Erratum to “Nonexistence of 2-reptile simplices”

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The paper “Nonexistence of 2-reptile simplices” of the first author [in *Discrete and Computational Geometry: Japanese Conference, JCDCG 2004*, Lecture Notes in Computer Science 3742, Springer, Berlin etc., pages 151–160, 2005] contains a (computational) error, found by the second author.

The error is this: At the end of the proof of Theorem 1, the matrix $\bar{A}_2^{-1}\bar{A}_1$ is considered, and it is claimed that its characteristic polynomial equals $(1-x)^{d-2}(x^2-2x+3)$. However, the characteristic polynomial actually equals $(1-x)^{d-2}(x^2+1)$, and its roots all have absolute value 1; thus, the desired contradiction is not reached using this matrix.

The proof can be corrected using the same approach, but considering another suitable expression in the matrices $\bar{A}_1$ and $\bar{A}_2$. Concretely, instead of $\bar{A}_2^{-1}\bar{A}_1$, we consider $\bar{A}_2\bar{A}_1$, which has the form (shown here for $d=5$)

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 1 \\ -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial $p(x)$ comes out as follows:

$$p(x) = \begin{cases} -x^d - x^{d-1} - \ldots - x^{(d+1)/2} + \frac{1}{4} & \text{for } d \text{ odd,} \\ x^d + x^{d-1} + \ldots + x^{d/2+1} + \frac{1}{4} & \text{for } d \text{ even.} \end{cases}$$

It remains to check that for every $d \geq 3$, $p(x)$ has a root with absolute value distinct from $2^{-2/d}$.

First let $d \geq 3$ be odd. Then $p(x)$ has at least one real root, and it is easily checked that neither $2^{-2/d}$ nor $-2^{-2/d}$ is a root.

For d even and at least 6, we use Lehmer’s criterion as stated in the paper (in the proof of Lemma 3). To this end, we first rewrite $p(x)$ to the form $p(x) = \frac{q(x)}{4-4x}$ with $q(x) = -4x^{d+1} + 4x^{d/2+1} - x + 1$. It suffices to show that $q(x)$ has a root strictly inside the circle $\Gamma' = \{z \in \mathbf{C} : |z| = 2^{-2/d}\}$. In other words, we want that for some $\beta < 2^{-2/d}$ the polynomial $g(z) := q(\beta z)$ has a root inside the unit circle Γ.

We have $g(z) = \sum_{i=0}^{d+1} a_i x^i = -4\beta^{d+1}z^{d+1} + 4\beta^{1+d/2}z^{1+d/2} - \beta z + 1$. Let us write $T(g)(z) = \overline{a}_0 g(z) - a_{d+1}z^{d+1}\overline{g(z^{-1})} = \sum_{i=0}^{d} b_i$. Then, for β sufficiently close to $2^{-2/d}$, we have $b_0 = 1 - (4\beta^{(d+1)})^2 \leq 1 - 2^{-4/d} + \varepsilon \leq 1 - 2^{-4/6} + \varepsilon < \frac{1}{2}$, while $|b_d| = 4\beta^{d+2} \geq \frac{1}{2}$. Thus,

$T^2(g)(0) = |b_0|^2 - |b_d|^2 < 0$, and so $g(z)$ indeed has a root inside Γ by Lehmer's criterion. This finishes the case of even $d \geq 6$.

Finally, for $d = 4$, we have $p(x) = x^4 + x^3 + \frac{1}{4}$, and it is easy to verify that $p(x)$ has a root with absolute value larger than $2^{-2/d} = 2^{-1/2}$. For example, we can use the *Gauss–Lucas theorem*, asserting that the roots of the derivative $p'(x)$ in the complex plane lie in the convex hull of the roots of $p(x)$. Since $p'(x) = x^2(4x + 3)$ has $-\frac{3}{4}$ as a root, $p(x)$ must also have a root with absolute value exceeding $\frac{3}{4} > 2^{-1/2}$. This finishes the proof that not all eigenvalues of $\bar{A}_2^{-1}\bar{A}_1$ have absolute value $2^{-2/d}$.