Chapter 1

Definitions and basic examples

Matroids were established as a generalization of different concepts in two areas of mathematics: the concept of graphs in the graph theory and the concept of vector spaces in the linear algebra. The matroid theory uses and extends results from both the graph theory and the linear algebra and provides a link between them. The pioneering work in the matroid theory was the paper of Whitney entitled "On abstract properties of linear dependence" [29] which appeared in the 1930's and since then the matroid theory flourished a lot. We do not aim to cover its whole area and refer the reader particularly to the classical monograph by Oxley on this subject [16] for topics not covered here.

In this chapter, we provide main motivations for introducing the notion of matroids, give basic examples and explain notation that we use.

1.1 Independent sets, circuits and bases

A *matroid* $\mathcal{M}$ is an ordered pair $(E,\mathcal{I})$ of a finite set E and a family $\mathcal{I}$ of subsets of E satisfying the following properties:

(I1) $\emptyset \in \mathcal{I}$

(I2) if $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$ and

(I3) if $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The third property (I3) is called *the independence augmentation axiom*. If $\mathcal{M}$ is a matroid $(E,\mathcal{I})$, then E is called the *ground set* of $\mathcal{M}$, the elements of E *elements* of $\mathcal{M}$ and the members of $\mathcal{I}$ *independent sets*. $\mathcal{M}$ is also referred to as a matroid on E and sets contained in $\mathcal{I}$ are also said to be independent in $\mathcal{M}$. Any subset of E not included in $\mathcal{I}$ is called *dependent*. We also write $E(\mathcal{M})$ and $\mathcal{I}(\mathcal{M})$ for the ground set and the family of independent sets of a matroid $\mathcal{M}$ or just E and $\mathcal{I}$ if $\mathcal{M}$ is clear from the context.

Let $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ be two matroids. A map $\varphi : E_1 \to E_2$ is an *isomorphism* of the matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ if φ is a bijection from E_1 to E_2 and $I_1 \in \mathcal{I}_1$ if and only if $\varphi(I_1) \in \mathcal{I}_2$.

An inclusion-wise minimal dependent set in a matroid $\mathcal{M}$, i.e., a set X such that $X \notin \mathcal{I}(\mathcal{M})$ but every proper subset if X is independent in $\mathcal{M}$, is called a *circuit* of $\mathcal{M}$; the family of circuits of $\mathcal{M}$ is denoted by $\mathcal{C}(\mathcal{M})$ or $\mathcal{C}$ if $\mathcal{M}$ is clear from the context. Those elements x of $\mathcal{M}$ such that $\{x\}$ is a circuit are called *loops*. Observe that the family of independent sets uniquely determine the family of circuits of $\mathcal{M}$ and vice versa.

The family of circuits of a matroid has several important properties which we summarize in the next lemma.

Lemma 1.1. *The family $\mathcal{C}$ of circuits of a matroid $\mathcal{M}$ has the following properties:*

(C1) $\emptyset \notin \mathcal{C}$

(C2) if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$, and

(C3) if $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Before proving the lemma, let us point out one particular less common notation which we use: if X is a set and x an element of X, then $X - x$ denotes the set $X \setminus \{x\}$. Similarly, we use $X + y$ for $X \cup \{y\}$.

Proof. The properties (C1) and (C2) immediately follow from the definition of a circuit. To prove (C3), assume that $(C_1 \cup C_2) - e$ does not contain any circuit and thus it is independent. By (C2), there is an element $f \in C_2 - C_1$ and $C_2 - f$ is independent. Let I be a maximal independent subset of $C_1 \cup C_2$ containing $C_2 - f$. Clearly, $f \notin I$. On the other hand, there exists $g \in C_1 \setminus I$ since $I \not\subseteq C_1$. As $f \in C_2 - C_1$, the elements f and g are distinct. Hence

$$|I| \leq |(C_1 \cup C_2) \setminus \{f, g\}| = |C_1 \cup C_2| - 2 < |(C_1 \cup C_2) - e| \, .$$

We now apply (I3) to the independent sets I and $(C_1 \cup C_2) - e$; the resulting independent set contradicts the maximality of I. $\square$

The property (C3) is called *the circuit elimination axiom*. We show that the properties (C1)–(C3) fully characterize those families of sets that can be families of circuits of a matroid.

Theorem 1.2. *Let E be a set and $\mathcal{C}$ a family of subsets of E satisfying the properties (C1), (C2) and (C3). Let $\mathcal{I}$ be a family of all subsets of E that contain no member of $\mathcal{C}$. Then, the pair $(E, \mathcal{I})$ is a matroid and $\mathcal{C}$ is its family of circuits.*

Proof. First, we prove that the family $\mathcal{I}$ satisfies (I1)–(I3). By (C1), $\emptyset \notin \mathcal{C}$ and thus $\emptyset \in \mathcal{I}$, i.e., (I1) holds. If I contains no member of $\mathcal{C}$, then any subset I' of I also contains no member of $\mathcal{C}$, i.e., (I2) holds.

