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Counting plane graphs: Perfect matchings, spanning cycles, and Kasteleyn's technique ☆

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ABSTRACT

We derive improved upper bounds on the number of crossing-free straight-edge spanning cycles (also known as Hamiltonian tours and simple polygonizations) that can be embedded over any specific set of N points in the plane. More specifically, we bound the ratio between the number of spanning cycles (or perfect matchings) that can be embedded over a point set and the number of triangulations that can be embedded over it. The respective bounds are $O(1.8181^N)$ for cycles and $O(1.1067^N)$ for matchings. These imply a new upper bound of $O(54.543^N)$ on the number of crossing-free straight-edge spanning cycles that can be embedded over any specific set of N points in the plane (improving upon the previous best upper bound $O(68.664^N)$). Our analysis is based on a weighted variant of Kasteleyn's linear algebra technique.

1. Introduction

In this paper we consider the problem of bounding the number of all crossing-free straight-edge spanning cycles that can be embedded over a specific set of points in the plane. That is, given a set S

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of N labeled points in the plane, we consider the number of spanning cycles that have a straight-edge planar embedding over S. We rely on Kasteleyn's linear-algebra technique [15], and on edge-flipping techniques that were developed in a previous paper by the authors [8]. No familiarity with [8] is necessary, since we re-introduce all the notions that we require from it. We now give a detailed and more formal definition of the problem.

A *planar graph* is a graph that can be embedded in the plane in such a way that its vertices are embedded as points and its edges are embedded as Jordan arcs that connect the respective pairs of points and can meet only at a common endpoint. A *crossing-free straight-edge graph* is a plane embedding of a planar graph such that its edges are embedded as non-crossing straight line segments. In this paper, we only consider crossing-free straight-edge graphs, and we also assume that the points of the vertex set S are in general position, that is, no three points are collinear. (For upper bounds on the number of graphs, this involves no loss of generality, because the number of graphs can only grow when a degenerate point set is slightly perturbed into general position.) For simplicity, we sometimes refer to such graphs as *plane graphs*.

We focus on upper bounding the maximal number of plane *spanning cycles* (also known as *Hamiltonian cycles*, *Hamiltonian tours*, and *simple polygonizations*) that can be embedded over a fixed set of points in the plane. For a set S of points in the plane, we denote by $\mathcal{C}(S)$ the set of all crossing-free straight-edge spanning cycles of S, and put $\mathsf{sc}(S) := |\mathcal{C}(S)|$. Moreover, we let $\mathsf{sc}(N) = \max_{|S|=N} \mathsf{sc}(S)$. The main goal of this paper is thus to obtain improved upper bounds on $\mathsf{sc}(N)$.

There are many similar variants of this problem, such as bounding the number of plane forests, spanning trees, triangulations, and general plane graphs. Recent work on some of these variants can be found in [1,8,22], and we try to keep a comprehensive list of the up-to-date upper and lower bounds in a dedicated webpage.[1] It seems that the case of spanning cycles is the most popular one, already considered in [2–6,17,24] and many other works. Moreover, spanning cycles were the first case for which bounds were published, namely the bounds $3/20 \cdot 10^{N/3} \leqslant \mathsf{sc}(N) \leqslant 2 \cdot 6^{N-2} \cdot (\lfloor N/2 \rfloor)!$ in [17]. A brief history of the steady progress on bounding the number of spanning cycles can be found in a dedicated webpage by Erik Demaine.[2] Currently, the best known lower bound is $\mathsf{sc}(N) = \Omega(4.642^N)$, due to García, Noy, and Tejel [6], and the previous upper bound is $\mathsf{sc}(N) = O(68.664^N)$ by Dumitrescu et al. [5]. We derive the improved bound $\mathsf{sc}(N) = O(54.543^N)$.

These problems have also been studied from an algorithmic point of view, where the goal is to derive algorithms for enumeration or counting of the plane graphs (or other graph types) that can be embedded over a given point set (such as in [13,19]). The combinatorial upper bounds are useful for analyzing the running times of such algorithms, and also answering questions such as "how many bits are required to represent a triangulation (or any other kind of plane graphs)?".

Our bound (as do some of the previous bounds) relies on *triangulations*. A triangulation of a set S of N points in the plane is a maximal plane graph on S (that is, no additional straight edges can be inserted without crossing some of the existing edges). For a set S of points in the plane, we denote by $\mathcal{T}(S)$ the set of all triangulations of S, and put $\mathsf{tr}(S) := |\mathcal{T}(S)|$. Moreover, we let $\mathsf{tr}(N) = \max_{|S|=N} \mathsf{tr}(S)$. Currently, the best known bounds for $\mathsf{tr}(N)$ are $\mathsf{tr}(N) < 30^N$ [22], and $\mathsf{tr}(N) = \Omega(8.65^N)$ [5].

The upper bound by Dumitrescu et al. [5] is obtained by proving that for every set S of N points in the plane $\mathsf{sc}(S) = O(2.2888^N) \cdot \mathsf{tr}(S)$. This has sharpened an earlier bound of Buchin et al. [4], who showed that every triangulation T of S contains at most $30^{N/4} \approx 2.3404^N$ spanning cycles (i.e., cycles whose edges belong to T), implying[3] that $\mathsf{sc}(S) < 2.3404^N \cdot \mathsf{tr}(S)$. Combining the above ratio with the bound $\mathsf{tr}(N) < 30^N$ directly implies the asserted bound. We derive our bound in a similar manner, showing that $\mathsf{sc}(S) = O(1.8181^N) \cdot \mathsf{tr}(S) = O(54.5430^N)$.

In spite of our improved bound, we strongly believe, and conjecture, that for every point set S (of size at least N_0, for some sufficiently large constant N_0) one has $\mathsf{sc}(S) < \mathsf{tr}(S)$, and perhaps even a much sharper ratio holds. The best lower bound for this ratio that we know of is obtained from the *double chain configuration*, presented in [6] (and depicted in Fig. 1). It is shown in [6] that when S is

[1] http://www.cs.tau.ac.il/~sheffera/counting/PlaneGraphs.html (version of December 2012).

