The Szemerédi–Trotter theorem using polynomials

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We apply the Guth–Katz method of "polynomial partitions" in yet another simple proof of the Szemerédi–Trotter theorem on point-line incidences.

1 Preliminaries on polynomials

We will consider mostly bivariate polynomials $f = f(x,y) = \sum_{i,j} a_{ij}x^iy^j \in \mathbb{R}[x,y]$. The **degree** of f is $\deg(f) = \max\{i+j : a_{ij} \neq 0\}$. Let $Z(f) = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\}$ be the **zero set** of f.

1.1 Lemma. *If ℓ is a line in $\mathbb{R}^2$ and $f \in \mathbb{R}[x,y]$ is of degree at most d, then either $\ell \subseteq Z(f)$, or $|\ell \cap Z(f)| \leq d$.*

Proof. Writing ℓ in parametric form $\{(u_1t+v_1, u_2t+v_2) : t \in \mathbb{R}\}$, we get that the points of $\ell \cap Z(f)$ are roots of the univariate polynomial $g(t) := f(u_1t+v_1, u_2t+v_2)$, which is of degree at most d. Thus, either g is identically 0, or it has at most d roots. $\square$

1.2 Lemma. *If $f \in \mathbb{R}[x,y]$ is nonzero and of degree at most d, then $Z(f)$ contains at most d distinct lines.*

Proof. We need to know that a nonzero bivariate polynomial (i.e., with at least one nonzero coefficient) does not vanish on all of $\mathbb{R}^2$. There are several ways of proving this—we leave it as a challenge for the reader to find one.

Now we fix a point $p \in \mathbb{R}^2$ not belonging to $Z(f)$. Let us suppose $Z(f)$ contains lines $\ell_1,\ldots,\ell_k$. We choose another line ℓ passing through p that is not parallel to any ℓ_i and not passing through any of the intersections $\ell_i \cap \ell_j$. (Such an ℓ exists since only finitely many directions need to be avoided.) Then ℓ is not contained in $Z(f)$ and it has k intersections with $\bigcup_{i=1}^k \ell_i$. Lemma 1.1 yields $k \leq d$. $\square$

2 The polynomial ham-sandwich theorem

We assume the **ham sandwich theorem** in the following discrete version: *Every d finite sets $A_1,\ldots,A_k \subset \mathbb{R}^k$ can be simultaneously bisected by a hyperplane.* Here a hyperplane h

bisects a finite set A if neither of the two open halfspaces bounded by A contains more than $\lfloor |A|/2 \rfloor$ points of A.

From this, it is easy to derive the **polynomial ham-sandwich theorem** (which we state for bivariate polynomials).

2.1 Theorem. *Let $A_1, \dots, A_t \subseteq \mathbb{R}^2$ be finite sets, and let d be an integer with $\binom{d+2}{2} - 1 \geq t$. Then there exists a nonzero polynomial $f \in \mathbb{R}[x, y]$ of degree at most d that simultaneously bisects all the A_i, where "f bisecting A_i" means that $f > 0$ in at most $\lfloor |A_i|/2 \rfloor$ points of A_i and $f < 0$ in at most $\lfloor |A_i|/2 \rfloor$ points of A_i.*

Proof. We note that $\binom{d+2}{2}$ is the number of monomials in a bivariate polynomial of degree d, or in other words, the number of pairs (i, j) of nonnegative integers with $i + j \leq d$. We set $k := \binom{d+2}{2} - 1$, and we let $\Phi \colon \mathbb{R}^2 \to \mathbb{R}^k$ be the **Veronese map** given by

$$\Phi(x, y) := \left(x^i y^j\right)_{(i,j):1 \leq i+j \leq d} \in \mathbb{R}^k.$$

(We think of the coordinates in $\mathbb{R}^k$ as indexed by pairs (i, j) with $1 \leq i + j \leq d$.)

Assuming, as we may, that $t = k$, we set $A'_i := \Phi(A_i)$, $i = 1, 2, \dots, k$, and we let h be a hyperplane simultaneously bisecting $A'_1, \dots, A'_k$. Then h can has an equation of the form $a_{00} + \sum_{i,j} a_{ij} z_{ij} = 0$, where $(z_{ij})_{(i,j):1 \leq i+j \leq d}$ are the coordinates in $\mathbb{R}^k$. It is easy to check that $f(x, y) := \sum_{i,j} a_{ij} x^i y^j$ is the desired polynomial. $\square$

3 Proof of the Szemerédi–Trotter theorem

For a finite set $P \subset \mathbb{R}^2$ and a finite set L of lines in $\mathbb{R}^2$, let $I(P, L)$ denote the number of **incidences** of P and L, i.e., of pairs (p, ℓ) with $p \in P$, $\ell \in L$, and $p \in \ell$.