To prove (I3), let I_1 and I_2 be members of $\mathcal{I}$ and $|I_1| < |I_2|$. Assume that (I3) fails for I_1 and I_2. Let I_3 be a subset of $I_1 \cup I_2$ contained in $\mathcal{I}$ with more elements than I_1 and $|I_1 \setminus I_3|$ minimal. As (I3) fails, $I_1 \setminus I_3$ is non-empty, in particular, there is an element $e \in I_1 \setminus I_3$. Fix such e.

Let $T_f = (I_3 - f) + e$ for an element $f \in I_3 \setminus I_1$. Since $T_f \subseteq I_1 \cup I_2$ and $|I_1 \setminus T_f| < |I_1 \setminus I_3|$, the choice of I_3 implies that $T_f \notin \mathcal{I}$. By the definition of $\mathcal{I}$, there exists a circuit $C_f \subseteq T_f$. Clearly, $e \in C_f$ but $f \notin C_f$.

If $C_f \cap (I_3 \setminus I_1) = \emptyset$ for some $f \in I_3 \setminus I_1$, then $C_f \subseteq ((I_1 \cap I_3) + e) - f \subseteq I_1$ which is impossible since I_1 is independent. Therefore, there is an element $g \in C_f \cap (I_3 \setminus I_1)$. Since $e \in C_f \cap C_g$, there is a circuit $C \subseteq (C_f \cup C_g) - e \subseteq I_3$ by (C3). which contradicts the choice of I_3. We conclude that $(E, \mathcal{I})$ is a matroid.

Finally, the definition of $\mathcal{I}$ and the property (C2) implies that the family $\mathcal{C}$ is the family of circuits of the matroid $(E, \mathcal{I})$. □

Lemma 1.1 and Theorem 1.2 yield the following.

Corollary 1.3. *A family $\mathcal{C}$ of subsets of a set E is a family of circuits of a matroid if and only if $\mathcal{C}$ satisfies the properties (C1), (C2) and (C3).*

We now introduce a notion dual to circuits. Inclusion-wise maximal independent sets are called *bases*; the name used is not a coincidence since bases of a matroid have a lot of common properties with bases of a vector space. Let us start with a simple fact that all bases of a matroid have the same size.

Proposition 1.4. *If B_1 and B_2 are two bases of a matroid $\mathcal{M}$, then $|B_1| = |B_2|$.*

Proof. Suppose that $|B_1| < |B_2|$. Then by (I3), there exists an element $e \in B_2 - B_1$ such that $B_1 + e$ is independent which contradicts the maximality of B_1. □

As in the case of circuits, the family of bases of a matroid $\mathcal{M}$ is denoted by $\mathcal{B}(\mathcal{M})$ or just $\mathcal{B}$ if $\mathcal{M}$ is clear from the context. We now give several important properties of bases of a matroid.

Lemma 1.5. *Let $\mathcal{M}$ be a matroid on E and let $\mathcal{B}$ be the family of its bases. The family $\mathcal{B}$ has the following properties:*

(B1) *$\mathcal{B}$ is non-empty, and*

(B2) *if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, then there is an element $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$.*

Proof. Since $\mathcal{I}(\mathcal{M})$ is non-empty by (I1), the family $\mathcal{B}$ must also be non-empty, i.e., the property (B1) holds.

In order to prove (B2), consider two bases B_1 and B_2 and an element $e \in B_1 \setminus B_2$. Both $B_1 - e$ and B_2 are independent sets and $|B_1 - e| < |B_2|$ by Proposition 1.4. (I3) implies that there exists $f \in B_2 \setminus (B_1 - e)$ such that $(B_1 - e) + f$ is independent. As $f \in B_2 \setminus B_1$ and $|(B_1 - e) + f| = |B_1|$, $(B_1 - e) + f$ is the sought base of $\mathcal{M}$ by Proposition 1.4. $\square$

The property (B2) from Lemma 1.5 is usually called *the base exchange axiom*. On the other hand, the properties (B1) and (B2) fully characterize families of sets that can be families of bases of a matroids. Before we prove this fact, we have to establish the following lemma.

