minor isomorphic to any graph of $excl(\mathcal{G})$. In particular, the number of excluded minors is finite for every proper minor-closed class of graphs.*

Theorems 6.13 and 6.18 could suggest that the same might be true for matroids. However, this is far from being true [17].

Theorem 6.20. *There is infinite family of matroids such that each of them is an excluded minor for $\mathbb{Q}$-representability. Moreover, there is such a family of matroids that each its member is representable over a field with characteristic two (different members can be representable over different fields).*

6.3 Regular matroids

In the final section of this chapter, we study regular matroids, i.e., matroids that can be represented over any field. A *totally unimodular matrix* is a matrix A over $\mathbb{R}$ such that every square submatrix of A has determinant in $\{0, 1, -1\}$. We say that a matroid $\mathcal{M}$ is *unimodular* if it can be represented by a totally unimodular matrix over $\mathbb{R}$. We show in this section that the classes of unimodular and regular matroids coincide.

We now describe a matrix operation called *pivoting* which we already used in the proof of Theorem 6.13. Let A be an $m \times n$-matrix and a_{st} a non-zero entry of it. The matrix A' obtained by pivoting on a_{st} is the matrix obtained from A by the following two operations (applied in the given order):

(i) multiply the s-th row with the inverse of a_{st}, and

(ii) subtract from the s'-th row, $s' \neq s$, the multiple of $a_{s't}$ of the s-th row.

An important property of pivoting is that it preserves unimodularity of a matrix.

Lemma 6.21. *Let A be a totally unimodular matrix. If a matrix B is obtained from A by pivoting on a non-zero entry a_{st} of A, then the matrix B is also totally unimodular.*

Proof. Let B' be a square submatrix of B, A' the corresponding submatrix of A, and J_r and J_c the indices of the rows and columns forming B'. If $s \in J_r$, then $|\det A'| = |\det B'|$. Hence, the determinant of B' is 0, +1 or −1. Otherwise, if $t \in J_c$, then the B' has an all-zero column and $\det B' = 0$. Hence, we may assume that $s \notin J_r$ and $t \notin J_c$. Let A'' and B'' be the submatrices of A and B formed by rows and columns indexed with $J_r \cup \{s\}$ and $J_c \cup \{t\}$. Clearly, $|\det A''| = |\det B''|$. Since the only non-zero of the t-th column of B'' is b_{st}, the determinants B' and B'' can differ in signs only. We conclude that the determinant of any square submatrix of B is 0, +1 and −1, i.e., the matrix B is totally unimodular. □

We now show that the class of unimodular matroids is closed under taking duals.

Theorem 6.22. *The dual of a unimodular matroid is unimodular.*

Proof. Let $\mathcal{M}$ be a unimodular matroid and A a totally unimodular matrix representing $\mathcal{M}$ over $\mathbb{R}$. By pivoting non-zero elements in the columns of A corresponding to a base of $\mathcal{M}$, we obtain a totally unimodular standard representation of $\mathcal{M}$, i.e., a totally unimodular matrix $[I_r|D]$ representing $\mathcal{M}$. By Theorem 2.14, the matrix $[D^T|I_{n-r}]$ is a standard representation of $\mathcal{M}^*$. Clearly, $[D^T|I_{n-r}]$ is totally unimodular and thus $\mathcal{M}^*$ is unimodular. □

Since deleting a column of a totally unimodular matrix does not affect its total unimodularity, Theorem 6.22 immediately yields.

Corollary 6.23. *Every minor of a unimodular matroid is unimodular.*

We now show another property of total unimodular matrices which is related to representation of binary matroids.