[2] http://erikdemaine.org/polygonization/ (version of December 2012).

[3] The implication comes from the fact that every spanning cycle, and in fact every plane graph, is contained in at least one triangulation; see Section 2.

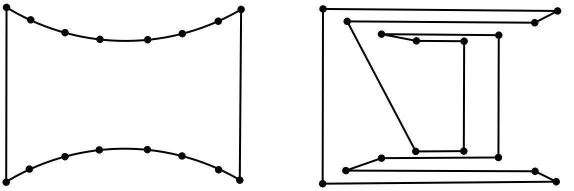


Fig. 1. Two spanning cycles embedded over a double chain point configuration.

a double chain configuration with N vertices, $\mathrm{tr}(S) = \Theta^*(8^N)$ and $\mathrm{sc}(S) = \Omega^*(4.64^N)$.[4] Thus, in this case, $\mathrm{sc}(S)/\mathrm{tr}(S) = \Omega^*(0.58^N)$. (It is stated in [1], albeit without proof, that $\mathrm{sc}(S) = O(5.61^N)$, so this example supports our conjecture.)

In Section 2 we go over the preliminaries required for our analysis. These include, among others, the edge-flip techniques used in [8]. Section 3 derives the bound $\mathrm{sc}(S) = O(12^{N/4}) \cdot \mathrm{tr}(S) = O(1.8613^N) \cdot \mathrm{tr}(S)$ for any set S of N points in the plane. As part of this derivation, we describe Kasteleyn's technique for counting perfect matchings and present a new way of applying it. The more advanced analysis, deriving the improved bound $\mathrm{sc}(S) = O(10.9247^{N/4}) \cdot \mathrm{tr}(S) = O(1.8181^N) \cdot \mathrm{tr}(S)$, is presented in Section 5. In Section 4 we use the same methods to prove an upper bound on the ratio between the number of plane perfect matchings and the number of triangulations, showing that $\mathrm{pm}(S) = O(1.1067^N) \cdot \mathrm{tr}(S)$ (where $\mathrm{pm}(S)$ is the number of crossing-free straight-edge matchings that can be embedded over the point set S).

While this paper constitutes a significant improvement over previous bounds, it is only one stepping stone towards the goal of establishing a sharp bound on $\mathrm{sc}(N)$, or of at least showing that $\mathrm{sc}(N) < \mathrm{tr}(N)$, as conjectured above. The interest in this paper, in our opinion, is in the technique that it employs, where it combines recent results on edge flippability in triangulations [8] with the beautiful (and fairly old) technique of Kasteleyn [15,16] that applies tools from linear algebra to derive upper bounds on the number of perfect matchings in planar graphs. Kasteleyn's technique has already been used recently in [4] for deriving bounds on $\mathrm{sc}(N)$, but the application in this paper is different, as it handles edge-weighted planar graphs. Instead of bounding the number of perfect matchings, it bounds the sum of their weights, where the weight of a matching is the product of the weights of its edges. This enhanced version allows us to "push" the technique much further and obtain our improved bounds. We hope that this enhanced tool will lead to further results in this area.

2. Preliminaries

In this section we establish some notations and lemmas that are required for the following sections.

Given two plane graphs G and H over the same point set S, if every edge of G is also an edge of H, we write $G \subseteq H$.

Hull edges and vertices (resp., interior edges and vertices) of a graph embedded on a point set S are those that are part of the boundary of the convex hull of S (resp., not part of the convex hull boundary).

Given a set S of N points in the plane, we denote by h the number of hull vertices of S, and put $n = N - h$, which is the number of interior vertices of S.

2.1. The support of a graph

Let us denote by $\mathrm{sc}_\triangle(N)$ the maximal number of plane spanning cycles that can be contained in any fixed triangulation of a set of N points in the plane. Moreover, denote the set of spanning cycles contained in a triangulation T by $\mathcal{C}(T)$, so $\mathrm{sc}_\triangle(N) = \max_{|S|=N,\, T\in\mathcal{T}(S)} |\mathcal{C}(T)|$.

[4] In the notations $O^*()$, $\Theta^*()$, and $\Omega^*()$, we neglect polynomial factors.

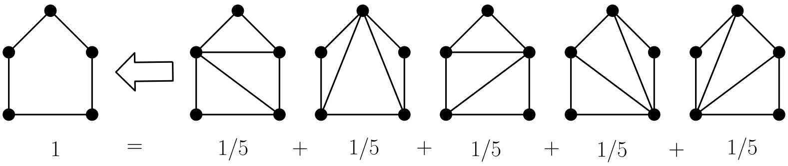


Fig. 2. A spanning cycle with a support of 5 that contributes 1 to the sum in (1).

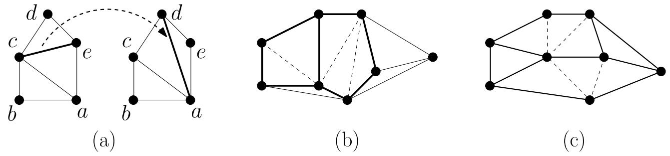


Fig. 3. (a) The edge *ce* can be flipped to the edge *ad*. (b) A set of three (dashed) ps-flippable edges which are diagonals of interior-disjoint convex quadrilateral and convex pentagon. (c) A convex decomposition with four bounded cells obtained by removing the dashed edges from the triangulation; here too the dashed edges are diagonals of pairwise disjoint convex faces, and form a set of ps-flippable edges.

Any spanning cycle (or, for that matter, any plane graph) is contained in at least one triangulation. Therefore, we can upper bound the number of spanning cycles of a set S of N points in the plane by going over every triangulation $T \in \mathcal{T}(S)$ and counting the number of spanning cycles contained in T. This implies the bound $\mathsf{sc}(N) \leqslant \mathsf{tr}(N) \cdot \mathsf{sc}_{\triangle}(N)$. Applying the bounds $\mathsf{tr}(N) < 30^N$ from [22] and $\mathsf{sc}_{\triangle}(N) \leqslant 30^{N/4}$ from [4], we obtain $\mathsf{sc}(N) < 30^{5N/4} \approx 70.21^N$.