Lemma 1.6. *Let E be a set and $\mathcal{B}$ be a family of subsets of E. If $\mathcal{B}$ satisfies the properties (B1) and (B2), then all the members of $\mathcal{B}$ have the same size.*

Proof. Suppose that there are two members of $\mathcal{B}$ with different sizes. Choose such two members B_1 and B_2, $|B_1| > |B_2|$ with minimal $|B_1 \setminus B_2|$. Fox every $e \in B_1 \setminus B_2$, (B2) yields the existence of an element $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$. Clearly, $|(B_1 - e) + f| = |B_1| > |B_2|$ and $|((B_1 - e) \cup f) - B_2| < |B_1 \setminus B_2|$ which is impossible by the choice of B_1 and B_2. $\square$

We are now ready to show that the properties (B1) and (B2) fully characterize families of bases of matroids.

Theorem 1.7. *Let E be a set and $\mathcal{B}$ be a family of subsets of E satisfying the properties (B1) and (B2). Let $\mathcal{I}$ be the family of subsets of E that are contained in some member of $\mathcal{B}$. Then, the pair $(E, \mathcal{I})$ is a matroid and $\mathcal{B}$ are its bases.*

Proof. The definition of $\mathcal{I}$ implies that the property (I2) holds. Since $\mathcal{B} \neq \emptyset$ by (B1), it also holds $\emptyset \in \mathcal{I}$ and the property (I1) is satisfied.

Suppose that (I3) fails, i.e., there are members I_1 and I_2 of $\mathcal{I}$ with $|I_1| < |I_2|$ such that for each $e \in I_2 \setminus I_1$, the set $I_1 + e$ is dependent.

By the definition of $\mathcal{I}$, there are bases B_1 and B_2 such that $I_1 \subseteq B_1$ and $I_2 \subseteq B_2$. Let B_2 be chosen that $|B_2 \setminus (I_2 \cup B_1)|$ is minimal possible. If there is an element $e \in B_2 \setminus (I_2 \cup B_1)$, then there would be an element $f \in B_1 \setminus B_2$ by (B2) such that $(B_2 - e) + f \in \mathcal{B}$. Since

$$|((B_2 - e) + f) \setminus (I_2 \cup B_1)| < |B_2 \setminus (I_2 \cup B_1)|,$$

this would contradict the choice of B_2. We conclude that $B_2 \setminus (I_2 \cup B_1) = \emptyset$, i.e., $B_2 \setminus B_1 = I_2 \setminus B_1$. The choice of I_1 and I_2 implies that $I_2 \setminus B_1 = I_2 \setminus I_1$ (otherwise, an element of $B_1 \cap I_2$ could be added to I_1 to form an independent set). The two equalities now combine to the following:

$$B_2 - B_1 = I_2 - I_1 . \tag{1.1}$$

We next show that $B_1 \setminus (I_1 \cup B_2)$ is empty. If not, there exists an element $e \in B_1 \setminus (I_1 \cup B_2)$. The property (B2) yields the existence of an element $f \in B_2 \setminus B_1$ such that $(B_1 - e) + f \in \mathcal{B}$. Clearly, $I_1 + f \subseteq (B_1 - e) \cup f$ is an independent set which is impossible since $f \in I_2 \setminus I_1$ by (1.1). Hence, $B_1 \setminus (I_1 \cup B_2) = \emptyset$ which implies that

$$B_1 \setminus B_2 = I_1 \setminus B_2 \subseteq I_1 \setminus I_2 \,. \tag{1.2}$$

By Lemma 1.6, both B_1 and B_2 have the same size and thus $|B_1 \setminus B_2| = |B_2 \setminus B_1|$. Consequently, (1.1) and (1.2) imply that $|I_1 \setminus I_2| \geq |I_2 \setminus I_1|$ which yields $|I_1| \geq |I_2|$. This contradicts the choice of I_1 and I_2. $\square$

Lemma 1.5 and Theorem 1.7 together yield the following characterization of families of subsets that can be a family of bases of a matroid.

Corollary 1.8. *Let E be a set and $\mathcal{B}$ a family of subsets of E. $\mathcal{B}$ is the family of bases of a matroid if and only if $\mathcal{B}$ satisfies conditions (B1) and (B2).*

1.2 The rank function

We define *the rank function* of a matroid $\mathcal{M}$ as a function $r_{\mathcal{M}}$ from $2^{E(\mathcal{M})}$ to non-negative integers where $r_{\mathcal{M}}(X)$ is defined to be the size of largest independent subset of X. By the property (I3), all inclusion-wise maximal independent subsets of X have the same size. Instead of $r(E(\mathcal{M}))$, we just write $r(\mathcal{M})$ and we also use $r(X)$ if $\mathcal{M}$ is clear from the context.

Lemma 1.9. *Let $\mathcal{M}$ be a matroid on a set E. The rank function r of $\mathcal{M}$ has the following properties:*

(R1) $0 \leq r(X) \leq |X|$ *for every* $X \subseteq E$,

(R2) $r(X) \leq r(Y)$ *for every* $X \subseteq Y \subseteq E$, *and*

(R3) $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ *for every* $X, Y \subseteq E$.