Lemma 6.24. *Let $\mathcal{M}$ be a binary matroid and $[I_r|D_1]$ a representation of $\mathcal{M}$ with all entries equal to 0, +1 or −1 over a field $\mathbb{F}$ with characteristic different from 2. If a matrix $[I_r|D_2]$ is obtained from $[I_r|D_1]$ by pivoting on a non-zero entry of D_1, then every entry of D_2 is equal to 0, +1 or −1.*

Proof. Assume that we have pivoted on an element in the s-th row and t-th column. Clearly, the entries of the s-th row and t-th column of $[I_r|D_2]$ are equal to 0, +1 or −1. Consider an entry in the i-th row and j-th column for $i \neq s$ and $j \neq t$. If $j \leq r$, the considered entry is clearly equal to 0, +1 or −1. Hence, we assume that $j > r$. Pivoting replaces the entry d_{ij} with $d_{ij} - (d_{it}/d_{st}) \cdot d_{sj}$. Since all entries of D_1 are equal to 0, +1 or −1, the difference $d_{ij} - (d_{it}/d_{st}) \cdot d_{sj}$ is equal to 0, +1 or −1 unless $|d_{st}d_{ij} - d_{it}d_{sj}| = 2$ in which case all the four entries d_{ij}, d_{it}, d_{sj} and d_{st} are non-zero and $|d_{st}d_{ij} - d_{it}d_{sj}|$ is the determinant of the matrix $\begin{pmatrix} d_{st} & d_{it} \\ d_{sj} & d_{ij} \end{pmatrix}$, which is a square submatrix of D_1.

Since the matroid $\mathcal{M}$ is binary, $[I_r|D_1^{\#}]$ is a representation of $\mathcal{M}$ over GF(2). However, the first r columns of $[I_r|D_1^{\#}]$ except for the s-th and the i-th columns and the t-th and the j-th columns are linearly dependent over GF(2), but the same columns of $[I_r|D_1]$ are linearly independent over $\mathbb{F}$ which is impossible. □

We are now ready to show that the classes of regular and unimodular matroids coincide.

Theorem 6.25. *The following statements are equivalent for every matroid $\mathcal{M}$:*

(i) $\mathcal{M}$ is unimodular.

(ii) $\mathcal{M}$ is regular.

(iii) $\mathcal{M}$ is binary and $\mathbb{F}$-representable for a field $\mathbb{F}$ of characteristic different from two.

Proof. Clearly, it is enough to prove that the statements are equivalent for matroids $\mathcal{M}$ with $r(\mathcal{M}) > 0$. Suppose that *(i)* holds, i.e., there is a totally unimodular matrix $[I_r|D]$ representing $\mathcal{M}$ over $\mathbb{R}$. Let X be a set of r elements of $\mathcal{M}$. The set X is a base of $\mathcal{M}$ if and only if the columns of $[I_r|D]$ corresponding to the elements of X are linearly independent. This is equivalent to the fact the determinant of the square submatrix of $[I_r|D]$ formed by these columns is non-zero which must be either +1 or −1 since $[I_r|D]$ is a totally unimodular matrix. However, the determinant of this matrix is non-zero when $[I_r|D]$ is viewed as a matrix over any field $\mathbb{F}$. Similarly, if X is not a base, the determinant of the square submatrix of $[I_r|D]$ formed by the columns corresponding to X is zero and it is zero over any field $\mathbb{F}$. We conclude that $[I_r|D]$ is an $\mathbb{F}$-representation of $\mathcal{M}$ for any field $\mathbb{F}$ and thus *(ii)* holds.

Since *(ii)* implies *(iii)* by the definition of regular matroids, it remains to prove that *(iii)* implies *(i)*.

Suppose that *(iii)* holds and $[I_r|D]$ is an $\mathbb{F}$-representation of $\mathcal{M}$ for a field $\mathbb{F}$ of characteristic different from two. Let us define a bipartite graph G such that the vertices of G correspond to rows and columns of D and a vertex corresponding to a row is adjacent to a vertex corresponding to a column if the corresponding entry of D is non-zero. Observe that by multiplying the rows and columns of $[I_r|D]$ with non-zero elements of $\mathbb{F}$, we can always assume that the entries of D corresponding to a fixed spanning forest (inclusion-wise maximal acyclic subgraph) T of G are all equal to 1. For every edge e_d not contained in T, we will argue that the corresponding entry d in D is equal to ± 1. The argument will proceed by the induction of the length ℓ of the fundamental cycle C_{e_d} of e_d with respect to a spanning forest in G. Recall that the fundamental cycle C_{e_d} of e_d is the unique cycle contained in the graph obtained from a spanning forest by adding the edge e_d.