3.1 Theorem (Szemerédi–Trotter). *We have $I(P, L) = O(m^{2/3} n^{2/3} + m + n)$ for every m-point P and every set L of n lines.*

Let us say that sets $P, Q \subset \mathbb{R}^2$ are **strictly separated** by a polynomial $f \in \mathbb{R}[x, y]$ if $P \cap Z(f) = Q \cap Z(f) = \emptyset$, and every segment pq, $p \in P$, $q \in Q$, intersects $Z(f)$.

Let $P \subset \mathbb{R}^2$ be an m-point set, and let $s > 1$ be a parameter. We say that $f \in \mathbb{R}[x, y]$ is an ***s*-partitioning polynomial** for P if the set $P \setminus Z(f)$ can be partitioned into disjoint subsets $P_1, \dots, P_t$ so that $t = O(s)$, $|P_i| \leq m/s$ for all i, and for every $i \neq j$, P_i and P_j are strictly separated by f.

3.2 Lemma (Polynomial partitioning lemma). *For every $s > 1$, every finite $P \subset \mathbb{R}^2$ admits an s-partitioning polynomial f of degree $O(\sqrt{s}\,)$.*

Proof. We inductively construct collections $\mathcal{P}_0, \mathcal{P}_1, \dots$, each consisting of disjoint subsets of P. We start with $\mathcal{P}_0 := \{P\}$. Having constructed $\mathcal{P}_j$ with at most 2^j sets, we use the polynomial ham-sandwich theorem to construct a polynomial f_j that bisects each of the sets of $\mathcal{P}_j$. Then for every class $Q \in \mathcal{P}_j$, we let Q' consist of the points of Q on which $f_j > 0$, Q'' consists of those points of Q where $f_j < 0$, and $\mathcal{P}_{j+1} := \bigcup_{Q \in \mathcal{P}_j} \{Q', Q''\}$ (empty sets ignored).

The sets in $\mathcal{P}_j$ have size at most $|P|/2^j$. We let $k := \lceil \log_2 s \rceil$; then the sets in $\mathcal{P}_k$ have size at most $|P|/s$ and they form the desired $P_1, \ldots, P_t$, where $t \le 2^k \le 2s$. We also set $f := f_1 f_2 \cdots f_k$.

Then f is an s-partitioning polynomial for P by the construction, and it remains to bound $\deg(f)$. By the polynomial ham sandwich theorem, for bisecting the at most 2^j sets in $\mathcal{P}_j$, a polynomial f_j of degree $O(\sqrt{2^j}\,)$ suffices. Thus, $\deg(f) = O\Big(\sum_{j=1}^{k} O(2^{j/2})\Big) = O(\sqrt{s}\,)$. □

Proof of the Szemerédi–Trotter theorem. For simplicity, we prove the theorem for $m = n$. We set $s := n^{2/3}$, and we let f be an s-partitioning polynomial for P. By the polynomial partitioning lemma, we may assume $r := \deg(f) = O(\sqrt{s}\,)$.

Let $P_1, \ldots, P_t$ be the sets as in the definition of an s-partitioning polynomial for P, and let $R := P \cap Z(f)$. Further let $L_0 \subset L$ consist of the lines of L contained in $Z(f)$; we have $|L_0| \le r$ by Lemma 1.2.

We decompose

$$I(P, L) = \sum_{i=1}^{t} I(P_i, L) + I(R, L_0) + I(R, L \setminus L_0).$$

We can immediately bound

$$I(R, L_0) \le |L_0| \cdot |R| \le |L_0| n \le nr = O(n^{4/3}),$$

and

$$I(R, L \setminus L_0) \le |L \setminus L_0| r = O(n^{4/3}),$$

since each line of $L \setminus L_0$ intersects $Z(f)$, and thus also R, in at most $r = \deg(f)$ points.

It remains to bound $\sum_{i=1}^{t} I(P_i, L)$. Let $L_i \subset L$ be the lines containing at least one point of P_i (the L_i are typically *not* disjoint). By Lemma 1.1, no line intersects more than $r+1$ of the P_i, and so $\sum_{i=1}^{t} |L_i| \le (r+1)n$.

Let us further divide L_i into L_i', the lines containing exactly one point of P_i, and L_i'', the lines containing at least two points of P_i.

We have $I(P_i, L_i'') \le |P_i|^2$, because for every $p \in P_i$, there are at most $|P_i| - 1$ lines that pass through p and contain at least one other point of P_i. Obviously, $I(P_i, L_i') \le |L_i'|$. Thus, we can estimate

$$\begin{aligned} \sum_{i=1}^{t} I(P_i, L) &= \sum_{i=1}^{t} I(P_i, L_i') + \sum_{i=1}^{t} I(P_i, L_i'') \le \sum_{i=1}^{t} |L_i'| + \sum_{i=1}^{t} |P_i|^2 \\ &\le (r+1)n + t\left(\frac{n}{s}\right)^2 = O(n^{4/3}). \end{aligned}$$

□