Proof. The properties (R1) and (R2) directly follow from the definition of the rank function. To prove (R3), fix subsets X and Y of E. Let $B_\cap$ be an inclusion-wise maximal independent subset of $X \cap Y$ and $B_\cup$ an inclusion-wise maximal independent subset of $X \cup Y$ such that $B_\cap \subseteq B_\cup$. Clearly, $B_\cup \cap X$ and $B_\cup \cap Y$ are independent subsets of X and Y, respectively. The properties (R1) and (R2) imply that $|B_\cup \cap X| \leq r(X)$ and $|B_\cup \cap Y| \leq r(Y)$. We now obtain that

$$\begin{aligned} r(X) + r(Y) &\geq |B_\cup \cap X| + |B_\cup \cap Y| \\ &= |(B_\cup \cap X) \cup (B_\cup \cap Y)| + |(B_\cup \cap X) \cap (B_\cup \cap Y)| \\ &= |B_\cup \cap (X \cup Y)| + |B_\cup \cap (X \cap Y)| \\ &= |B_\cup| + |B_\cap| = r(X \cup Y) + r(X \cap Y) \,. \end{aligned}$$

$\square$

The property (R3) from the lemma is an important property of the rank function and is known as the *submodularity.*

As with the previously studied properties of circuits and bases of matroids, we shall see that the properties (R1), (R2) and (R3) characterize functions that are rank functions of matroids. Before we prove this formally, we need to establish an auxiliary lemma.

Lemma 1.10. *Let E be a finite set and r a function mapping 2^E to non-negative integers that has the properties (R1), (R2) and (R3). If X and Y are subsets of E such that $r(X+y) = r(X)$ for every $y \in Y \setminus X$, then $r(X \cup Y) = r(X)$.*

Proof. The proof proceeds by the induction on the size of $Y \setminus X$ which is further denoted by k. Let $Y \setminus X = \{y_1, y_2, \ldots, y_k\}$. If $k = 1$, the lemma clearly holds.

Assume that $k \geq 2$. Using the induction assumption, we infer from (R2) and (R3) that

$$\begin{aligned} r(X) + r(X) &= r(X \cup \{y_1, y_2, \ldots, y_{k-1}\}) + r(X + y_k) \\ &\geq r(X \cup \{y_1, y_2, \ldots, y_k\}) + r(X) \geq r(X) + r(X)\,. \end{aligned}$$

Since the first and last terms of the inequality are the same, the inequalities are in fact equalities and $r(X \cup \{y_1, y_2, \ldots, y_k\}) = r(X)$. □

We now show that the properties (R1), (R2) and (R3) guarantees that the considered function is the rank function of a matroid.

Theorem 1.11. *Let E be a finite set and r a function mapping 2^E to non-negative integers that has the properties (R1), (R2) and (R3). Let $\mathcal{I}$ be a family of all subsets X of E with $r(X) = |X|$. Then, the pair $(E, \mathcal{I})$ is a matroid and r is its rank function.*

Proof. By (R1), $r(\emptyset) = 0 = |\emptyset|$ which implies that $\emptyset \in \mathcal{I}$, i.e., (I1) holds. Suppose that $I \in \mathcal{I}$, i.e., $r(I) = |I|$, and consider a subset $I' \subseteq I$. The property (R3) implies that

$$\begin{aligned} r(I' \cup (I \setminus I')) + r(I' \cap (I \setminus I')) &\leq r(I') + r(I \setminus I') \\ r(I) + r(\emptyset) &\leq r(I') + r(I \setminus I')\,. \end{aligned}$$

Since $r(I') \leq |I'|$ and $r(I \setminus I') \leq |I \setminus I'|$ by (R1), we obtain that

$$|I| \leq r(I') + r(I \setminus I') \leq |I'| + |I \setminus I'| = |I|\,.$$

Since the first and the last terms are equal, the inequalities must be equalities. We conclude that $r(I') = |I'|$, i.e., $I' \in \mathcal{I}$, which proves (I2).

It remains to verify that $\mathcal{I}$ also satisfies (I3). Consider two sets I_1 and I_2 contained in $\mathcal{I}$ such that $|I_1| < |I_2|$. Assume that $I_1 + e$ does not belong to $\mathcal{I}$ for every $e \in I_2 \setminus I_1$. Hence, for every $e \in I_2 \setminus I_1$, it holds that

$$|I_1| + 1 > r(I_1 + x) \geq r(I_1) = |I_1|$$

which implies $r(I_1 + e) = |I_1|$. By Lemma 1.10, it holds that $r(I_1) = r(I_1 \cup I_2)$. Since $r(I_1 \cup I_2) \geq r(I_2)$ by (R2), we obtain that

$$r(I_2) \leq r(I_1) = |I_1| < |I_2|$$

which implies that I_2 should not be contained in the family $\mathcal{I}$.

