the Ramanujan conjecture. Eichler's proof relies on Weil's theorem mentioned in the previous section. The non-bipartite graphs G constructed in this manner satisfy a somewhat stronger assertion than $\lambda(G) \leq 2\sqrt{d-1}$. In fact, besides their largest eigenvalue d, they do not have eigenvalues whose absolute value exceed $2\sqrt{d-1}$. This fact implies some strong pseudo-random properties, as shown in the next results.

Theorem 9.2.4 *Let $G = (V, E)$ be a d-regular graph on n vertices, and suppose the absolute value of each of its eigenvalues but the first one is at most λ. For a vertex $v \in V$ and a subset B of V denote by $N(v)$ the set of all neighbours of v in G, and let $N_B(v) = N(v) \cap B$ denote the set of all neighbours of v in B. Then, for every subset B of cardinality bn of V;*

$$\sum_{v \in V} (|N_B(v)| - bd)^2 \leq \lambda^2 b(1-b)n.$$

Observe that in a random d-regular graph each vertex v would tend to have about bd neighbours in each set of size bn. The above theorem shows that if λ is much smaller than d then for most vertices v, $N_B(v)$ is not too far from bd.

Proof. Let A be the adjacency matrix of G and define a vector $f : V \mapsto R$ by $f(v) = 1 - b$ for $v \in B$ and $f(v) = -b$ for $v \notin B$. Clearly $\sum_{v \in V} f(v) = 0$, i.e., f is orthogonal to the eigenvector of the largest eigenvalue of A. Therefore

$$(Af, Af) \leq \lambda^2 (f, f).$$

The right hand side of the last inequality is $\lambda^2(bn(1-b)^2 + (1-b)nb^2) = \lambda^2 b(1-b)n$. The left hand side is

$$\sum_{v \in V} ((1-b)|N_B(v)| - b(d - |N_B(v)|))^2 = \sum_{v \in V} (|N_B(v)| - bd)^2.$$

The desired result follows. ■

Corollary 9.2.5 *Let $G = (V, E), d, n$ and λ be as in Theorem 9.2.4. Then for every two sets of vertices B and C of G, where $|B| = bn$ and $|C| = cn$ we have:*

$$|e(B, C) - cbdn| \leq \lambda \sqrt{bc}\, n.$$

Proof. By Theorem 9.2.4

$$\sum_{v \in C} (|N_B(v)| - bd)^2 \leq \sum_{v \in V} (|N_B(v)| - bd)^2 \leq \lambda^2 b(1-b)n.$$

Thus, by the Cauchy Schwarz inequality;

$$|e(B, C) - cbdn| \leq \sum_{v \in C} |N_B(v) - bd|$$

$$\leq \sqrt{cn}(\sum_{v\in C}(|N_B(v)|-bd)^2)^{1/2} \leq \sqrt{cn}\lambda\sqrt{b(1-b)n} \leq \lambda\sqrt{bc}\,n.$$

■

The special case $B = C$ gives the following result. A slightly stronger estimate is proved in a similar way in Alon and Chung (1988) .

Corollary 9.2.6 *Let $G = (V, E), d, n$ and λ be as in Theorem 9.2.4. Let B be an arbitrary set of bn vertices of G and let $e(B) = \frac{1}{2}e(B, B)$ be the number of edges in the induced subgraph of G on B. Then*

$$|e(B) - \frac{1}{2}b^2dn| \leq \frac{1}{2}\lambda bn.$$

A *walk of length l* in a graph G is a sequence $v_0, \ldots, v_l$ of vertices of G, where for each $1 \leq i \leq l$, $v_{i-1}v_i$ is an edge of G. Obviously, the total number of walks of length l in a d-regular graph on n vertices is precisely $n \cdot d^l$. Suppose, now , that C is a subset of, say, $n/2$ vertices of G. How many of these walks do not contain any vertex of C? If G is disconnected it may happen that half of these walks avoid C. However, as shown byAjtai, Komlós and Szemerédi (1987) , there are many fewer such walks if all the eigenvalues of G but the largest are small. This result and some of its extensions have several applications in theoretical computer science, as shown in the above mentioned paper (see also Cohen and Wigderson (1989)). We conclude this section by stating and proving the result and one of its applications.

Theorem 9.2.7 *Let $G = (V, E)$ be a d-regular graph on n vertices, and suppose that each of its eigenvalues but the first one is at most λ. Let C be a set of cn vertices of G. Then, for every l, the number of walks of length l in G that avoid C does not exceed $(1-c)n((1-c)d + c\lambda))^l$.*

Proof. Let A be the adjacency matrix of G and let A' be the adjacency matrix of its induced subgraph on the complement of C. We claim that the maximum eigenvalue of A' is at most $(1-c)d + c\lambda$. To prove this claim we must show that for every vector $f : V \mapsto R$ satisfying $f(v) = 0$ for each $v \in C$ and $\sum_{v\in V} f(v)^2 = 1$, the inequality $(Af, f) \leq (1-c)d + c\lambda$ holds. Let $f_1, f_2, \ldots, f_n$ be an orthonormal basis of eigenvectors of A, where f_1 is the eigenvector of λ_i, $\lambda_1 = d$ and each entry of f_1 is $1/\sqrt{n}$. Then $f = \sum_{i=1}^n c_i f_i$, where $\sum_{i=1}^n c_i^2 = 1$ and

$$c_1 = \sum_{v\in V} f(v)/\sqrt{n} = \sum_{v\in V-C} f(v)/\sqrt{n}$$

$$\leq (\sum_{v\in V-C} f(v)^2)^{1/2}((1-c)n/n)^{1/2} = \sqrt{1-c}\,,$$

where here we used the Cauchy-Schwarz Inequality. Therefore $\sum_{i=2}^n c_i^2 = c$ and

$$(Af, f) = \sum_{i=1}^n c_i^2\lambda_i \leq (1-c)d + c\lambda,$$