This bounding method seems rather weak since it potentially counts some spanning cycles many times. For example, consider a spanning cycle of the double-chain configuration consisting of two convex chains facing each other, as depicted in the left-hand side of Fig. 1. García, Noy, and Tejel [6] show that such a spanning cycle is contained in $\Theta^*(8^N)$ triangulations of its point set. Therefore, the above method will count this spanning cycle $\Theta^*(8^N)$ times. In this case the above analysis method will be grossly over-counting because, as stated in [1], this point set has only $O(5.61^N)$ spanning cycles.

In order to deal with this inefficiency, we define the notion of *support* (the same notion was also used in [5,8,22,23,25]). Given a plane graph G embedded over a set S of points in the plane, we say that G has a support of x if G is contained in (exactly) x triangulations of S; we write $\mathsf{supp}(G) = x$. Notice that

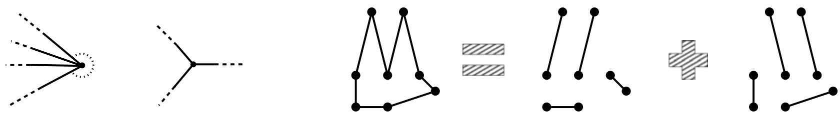
$$\mathsf{sc}(S) = \sum_{T \in \mathcal{T}(S)} \sum_{C \in \mathcal{C}(T)} \frac{1}{\mathsf{supp}(C)}, \tag{1}$$

because every spanning cycle C contributes exactly one to the right-hand side of the equation (it appears in $\mathsf{supp}(C)$ terms of the first sum, and contributes $1/\mathsf{supp}(C)$ in every appearance); an example is depicted in Fig. 2. We will use (1) to obtain better upper bounds for $\mathsf{sc}(N)$, by showing that, on average, $\mathsf{supp}(C)$ is large.

2.2. Ps-flippable edges

An edge in a triangulation is said to be *flippable*, if its two incident triangles form a convex quadrilateral. A flippable edge can be *flipped*, that is, removed from the graph of the triangulation and replaced by the other diagonal of the resulting quadrilateral. Such an operation is depicted in Fig. 3(a), where the edge *ce* can be flipped to the edge *ad*.

In [8], we present the concept of *pseudo simultaneously flippable edges* (or *ps-flippable edges*, for short). Given a triangulation T, we say that a subset F of its edges is a set of ps-flippable edges if the edges of F are diagonals of interior-disjoint convex polygons (whose boundaries are also parts of T). For example, in Fig. 3(b), the three dashed edges form a set of ps-flippable edges, since they are



(a) (b)

Fig. 4. (a) A vertex is valid if and only if it is not a reflex vertex of any face. (b) In a point set of an even size, every spanning cycle is the union of two edge-disjoint perfect matchings.

diagonals of interior-disjoint convex quadrilateral and convex pentagon (another set of ps-flippable edges, in a different triangulation, is depicted in Fig. 3(c)).

Ps-flippable edges are related to *convex decompositions*. A convex decomposition of a point set S is a crossing-free straight-edge graph D on S such that (i) D includes all the hull edges, (ii) each bounded face of D is a convex polygon, and (iii) no point of S is isolated in D. See Fig. 3(c) for an illustration. For additional information about convex decompositions, see, for example, [10,20]. Notice that if T is a triangulation that contains D, the edges of $T \setminus D$ form a set of ps-flippable edges, since they are the diagonals of the interior-disjoint convex polygons of D (again, consider the dashed edges in Fig. 3(c) for an illustration). Thus, finding a large set of ps-flippable edges in a triangulation T is equivalent to finding a convex decomposition with a small number of faces (or edges) in T (though the bounds in [10,20] are not directly related to the bounds stated below).

The two following lemmas are proven in [8].

Lemma 2.1. *Every triangulation T over a set of N points in the plane contains a set F of $N/2-2$ ps-flippable edges. Also, there are triangulations with no larger sets of ps-flippable edges.*

Lemma 2.2. *Consider a triangulation T, a set F of $N/2-2$ ps-flippable edges in T, and a graph $G \subseteq T$. If G does not contain j edges from F then* $\mathrm{supp}(G) \geqslant 2^j$.

Proof sketch. Consider the set $F' = F \setminus G$ of j ps-flippable edges (every subset of a set of ps-flippable edges is also a set of ps-flippable edges). The convex faces of $T \setminus F'$ can be triangulated in at least 2^j ways (the actual number, which is a product of Catalan numbers [26, Section 5.3], attains this minimum when every edge of F' is a diagonal of a distinct quadrangular face of $T \setminus F'$), and each of the resulting triangulations contains G. See [8] for more details. □

We now describe another property of convex decompositions (not discussed in [8]). Consider a set S of points in the plane and a crossing-free straight-edge graph G embedded on S. We say that an interior point $p \in S$ has a *valid triple of edges in G* if there exist three points $a, b, c \in S$ such that p is contained in the convex hull of $\{a, b, c\}$ and the edges ap, bp, and cp belong to G. For simplicity, we refer to vertices with valid triples as *valid* (with respect to G), and to the other interior vertices as *non-valid*. See Fig. 4(a) for an illustration.