To complete the proof of the theorem, we also need to show that r is the rank function $r_{\mathcal{M}}$ of the matroid $\mathcal{M} = (E, \mathcal{I})$. Consider $X \subseteq E$. If X is independent in $\mathcal{M}$, then $r(X) = |X| = r_{\mathcal{M}}(X)$. Otherwise, let I be a maximal subset of X independent in $\mathcal{M}$. Since $I + x \notin \mathcal{I}$ for every $x \in X \setminus I$, Lemma 1.10 yields that $r(X) = r(I) = r_{\mathcal{M}}(I)$. Hence, r and $r_{\mathcal{M}}$ coincide, i.e., r is the rank function of $\mathcal{M}$. □

Lemma 1.9 and Theorem 1.11 imply that the properties (R1), (R2) and (R3) fully characterize those functions that are the rank function of a matroid.

Corollary 1.12. *Let E be a set. A non-negative integer function r on 2^E is the rank function of a matroid on E if and only if r satisfies properties (R1), (R2) and (R3).*

It is not hard to give characterizations of independent sets, bases, and circuits of matroids by their rank functions. We state this in the following proposition.

Proposition 1.13. *Let $\mathcal{M}$ be a matroid with the rank function r. A set $X \subseteq E(\mathcal{M})$*

(i) is independent if and only if $|X| = r(X)$,

(ii) is a base if and only if $|X| = r(X) = r(\mathcal{M})$, and

(iii) is a circuit if and only if X is non-empty and $r(X - x) = |X| - 1 = r(X)$ for every $x \in X$.

The set $H \subset E$ is called *a hyperplane* of the matroid $\mathcal{M}$ if it is an inclusion-wise maximal set such that $r(H) < r(\mathcal{M})$. It is easy to see that the properties of the rank function imply that the rank of any hyperplane is $r(\mathcal{M}) - 1$.

1.3 Vector and affine matroids

Let us start with examples of matroids stemming from linear algebra which will justify the use of some terms from linear algebra in the matroid theory. Consider a multiset E of vector of a vector space. A subset of E is called independent if it is linearly independent. It is clear that conditions (I1)–(I3) are satisfied. Such a matroid is called *the vector matroid*. This example is captured in the following proposition.

Proposition 1.14. *Let E be the set of columns of an $m \times n$ matrix A over a field $\mathbb{F}$, and let $\mathcal{I}$ be the family of subsets X of E that are linearly independent in the m-dimensional vector space over $\mathbb{F}$. The pair $(E, \mathcal{I})$ is a matroid.*

The definition of *affine matroids* is similar to vector matroids but we use affine independence of vectors instead of the linear independence. Recall that a multiset $\{v_1, v_2, \ldots, v_k\}$ of d-dimensional vectors over a field $\mathbb{F}$ is *affinely dependent* if $k \geq 1$ and there are elements $a_1, a_2, \ldots, a_k$ of $\mathbb{F}$ such that $\sum_{i=1}^{k} a_i v_i = 0$, $\sum_{i=1}^{k} a_i = 0$ and all the elements $a_1, \ldots, a_k$ are not zero. The multiset is *affinely independent* if it is not affinely dependent.

We now give an example of matroids based on the notion of affine independence.

Proposition 1.15. *Let E be the set of columns of an $m \times n$ matrix A over a field $\mathbb{F}$, and let $\mathcal{I}$ be the family of subsets X of E that are affinely independent in the m-dimensional vector space over $\mathbb{F}$. The pair $(E, \mathcal{I})$ is a matroid.*

Matroids $(E, \mathcal{I})$ of the type given in the previous proposition are called *affine matroids* on E. To familiarize with this notion, we give two particular examples of such matroids.

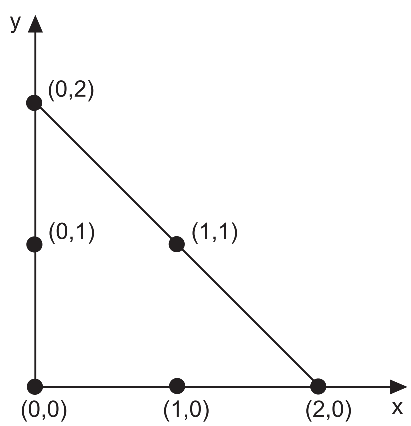

Figure 1.1: An affine matroid of rank 3.

