Exercise 1

February, 2006

1. A character χ of an Abelian group G is a homomorphism from a group G to the multiplicative group $\mathbb{C}^{(\times)}$. Let f, g be arbitrary characters of a finite group G.
   (a) Show that $\forall x \in G : f(x)$ is a root of unity.
   (b) Show that $\bar{f}$, $fg(x) = f(x)g(x)$ and $f^{-1}(x) = f(x)^{-1}$ are characters.
   (c) Show that characters are orthonormal, i.e.:
   $$\frac{1}{|G|} \sum_{x \in G} f(x)\bar{g}(x) = \delta_{f,g}$$
2. Let $f : \mathbb{T} \to \mathbb{C}$ be continuous. What are the Fourier coefficients of Ref and Imf?
3. Let G be a finite Abelian group.
   (a) For a function $f : G \to \mathbb{R}$, define $f_y(x) = f(x - y)$. Calculate $\hat{f}_y$ in terms of $\hat{f}$.
   (b) Convolution is defined by $f * g(y) = \frac{1}{|G|} \sum_x f(x - y)g(x)$. Convolution is a linear operator, and therefore can be described as matrix multiplication, i.e. there exists a matrix F^* such that $F^* \cdot g = f * g$. Explicitly define F^*.
   (c) What are the eigenvectors and eigenvalues of F^*?
4. (a) The Cesáro operator $C = C_1$, from series to series, is defined as follows:
   $$C(\{a_n\})_i = \frac{1}{i+1} \sum_{j=0}^{i} a_j$$
   For every t, give a series $\{a_n\}$ such that $\{a_n\}$ does not converge in the C_t sense but does converge in the C_{t+1} sense, i.e., t applications of the Cesàro operator do not yield a convergent series but $t+1$ applications do.
   (b) Is there a series $\{a_n\}$ such that for any fixed t, the series $\{a_n\}$ does not converge in the C_t sense, but the series $\{b_n\}$ defined by setting b_t to be the tth element in the tth Cesàro summation does converge?

(c) Given a matrix $(a_{ij}) = A \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$, define

$$\lim_A s_n = \lim_{i\to\infty} \sum_j a_{ij} s_j$$

assuming that all the sums are defined and the limit exists. A matrix A is called a *regular* if $\lim_A s_n = \lim_n s_n$ whenever the latter is defined. Show that following conditions are required for regularity:

- $\forall j : \lim_{i\to\infty} a_{ij} = 0$
- $\lim_i \sum_{j=0}^{\infty} a_{ij} = 1$

Note: with the addition of the following condition:

- There is some fixed K such that for every i,

$$\sum_{j=0}^{n} |a_{ij}| < K$$

we get equivalence.

(d) Show that if the Cesàro limit gof $\{\sum_n a_n\}$ exists, then the Abel sum

$$\lim_{x\to 1^-} \sum_n a_n x^n$$

also exists and is equal to the Cesàro limit.