Lemma 2.3. *Let S be a set of points in the plane and let G be a crossing-free straight-edge graph over S that contains all the edges of the convex hull of S. Then G is a convex decomposition of S if and only if every interior vertex of S is valid with respect to G.*

Proof. An interior vertex v is a reflex vertex of some face of G if and only if v is non-valid (see Fig. 4(a)). The lemma follows by observing that G is a convex decomposition if and only if no bounded face of G has a reflex vertex. □

2.3. Spanning cycles and perfect matchings

Our analysis, as most of the previous works dealing with the number of spanning cycles, heavily relies on the number of plane perfect matchings on S (for example, see [4,5,24]). To see the connection between the two problems, notice that if $|S|$ is even, every spanning cycle C is the union of two

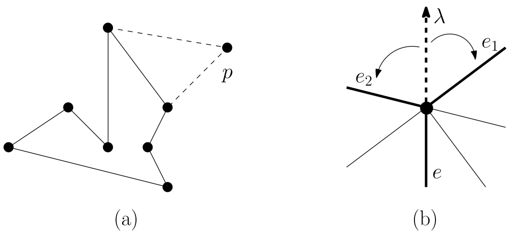


Fig. 5. (a) We can always connect a new vertex p outside the convex hull of S to two endpoints of some edge of the spanning cycle. (b) The set $\{e, e_1, e_2\}$ is a valid triple of edges.

edge-disjoint perfect matchings on S; namely, the matching consisting of the even-indexed edges of C, and the matching consisting of the odd-indexed edges. An illustration of this property is depicted in Fig. 4(b). Denote by $\mathcal{M}(S)$ the set of all plane perfect matchings on S, and put $\mathrm{pm}(S) = |\mathcal{M}(S)|$. We also set $\mathrm{pm}(N) = \max_{|S|=N} \mathrm{pm}(S)$. Hence, a simple upper bound on $\mathrm{sc}(S)$ is $\mathrm{pm}(S)^2$. In general, the union of two edge-disjoint perfect matchings is not always a spanning cycle, but it is a cover of S by vertex-disjoint even-sized cycles.

To deal with point sets of odd size, we use the following lemma:

Lemma 2.4. *Let $c > 1$ be a constant such that every set S of an even number of points in the plane satisfies $\mathrm{sc}(S) = O(c^{|S|})$. Then $\mathrm{sc}(S) = O(c^{|S|})$ also holds for sets S of an odd number of points.*

Proof. Consider a set S of N points in the plane, where N is odd. Pick a new point p outside the convex hull of S, and put $S' = S \cup \{p\}$. Let C be a plane spanning cycle of S. Then there exists an edge $e = vu$ of C such that p can be connected to the two endpoints u, v of e without crossing C (e.g., see Fig. 5(a)). Indeed, this is a variant of the property, noted in [7], that every finite collection of non-crossing straight segments in the plane contains a segment e such that no other segment lies vertically above any point of e (see also [18, Section 8.7]). By replacing e with the edges vp and pu, we obtain a crossing-free spanning cycle of S'. This implies that we can map every spanning cycle of S to a distinct spanning cycle of S', and thus, $\mathrm{sc}(S) \leqslant \mathrm{sc}(S')$. The lemma then follows since $\mathrm{sc}(S') = O(c^{N+1}) = O(c^N)$. $\square$

Bounding the number of perfect matchings on S within a fixed triangulation T can be done by the beautiful linear-algebra technique of Kasteleyn [15], described in detail in [16, Section 8.3]; see Section 3 for more details. Buchin et al. [4] have used this technique to show that any triangulation T of S contains at most $6^{N/4}$ perfect matchings, and at most $30^{N/4} \approx 2.3403^N$ spanning cycles. We also note that Sharir and Welzl [24] showed that $\mathrm{pm}(S) = O(10.05^N)$, completely bypassing the approach of counting matchings (or other graphs) *within* a triangulation. While this bound is fairly small, it does not seem to be useful for obtaining a good bound on $\mathrm{sc}(N)$. For example, using the inequality $\mathrm{sc}(N) \leqslant \mathrm{pm}(N)^2$, noted above, only gives the rather weak bound $\mathrm{sc}(N) = O(101.01^N)$.

2.4. Kasteleyn's technique

We now present a brief overview of Kasteleyn's technique. While Kasteleyn originally developed it to study physical phenomena (such as *dimers* in a square lattice [14]), his methods are also useful when studying perfect matchings in general graphs.

Given an oriented graph[5] $\vec{G} = (V, E)$ with no anti-parallel edges (i.e., edges between the same pair of vertices but in opposite directions), we define the following adjacency matrix $B_{\vec{G}} = (b_{ij})_{N \times N}$ of $\vec{G}$,

[5] We follow here the notation used in [16] to denote a digraph obtained from an underlying undirected graph by giving each of its edges an orientation.

$$b_{ij} = \begin{cases} 1, & \text{if } e = (i, j) \in E, \\ -1, & \text{if } e = (j, i) \in E, \\ 0, & \text{otherwise} \end{cases}$$

(where $N = |V|$, and the rows and columns of $B_{\vec{G}}$ correspond to an arbitrary fixed enumeration of the vertices). Further details and a proof of the following theorem can be found in [16, Chapter 8].

Theorem 2.5 *(Kasteleyn's theorem). Let G be a planar graph. Then there exists an orientation $\vec{G}$ of G such that the number of perfect matchings of G is $\sqrt{|\det B_{\vec{G}}|}$.*

3. An upper bound on the number of spanning cycles

In this section we first review an enhanced variant of Kasteleyn's technique and then use it to derive the following upper bound on $\mathsf{sc}(S)$.

Theorem 3.1. *For any set S of N points in the plane,*

$$\mathsf{sc}(S) = O\left(12^{N/4}\right) \cdot \mathsf{tr}(S) = O\left(1.8613^N\right) \cdot \mathsf{tr}(S) = O\left(55.839^N\right).$$

This bound is slightly weaker than the one stated in the introduction, but its proof is considerably simpler; the improved bound is derived in Section 5.