We start with affine matroids of rank at most 3 over $\mathbb{R}$. Let E be a multiset of vectors of $\mathbb{R}^2$. Let us view these vectors as points in the plane. A subset $A \subseteq E$ is affinely dependent if it contains two identical points, three collinear points or four points. An example of such a matroid can be found in Figure 1.1. The ground set of this matroid is $E = \{(0,0), (1,0), (2,0), (0,1), (0,2), (1,1)\}$ and a subset $A \subseteq E$ is dependent if $|A| \geq 4$ or A is one of the sets $\{(0,0), (1,0), (2,0)\}$, $\{(0,0), (0,1), (0,2)\}$ and $\{(0,2), (1,1), (2,0)\}$.

It is also easy to geometrically describe affine matroids of rank four. Again, we view the vectors of $\mathbb{R}^3$ as points in the space. A subset A is affinely dependent if it contains two identical points, three collinear points, four coplanar points, or five or more points.

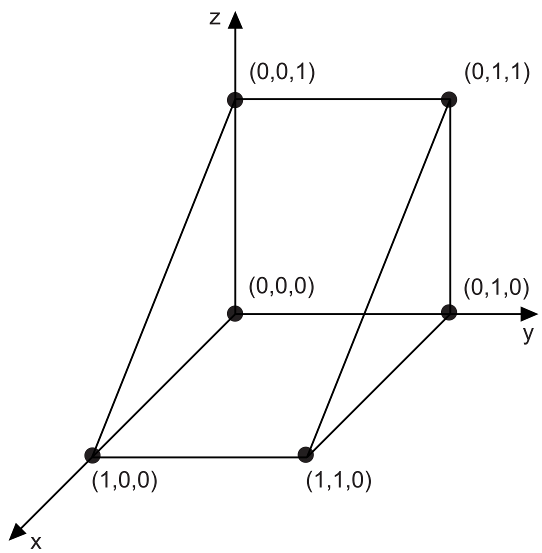

Figure 1.2: An affine matroid of rank 4.

Matroids of rank at most four can be visualized by *diagrams* drawn according to the rules we now describe. All loops are represented outside the figure. For each element that is not a loop, there is a point in the diagram labeled by this element. Parallel elements are represented by multiple labels of a single point. If three elements form a circuit, the corresponding points are collinear in the diagram. Likewise, if four points form a circuit, the corresponding points are coplanar. In Figure 1.2, there is a diagram of an affine matroid of rank 4 with three dependent sets of size four, $\{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}$, $\{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\}$ and $\{(1,0,0),(1,1,0),(0,0,1),(0,1,1)\}$. Any subset of the ground set of size at least five is also dependent.

An important example of a matroid is *the Fano matroid* . It is closely related to affine matroids but its construction is based on *a projective plane* of rank 2. Let E be the set of points of the Fano plane, i.e., the projective plane over GF(2). A subset X of E is independent if the points of X are independent in the Fano plane, i.e., they are not collinear. The Fano matroid is denoted by F_7. Its diagram can be found in Figure 1.3; note that the collinearity of points $\{2,4,6\}$ cannot be depicted by a straight line. Hence, diagrams with non-straight lines and twisted planes are used to represent matroids which are not affine matroids over $\mathbb{R}$.

Not all diagrams with points, lines and planes are geometric visualizations of matroids. As an example, consider the diagram in Figure 1.4 which does not represent a matroid. To see this, consider sets $X = \{1,2,3,6,7\}$ and $Y =$

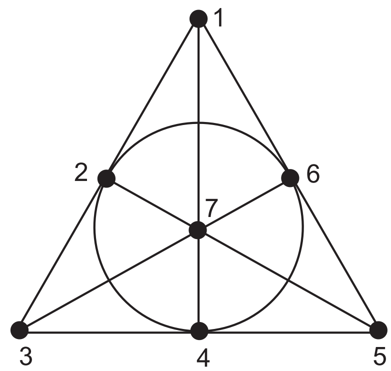

Figure 1.3: The Fano matroid.

$\{1,4,5,6,7\}$ and observe that $r(X) = r(Y) = 3$, $r(X \cup Y) = 4$ and $r(X \cap Y) = 3$ violating the submodularity of the rank function. Note that if points $\{1,6,7\}$ are changed to be collinear in Figure 1.5, we obtain a diagram of a matroid of rank 4.