There are exactly $\ell/2$ rows and columns of D corresponding to the vertices of C_{e_d}. Let D_d be the submatrix corresponding to these rows and columns. In D_d, each row and each column contains at least two non-zero entries, those corresponding to the edges of C_{e_d}.

Assume first that the submatrix D_d contains non-zero entries not corresponding to the edges of C_{e_d}. Since the edge $e_{d'}$ for every such entry d' is a chord of C_{e_d}, it holds that d' is either $+1$ or -1 by the induction. This allows us to modify the spanning forest T to a spanning forest T', which does not contain e_d, by multiplying rows and columns by $+1$ and -1 only in such a way that the fundamental cycle of e_d with respect to T' is shorter. By the induction, the entry d is either $+1$ or -1. Hence, we can assume that D_d has exactly two non-zero entries in each row and in each column.

Evaluating the determinant of D_d, we obtain that $\det(D_d) \in \{d+1, d-1, -d+1, -d-1\}$. Since $\mathcal{M}$ is binary, $[I_r|D^{\#}]$ is a GF(2)-representation for $\mathcal{M}$ and thus the columns of D_d correspond to a circuit of $\mathcal{M}$. Therefore, $\det(D_d) = 0$ which implies that d is either $+1$ or -1.

We now show that the matrix $[I_r|D]$ represents $\mathcal{M}$ over $\mathbb{R}$ and it is totally unimodular. Recall that the matrix $[I_r|D]$ represents $\mathcal{M}$ over $\mathbb{F}$ and the matrix $[I_r|D^{\#}]$ represents $\mathcal{M}$ over GF(2). In order to archive our goal, we have to show that the determinant of every regular square submatrix of $[I_r|D]$ over $\mathbb{F}$ is $+1$ or -1 over $\mathbb{R}$ and the determinant of every singular square submatrix over $\mathbb{F}$ is zero over $\mathbb{R}$.

Let us consider a square submatrix D' of $[I_r|D]$. If D' is 1×1-matrix, its only entry is 0, $+1$ or -1 and the claim follows. If D' has no non-zero entry, its determinant is equal to zero both over $\mathbb{F}$ and $\mathbb{R}$. Otherwise, we can pivot over any non-zero element of D' to obtain a unit column vector. Note that this pivoting results in the same matrix both over $\mathbb{F}$ and $\mathbb{R}$ by Lemma 6.24. Let D'' be the matrix obtained from D' by deleting the row containing the only non-zero entry of the unit column and the unit column. Clearly, $|\det(D')| = |\det(D'')|$. Since $\det(D'')$ is equal to 0, $+1$ or -1 by the induction, the determinant of D' is also equal to 0, $+1$ and -1. Moreover, the induction yields that D'' is singular over $\mathbb{F}$ if and only if it is singular over $\mathbb{R}$ which implies that D' is singular over $\mathbb{F}$ if and only if it is singular over $\mathbb{R}$. We conclude that $[I_r|D]$ is a totally unimodular matrix which represents $\mathcal{M}$ over $\mathbb{R}$. The proof of the theorem is now completed. □

Theorem 6.25 immediately yields the following.

Corollary 6.26. *A matroid $\mathcal{M}$ is regular if and only if it is binary and ternary.*

Theorems 6.13 and 6.18 together with Corollary 6.26 implies that a matroid is regular if and only if it does not contain any of the matroids $U_{2,4}$, $U_{2,5}$, $U_{3,5}$, F_7 and F_7^* as a minor. Since $U_{2,4}$ is a minor of both matroids $U_{2,5}$ and $U_{3,5}$, we can obtain a list of excluded minors for regular matroids.

Theorem 6.27. *A matroid is regular if and only if it has no minor isomorphic to any of the matroids $U_{2,4}$, F_7, and F_7^*.*