Proof of Theorem 3.1. First, by Lemma 2.4, we may assume that N is even. Consider a triangulation T of S. As already observed, every spanning cycle contained in T is the union of two edge-disjoint perfect matchings of T. Given a plane graph G, we denote by $\mathcal{M}(G)$ the set of all perfect matchings of G. Recalling (1), we have

$$\mathsf{sc}(S) \leqslant \sum_{T \in \mathcal{T}(S)} \sum_{\substack{M_1, M_2 \in \mathcal{M}(T) \\ M_1, M_2 \text{ edge-disjoint}}} \frac{1}{\mathsf{supp}(M_1 \cup M_2)}.$$

(The inequality comes from the fact that not every pair M_1, M_2 of matchings, as in the sum, necessarily yields a spanning cycle.) Let us fix the "first" perfect matching $M_1 \subset T$; as mentioned above, Buchin et al. [4] prove that $|\mathcal{M}(T)| \leqslant 6^{N/4}$, so there are at most $6^{N/4}$ choices of M_1. Next, we construct a convex decomposition D such that $M_1 \subset D \subset T$, as follows. We start with M_1 and add all the missing hull edges; let us denote the resulting graph as D'. By Lemma 2.3, it suffices to add edges to D' so as to ensure that every interior point $p \in S$ is connected in D to (at least) three points $a, b, c \in S$, such that p is inside the convex hull of $\{a, b, c\}$. Every interior vertex p of S has degree 1 in D', so we start by setting $D := D'$, and then, for each interior point $p \in S$, we add to D two additional edges of T incident to p, so as to create a valid triple. To do so, let e be the edge of D' (that is, of M_1) incident to p, and let λ be the ray emanating from p in the opposite direction. Let e_1 (resp., e_2) be the first edge of T incident to p encountered in clockwise (resp., counterclockwise) direction from λ; see Fig. 5(b). Then, as is easily checked, $\{e, e_1, e_2\}$ is a valid triple of edges, and we add e_1, e_2 to D. After applying this step to each interior point p, the resulting graph D is indeed a convex decomposition of S.

We denote by F the set of edges that are in T but not in D. The edges of F are diagonals of interior-disjoint convex polygons, and thus F is a set of ps-flippable edges. By Euler's formula, the triangulation T contains $3N - 2h - 3$ interior edges, and D contains at most $2n + N/2$ interior edges (at most $N/2$ edges of M_1 and at most $2n$ added edges to form n valid triples). Therefore,

$$|F| \geqslant 3N - 2h - 3 - (2n + N/2) = N/2 - 3.$$

Remark. Note the strength of this bound: Lemma 2.1 has a rather involved proof, given in [8], and it yields a set of $N/2 - 2$ ps-flippable edges in the entire triangulation. In contrast, here we get the same number (minus 1) after we remove from T an arbitrary perfect matching, with a considerably simpler analysis. Thus the significance of the analysis in [8] (giving the proof of Lemma 2.1) is only for triangulations which contain no perfect matching on S. For example, any triangulation with more than $N/2$ interior vertices of degree 3 cannot contain a perfect matching, since, as is easily checked, two interior vertices of degree 3 cannot share an edge.

Without loss of generality, we assume that F consists of *exactly* $N/2-3$ edges. We now proceed to bound the number of ways to choose the second matching M_2 while taking the supports of the resulting graphs $M_1 \cup M_2$ into account. Since M_1 and M_2 have to be edge-disjoint, we can remove the $N/2$ edges of M_1 from T, and remain with a subgraph T' that has fewer than $5N/2$ edges. Next, we define a weight function μ over the edges of T', such that every edge in F has a weight of 1 and the remaining edges have weight 1/2. We define the weight $\mu(M_2)$ of a perfect matching $M_2 \subset T'$ as the product of the weights of its edges. Therefore, if M_2 contains exactly j edges of F, then $\mu(M_2) = (1/2)^{N/2-j}$. Moreover, for such a matching M_2, we have $|F \setminus M_2| = N/2-3-j$. Clearly, $F \setminus M_2$ is also a set of ps-flippable edges, none of which belongs to $M_1 \cup M_2$. By combining this with Lemma 2.2, we have

$$\frac{1}{\mathsf{supp}(M_1 \cup M_2)} \leqslant \frac{1}{2^{N/2-3-j}} = 8\mu(M_2),$$

which implies that, given a specific triangulation T and a specific perfect matching $M_1 \subset T$,

$$\sum_{M_2 \in \mathcal{M}(T')} \frac{1}{\mathsf{supp}(M_1 \cup M_2)} \leqslant 8 \sum_{M_2 \in \mathcal{M}(T')} \mu(M_2), \tag{2}$$

with $T' = T \setminus M_1$, as above.

Kasteleyn's technique: An enhanced version. We now apply an extension of Kasteleyn's technique to estimate the sum in the right-hand side of (2). Here is a brief overview of the technique being used (where instead of the original technique, presented in Section 2.4, we apply a weighted extension of it). Given an oriented graph $\vec{G} = (V, E)$ with no anti-parallel edges and a weight function μ over the edges, we define the following weighted adjacency matrix $B_{\vec{G},\mu} = (b_{ij})_{N\times N}$ of $(\vec{G}, \mu)$,

$$b_{ij} = \begin{cases} \mu(e), & \text{if } e=(i,j)\in E, \\ -\mu(e), & \text{if } e=(j,i)\in E, \\ 0, & \text{otherwise} \end{cases}$$

(where $N = |V|$, and the rows and columns of $B_{\vec{G},\mu}$ correspond to an arbitrary fixed enumeration of the vertices).

An easy extension of Kasteleyn's theorem states that every planar graph G can be oriented into some digraph $\vec{G}$ such that, for any real-valued weight function μ on its edges, we have

$$\left(\sum_{M \in \mathcal{M}(G)} \mu(M)\right)^2 = \left|\det(B_{\vec{G},\mu})\right| \tag{3}$$

(recall that $\mu(M) = \prod_{e \in M} \mu(e)$). Recall that in the "pure" form of Kasteleyn's theorem $\mu \equiv 1$ (i.e., G is unweighted) and the left-hand side of (3) is just the squared number of perfect matchings in G. The extension (3) to weighted graphs is given as Exercise 8.3.9 in [16, Section 8.3].