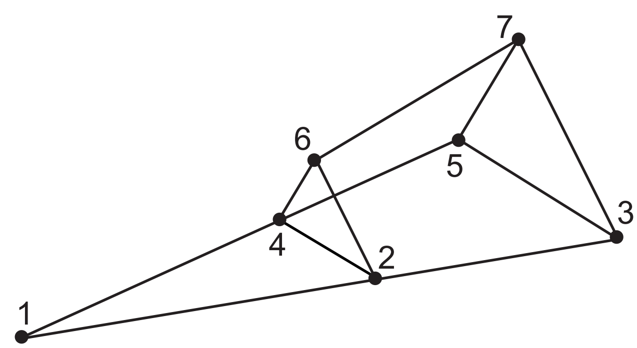

Figure 1.4: A diagram not representing a matroid.

As not every diagram represents a matroid, it is desirable to have a characterization of diagrams representing matroids. Clearly, every line contains at least two points and any two distinct points lie on a line. Observe that two-element lines can be omitted in the diagram as they correspond to parallel elements. Moreover, any plane contains at least three non-collinear points and any three distinct non-collinear points lie in a common plane (three-element planes can be omitted). For diagrams of matroids of rank at most three, there is only one additional rule: any two distinct lines meet in at most one point.

For diagrams representing matroids of rank four, there are three rules instead of the last one given in the previous paragraph.

i Any line not lying in the plane intersects it in at most one point.
ii Any two intersecting lines lie in the common plane.

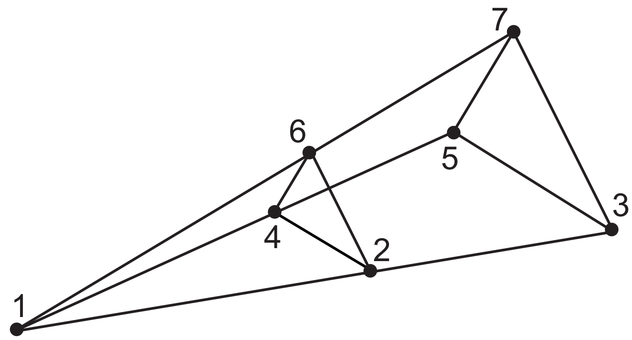

Figure 1.5: A modification of the diagram from Figure 1.4 which represents a matroid.

iii Any two planes meeting in more than two points do so in a line.

1.4 Representability

Vector matroids described in Proposition 1.14 provide a link between linear algebra and the matroid theory. A matroid given in a different way can be isomorphic to a vector matroid. This leads us to the notion of a representation.

Let $\mathbb{F}$ be any field and let A be an $m \times n$ matrix over $\mathbb{F}$. If the vector matroid given by A is isomorphic to a matroid $\mathcal{M}$, then A is a *representation* of $\mathcal{M}$ over $\mathbb{F}$. A matroid with a fixed representation over $\mathbb{F}$ is said to be *represented*. In general, the representation of $\mathcal{M}$ over $\mathbb{F}$ is not unique. In particular, any of the following operations yields to another representation of $\mathcal{M}$ over $\mathbb{F}$:

i Interchange two rows of A.

ii Multiply a row of A by a non-zero element of $\mathbb{F}$.

iii Replace a row of A by the sum of this row and another one.

iv Delete a zero row of A (unless it is the only row of A).

v Interchange two columns of A (note that this affects the isomorphism between the vector matroid and $\mathcal{M}$).

vi Multiply a column of A by a non-zero element of $\mathbb{F}$.

vii Replace each entry of A by its image under a fixed automorphism of $\mathbb{F}$.

We will later see that not every matroid can be represented. The matroids that have a representation over some field are called *representable matroids*.

Let A be a representation of a matroid $\mathcal{M}$ such that the number of rows of A is equal to the rank r of $\mathcal{M}$. It is well known (and easy to check) that a non-zero matrix A can be transformed by a sequence of operations i–viii into a matrix of the form $[I_r|D]$ where I_r is the $r \times r$ identity matrix and D is an $r \times (n - r)$

$$\begin{array}{cc} e_1 \dots e_r & e_{r+1} \dots e_n \end{array}$$
$$\left(\begin{array}{c|c} \quad I_r \quad & \quad D \quad \end{array} \right)$$

Figure 1.6: A standard representation of a matroid.

matrix over $\mathbb{F}$ (see Figure 1.6). Such a matrix is called *a standard representation* of $\mathcal{M}$ over $\mathbb{F}$.

1.5 Graphic matroids

In this section, we introduce matroids stemming from the graph theory. Let $G = (V, E)$ be a graph (with loops and parallel edges allowed). *The cycle matroid*, or *the graphic matroid*, $\mathcal{M}(G)$ of the graph G is the matroid on the ground set E with a subset $X \subseteq E$ independent if it is acyclic in G. Circuits of $\mathcal{M}(G)$ can be described using graph-theoretic terms: the family of circuits of $\mathcal{M}(G)$ is precisely the set of edges of cycles of G. A matroid $\mathcal{M}$ is called *graphic* if it is isomorphic to $\mathcal{M}(G)$ for some graph G.

