Exercise 7

February, 2006

1. Let $f : \{0,1\}^n \to \mathbb{R}$. Define $\tau(p) = \|f\|_p^p$. Prove that τ is log-convex, i.e.:
$$\tau\left(\frac{p+q}{2}\right) \leq \sqrt{\tau(p)\tau(q)}$$

2. On the $\mathbb{Z}^2$ lattice, every edge is assigned the weight 1 with probability p and 2 with probability $q = 1 - p$. What is the distribution of $d((0,0),(3,0))$?

3. Let $f : \{0,1\}^n \to \{0,1\}$.
   (a) Define a monotonization operator M_i along the ith coordinate.
   (b) Show that monotonization of f along the ith coordinate does not spoil monotonicity along the jth coordinate.
   (c) Show that the $\text{Inf}(M_i f) \leq \text{Inf}(f)$.
   (d) What is $\text{Inf}_i(f)$ in terms of the Fourier expansion of f, assuming f is monotone?
   (e) Can you generalize monotonization to the case of the solid cube ($[0,1]^n \to \{0,1\}$)?

4. On the $\mathbb{Z}^2$ lattice, each edge exists with probability p. Show that there is a probability p_c such that if $p > p_c$, then there is an infinite connected component with probability 1, and if $p < p_c$, then the probability is 0.

5. Let $f : \{0,1\}^n \to \mathbb{R}$, and let $f_i(x) = \frac{f(x)-f(x\oplus i)}{2}$. Show that
$$\text{Var}(f) = 2\sum_i \int_0^1 \|T_\epsilon f_i\|_2^2 \text{d}\epsilon$$