We denote by b_i the column vectors of B, for $1 \leqslant i \leqslant N$, and estimate the above determinant using Hadamard's inequality (e.g., see [9, Theorem 2.5.4])

$$\left|\det(B_{\vec{G},\mu})\right| \leqslant \prod_{i=1}^{N} \|b_i\|_2. \tag{4}$$

Applying the above machinery to our plane graph T' (i.e., using (3) and (4)), with the edge weights μ as defined above, we have

$$\sum_{M_2\in\mathcal{M}(T')}\mu(M_2)=\sqrt{\left|\det(B_{\vec{T}',\mu})\right|}\leqslant\sqrt{\prod_{i=1}^{N}\|b_i\|_2}=\left(\prod_{i=1}^{N}\|b_i\|_2^2\right)^{1/4}$$
$$\leqslant\left(\frac{1}{N}\sum_{i=1}^{N}\|b_i\|_2^2\right)^{N/4}=\left(\frac{2}{N}\sum_{e\in T'}\mu(e)^2\right)^{N/4} \tag{5}$$

(where we have used the arithmetic–geometric mean inequality and the fact that every edge of $\vec{T}'$ has two corresponding matrix entries). The bound $6^{N/4}$ on the number of perfect matchings in a triangulation T is obtained in [4] by applying the unweighted version of Kasteleyn's theorem to the entire T. In this case $\sum_{e\in T}\mu(e)^2$ is the number of edges of T, which is at most $3N$, and the bound follows.

By noting that

$$\left|T'\setminus F\right|\leqslant 5N/2-(N/2-3)=2N+3,$$

and combining this with (2) and (5), we obtain

$$\sum_{M_2\in\mathcal{M}(T')}\frac{1}{\operatorname{supp}(M_1\cup M_2)}\leqslant 8\cdot\left(\frac{2}{N}\sum_{e\in T'}\mu(e)^2\right)^{N/4}$$
$$\leqslant 8\cdot\left(\frac{2}{N}\cdot\left(1^2\cdot(N/2-3)+\left(\frac{1}{2}\right)^2\cdot(2N+3)\right)\right)^{N/4}$$
$$=O\left(2^{N/4}\right). \tag{6}$$

Recalling once again that a triangulation contains at most $6^{N/4}$ perfect matchings [4] (that is, there are $6^{N/4}$ ways of choosing M_1), and combining this with (6), we obtain

$$\operatorname{sc}(S)\leqslant\sum_{T\in\mathcal{T}(S)}\sum_{\substack{M_1,M_2\in\mathcal{M}(T)\\ M_1,M_2\text{ edge-disjoint}}}\frac{1}{\operatorname{supp}(M_1\cup M_2)}\leqslant\sum_{T\in\mathcal{T}(S)}6^{N/4}\cdot O\left(2^{N/4}\right)$$
$$=O\left(12^{N/4}\right)\cdot\operatorname{tr}(S),$$

as asserted. □

As already noted, by applying a more complex analysis, we will obtain in Section 5 a slightly better bound.

4. Perfect matchings and triangulations

In this section we apply the approach of the previous section to derive an upper bound on the ratio between the number of plane perfect matchings and the number of triangulations. As already mentioned in Section 2, Kasteleyn's technique implies that a triangulation of a set of N points can contain at most $6^{N/4}$ perfect matchings (see [4]). This implies that for every set S of N points in the plane, $\operatorname{pm}(S)\leqslant 6^{N/4}\cdot\operatorname{tr}(S)\approx 1.5651^N\cdot\operatorname{tr}(S)$. We will improve this bound, using lower bounds on the supports of perfect matchings, in a manner similar to that in Section 3.

Before proceeding, we note the following lower bound on the ratio $\operatorname{pm}(S)/\operatorname{tr}(S)$. Let S be a *double circle configuration*, depicted in Fig. 6, consisting of N points (see [11] for a precise definition). An inclusion–exclusion argument implies that $\operatorname{tr}(S)=12^{N/2}\approx 3.464^N$ (see [11,21]). On the other hand, Aichholzer et al. [1] proved that $\operatorname{pm}(S)=\Theta^*(2.2^N)$. Therefore, in this case, $\operatorname{pm}(S)/\operatorname{tr}(S)\approx\Theta^*(0.635^N)$.

We now present an improved upper bound for this ratio.

Theorem 4.1. *For any set S of N points in the plane,*

$$\operatorname{pm}(S)\leqslant 8\cdot(3/2)^{N/4}\cdot\operatorname{tr}(S)=O\left(1.1067^N\right)\cdot\operatorname{tr}(S).$$

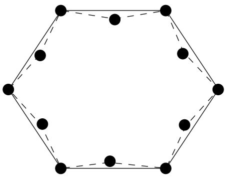


Fig. 6. A set of 12 points in a double circle configuration.

Proof. The exact value of $\mathsf{pm}(S)$ is

$$\mathsf{pm}(S) = \sum_{T \in \mathcal{T}(S)} \sum_{M \in \mathcal{M}(T)} \frac{1}{\mathsf{supp}(M)}. \tag{7}$$

Consider a triangulation $T \in \mathcal{T}(S)$ and a perfect matching $M \subseteq T$. As shown in the proof of Theorem 3.1, there exists a set of $N/2 - 3$ ps-flippable edges in $T \setminus M$. Therefore, by Lemma 2.2, the support of M is at least $2^{N/2-3}$. Combining this with (7) implies

$$\mathsf{pm}(S) \leqslant \sum_{T \in \mathcal{T}(S)} \sum_{M \in \mathcal{M}(T)} \frac{1}{2^{N/2-3}} \leqslant \sum_{T \in \mathcal{T}(S)} \frac{6^{N/4}}{2^{N/2-3}} = 8 \cdot (3/2)^{N/4} \cdot \mathsf{tr}(S). \qquad \square$$

Combining Theorem 4.1 with the bound $\mathsf{tr}(N) < 30^N$ from [22] implies $\mathsf{pm}(N) = O(33.201^N)$. As already mentioned above, this does not imply a new bound on $\mathsf{pm}(N)$, since Sharir and Welzl [24] showed that $\mathsf{pm}(N) = O(10.05^N)$, bypassing the approach of counting matchings *within* a triangulation. We are not aware of any construction for which $\mathsf{pm}(S) \geqslant \mathsf{tr}(S)$, and offer the conjecture that there exists a constant $c < 1$ such that $\mathsf{pm}(S) = O(c^{|S|} \cdot \mathsf{tr}(S))$ for every finite set S of points in the plane. (See also the conjecture concerning spanning cycles made in the introduction, which is probably stronger than this conjecture.)