We now observe that every graphic matroid is representable. Consider a graph G with incidence matrix I_G, i.e., I_G is an $n \times m$ matrix where n is the number of vertices of G and m is the number of edges such that the rows of I_G one-to-one correspond to vertices of G, the columns of I_G one-to-one correspond to edges of G and the entries in each column equal to 1 correspond to the end-vertices of the corresponding edge; other entries are equal to zero. The columns corresponding to loops of G have only zero entries. The oriented incidence matrix O_G is obtained from the usual incidence matrix I_G by replacing one '1' in each column by -1, i.e., the inverse element of 1 under the additive operation in a considered field. Over fields of characteristic two, the matrix O_G contains only entries equal to 0 and 1 as $1 = -1$ for such fields.

The matrix O_G is a representation of $\mathcal{M}(G)$ over any field as stated in the next proposition.

Proposition 1.16. *Every graphic matroid is representable over any field; in particular, the oriented incidence matrix O_G of a graph G is a representation of the matroid $\mathcal{M}(G)$.*

We finish this section with a proposition on the rank function of graphic matroids.

Proposition 1.17. *Let $G = (V, E)$ and let $\mathcal{M} = \mathcal{M}(G)$. The rank function of $\mathcal{M}$ satisfies,*

$$r_{\mathcal{M}}(X) = |V(G[X])| - c(G[X])$$

for any $X \subseteq E$ where $G[X]$ is a spanning subgraph (V, X) of G containing exactly the edges of X and $c(G[X])$ denotes the number of components of $G[X]$.

Proof. The rank function $r_{\mathcal{M}}(X)$ is defined as the number of edges of a maximal acyclic subgraph T_X of $G[X]$. For every component of $G[X]$, T_X contains one of its spanning trees, i.e., there are exactly $n-1$ edges of that component where n is the number of its vertices. Summing over all the components of $G[X]$, we obtain the formula from the statement. $\square$

1.6 Other examples

In this section, we give several other constructions of matroids. We let the proofs as straightforward exercises for readers. Let us start with the most simple one. A matroid is *the uniform matroid* $U_{m,n}$ which is described in the next proposition.

Proposition 1.18. *Let $m \leq n$ be non-negative integers, E any n-element set and $\mathcal{I}$ the family of all subsets X of E with $|X| \leq m$. The pair $U_{m,n} = (E, \mathcal{I})$ forms a matroid on E.*

A more complex example of matroids are *transversal matroids* described in the next proposition.

Proposition 1.19. *Let $A_1, \ldots, A_k$ be a family of disjoint sets and let $E = A_1 \cup \cdots \cup A_k$. Let $\mathcal{I}$ be a family of subsets X of E such that X contains at most one element from each of the sets $A_1, \ldots, A_k$. The pair $(E, \mathcal{I})$ is a matroid on E.*

We finish this chapter with a construction of matroids called the *matroid union*, also known as the *direct sum* of matroids which allows us to form larger matroids using matroids with disjoint ground sets.

Proposition 1.20. *Let $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ be two matroids with disjoint ground sets. Let $\mathcal{I}$ be a family of subsets X of $E_1 \cup E_2$ such that $X \cap E_1$ is independent in $\mathcal{M}_1$ and $X \cap E_2$ in $\mathcal{M}_2$. The pair $(E_1 \cup E_2, \mathcal{I})$ is a matroid on $E_1 \cup E_2$.*

Let us remark that the matroid $(E_1 \cup E_2, \mathcal{I})$ from Proposition 1.20 is usually denoted by $\mathcal{M}_1 \oplus \mathcal{M}_2$ and is called the union of $\mathcal{M}_1$ and $\mathcal{M}_2$. We finish this section with several basic properties of matroid unions. All of them directly follow from Proposition 1.20.

Proposition 1.21. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matroids with disjoint ground sets. The matroid union $\mathcal{M}_1 \oplus \mathcal{M}_2$ satisfies the following:*

(i) $\mathcal{C}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \mathcal{C}(\mathcal{M}_1) \cup \mathcal{C}(\mathcal{M}_2)$,

(ii) $\mathcal{B}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \{B_1 \cup B_2 \mid B_1 \in \mathcal{B}(\mathcal{M}_1), B_2 \in \mathcal{B}(\mathcal{M}_2)$, *and*

(iii) $r_{\mathcal{M}_1 \oplus \mathcal{M}_2}(X) = r_{\mathcal{M}_1}(X \cap E(\mathcal{M}_1)) + r_{\mathcal{M}_2}(X \cap E(\mathcal{M}_2))$ *for every subset X of the ground set of $\mathcal{M}_1 \oplus \mathcal{M}_2$.*