5. An improved bound

In this section we present a more complex analysis for the number of spanning cycles, obtaining a slightly better bound than the one presented in Section 3. The analysis has three parts, each presented in a separate subsection.

Let us denote the number of interior vertices of degree 3 in the triangulation T as $v_3(T)$, and the number of flippable edges in T as $\mathsf{flip}(T)$. In Section 5.1 we give an upper bound for $\sum_{C \in \mathcal{C}(T)} \frac{1}{\mathsf{supp}(C)}$ that depends on $v_3(T)$. In Section 5.2 we give an alternative upper bound that depends on $\mathsf{flip}(T)$. Finally, in Section 5.3 we combine these two bounds to obtain

$$\mathsf{sc}(N) = O\left(1.8181^N\right) \cdot \mathsf{tr}(N) = O\left(54.543^N\right).$$

5.1. A $v_3(T)$-sensitive bound

In this subsection we derive the following bound, which is a function of N and $v_3(T)$.

Lemma 5.1. *Let T be a triangulation over a set S of $N \geqslant 6$ points in the plane, such that N is even and S has a triangular convex hull; also, let $v_3(T) = tN$. Then*

$$\sum_{C \in \mathcal{C}(T)} \frac{1}{\mathsf{supp}(C)} < 8\left(\frac{3}{2^t}\left(\frac{(2-t)(2-t/2)}{(1-t)^2}\right)^{1-t}\right)^{N/4}.$$

Proof. As before, we treat every spanning cycle as the union of two edge-disjoint perfect matchings $M_1, M_2 \in \mathcal{M}(T)$. We start by bounding the number of ways to choose the first perfect matching M_1.

For this, we use the standard variant of Kasteleyn's technique, with the weight function $\mu \equiv 1$ (i.e., the underlying graph G is unweighted).

Recall the inequality $\sum_{M \in \mathcal{M}(T)} \mu(M) \leqslant (\prod_{i=1}^{N} \|b_i\|_2^2)^{1/4}$ obtained in Eq. (5), where the b_i's (for $1 \leqslant i \leqslant N$) are the column vectors of the (signed) adjacency matrix of the oriented graph $\vec{T}$. Substituting $\mu \equiv 1$, the left-hand side becomes the number of perfect matchings in T, and the squared l_2-norm of each column vector is the degree of the vertex corresponding to that column. Since every column that corresponds to a vertex of degree 3 has a squared norm of 3, the product of the squared norms of these columns is $3^{v_3(T)} = 3^{tN}$.

For the remaining $N - v_3(T)$ columns, we use, as in Section 3, the arithmetic–geometric mean inequality to bound the product of their squared norms (as in Eq. (5)). This yields the bound

$$\left(\frac{X}{N - v_3(T)}\right)^{(N - v_3(T))/4} = \left(\frac{X}{N(1-t)}\right)^{(N(1-t))/4}, \tag{8}$$

where X is the sum of the degrees of all vertices other than those counted in $v_3(T)$. The sum of the degrees over the vertices of any specific triangulation is smaller than $6N$, and the sum of the degrees of the interior degree-3 vertices in T is $3v_3(T)$. Therefore, we have

$$X < 6N - 3v_3(T) = 3N(2-t). \tag{9}$$

Combining (8), (9), and the product of the squared norms that correspond to interior vertices of degree 3, implies that the number of ways to choose M_1 is less than

$$\left(3^t \cdot \left(\frac{3N(2-t)}{N(1-t)}\right)^{1-t}\right)^{N/4} = \left(3 \cdot \left(\frac{2-t}{1-t}\right)^{1-t}\right)^{N/4}. \tag{10}$$

Next, let us fix a specific perfect matching $M_1 \in \mathcal{M}(T)$. As shown at the beginning of the proof of Theorem 3.1, there exists a set F of $N/2 - 3$ ps-flippable edges in T, none of which belongs to M_1.

We continue as in the proof of Theorem 3.1, by assigning a weight of 1 to the edges of F and a weight of 1/2 to the rest of the edges of $T \setminus M_1$, recalling (2), and then applying (the weighted) Kasteleyn's technique to obtain the bound

$$\sum_{M_2 \in \mathcal{M}(T')} \frac{1}{\mathsf{supp}(M_1 \cup M_2)} \leqslant 8 \sum_{M_2 \in \mathcal{M}(T \setminus M_1)} \mu(M_2) \leqslant 8\left(\prod_{i=1}^{N} \|b'_i\|_2^2\right)^{1/4},$$

where b'_i are the column vectors of the oriented weighted adjacency matrix of $T \setminus M_1$.

An interior vertex v of degree 3 in T has only two edges adjacent to it in $T \setminus M_1$, both not in F (since an edge adjacent to an interior vertex of degree 3 cannot be flippable). Therefore, the squared norm of a matrix column that corresponds to such a vertex is $(1/2)^2 + (1/2)^2 = 1/2$, and the product of the squared norms of all such columns is $1/2^{v_3(T)} = 1/2^{tN}$.

For the remaining $N - v_3(T)$ columns, we may once again use the arithmetic–geometric mean inequality to obtain a bound similar to the one in (8). Namely

$$\left(\frac{Y}{N(1-t)}\right)^{(N(1-t))/4}, \quad \text{where } Y = \sum \|b'_i\|_2^2,$$

and the sum is over the $N - v_3(T)$ vertices of $T \setminus M_1$ which are not of degree 3 in T. Each such vertex contributes to Y the sum of the squared weights of its incident edges. The estimate for Y will therefore be different, since (i) some of the edges of T were removed, and (ii) the weight function μ is not identically 1. The edges of F have remained and still have a weight of 1 each, so they contribute at most $2 \cdot (N/2 - 3) \cdot 1 < N$ to Y. Every other edge contributes $2 \cdot 1/4 = 1/2$ if it is not incident to an interior vertex of degree 3 in T, and 1/4 otherwise. Since a triangulation has fewer than $3N - 3$ edges, there are fewer than $2N$ edges in $T \setminus \{F \cup M_1\}$, and we get

$$Y < N + \bigl(2N - 2v_3(T)\bigr) \cdot \frac{1}{2} + 2v_3(T) \cdot \frac{1}{4} = 2N - \frac{v_3(T)}{2} = N(2 - t/2).$$

By combining this with the rest of the squared norms and with the present version of (8), we have

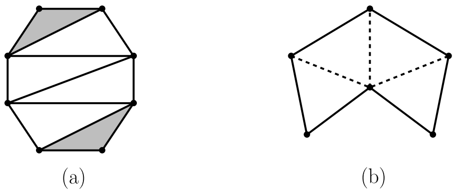


Fig. 7. (a) The shaded faces are the two ears of the triangulated polygon. (b) A polygon with three flippable diagonals, two of which form a ps-flippable set, and with $5 = 2^2(5/4)^1$ triangulations.

$$\sum_{M_2 \in \mathcal{M}(T \setminus M_1)} \frac{1}{\mathsf{supp}(M_1 \cup M_2)} < 8\left(\frac{1}{2^t} \cdot \left(\frac{N(2-t/2)}{N(1-t)}\right)^{1-t}\right)^{N/4} = 8\left(\frac{1}{2^t} \cdot \left(\frac{2-t/2}{1-t}\right)^{1-t}\right)^{N/4}. \tag{11}$$

Finally, to complete the proof, we combine (10) and (11), and obtain

$$\begin{aligned}\sum_{C \in \mathcal{C}(T)} \frac{1}{\mathsf{supp}(C)} &\leqslant \sum_{\substack{M_1, M_2 \in \mathcal{M}(T) \\ M_1, M_2 \text{ edge-disjoint}}} \frac{1}{\mathsf{supp}(M_1 \cup M_2)} \\ &< \left(3 \cdot \left(\frac{2-t}{1-t}\right)^{1-t}\right)^{N/4} \cdot 8\left(\frac{1}{2^t} \cdot \left(\frac{2-t/2}{1-t}\right)^{1-t}\right)^{N/4} \\ &= 8\left(\frac{3}{2^t}\left(\frac{(2-t)(2-t/2)}{(1-t)^2}\right)^{1-t}\right)^{N/4}. \qquad \square\end{aligned}$$

Remark. Notice that in the worst case (i.e., when $t = 0$) we obtain the same asymptotic value as in our initial bound of $12^{N/4}$. Similarly, the bound in (10) becomes $6^{N/4}$ when $t = 0$, as in Buchin et al. [4].

5.2. *A* flip(T)*-sensitive bound*

Hurtado, Noy, and Urrutia [12] proved that $\mathsf{flip}(T) \geqslant N/2 - 2$, and that this bound is tight in the worst case (the upper bound is also implied by Lemma 2.1; see also [8]). In this subsection we obtain a bound on $\sum_{C \in \mathcal{C}(T)} \frac{1}{\mathsf{supp}(C)}$ as a function of $\mathsf{flip}(T)$, which improves our initial bound of $12^{N/4}$ when $\mathsf{flip}(T)$ is larger than $N/2$ by some positive fraction of N.

We start by mentioning two basic properties of triangulated simple (not necessarily convex) polygons.

Ears. Given a triangulated polygon P, an *ear* of P is a bounded face of the triangulation with two of its edges on the boundary of P. It can easily be shown that every triangulated simple polygon with at least four edges contains at least two ears (whose boundary edges are all distinct). For example, the two ears of the triangulated polygon in Fig. 7(a) are shown shaded.

Catalan numbers. The N-th Catalan number is $C_N = \frac{1}{N+1}\binom{2N}{N}$. It is well known that a convex polygon with N vertices has C_{N-2} triangulations (e.g., see [26, Section 5.3]). Therefore, the number of triangulations of a convex polygon with $d \geqslant 1$ diagonals is

$$C_{d+1} = \frac{1}{d+2}\binom{2d+2}{d+1} \geqslant 2^d\left(\frac{5}{4}\right)^{d-1} \tag{12}$$

(the inequality can be easily verified by induction). Note that equality holds for $d = 1$ and $d = 2$.

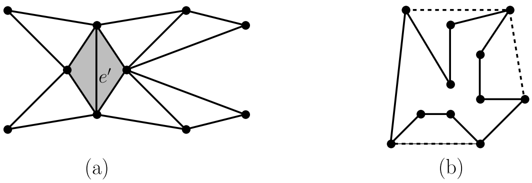


Fig. 8. (a) Removing the two (shaded) triangles incident to e' produces at most four triangulated sub-polygons. (b) A spanning cycle partitions the convex hull of the point set into interior-disjoint polygons.

We define c_{gon} as the maximum real number satisfying the following property. Every simple polygon P that has a triangulation T_P with k of its diagonals flippable and with $l \leqslant k$ of these diagonals forming a ps-flippable set, has at least $2^l c_{\text{gon}}^{k-l}$ triangulations. Notice that the triangulations under consideration, including T_P, are triangulations of the polygon P, and not of its vertex set. Note also that we can have the equality $l = k$ only when P is convex.

Lemma 5.2. *Let $\alpha \approx 1.17965$ be the unique real root of the polynomial $1 + 4x^2 - 4x^3$. Then $\alpha \leqslant c_{\text{gon}} \leqslant 5/4$. That is, the left inequality means that every simple polygon P that has a triangulation T_P with k of its diagonals flippable and with $l \leqslant k$ of these diagonals forming a ps-flippable set, has at least $2^l \alpha^{k-l}$ triangulations.*

Proof. Fig. 7(b) depicts a polygon that implies the upper bound.

We prove the lower bound by induction on l and k. For the base case of this induction, notice that when $l = k$, P has at least $C_{k+1} \geqslant 2^k = 2^l \alpha^{k-l}$ triangulations.

Next, consider a polygon P and a triangulation T_P of P, such that T_P contains a maximal set F of l ps-flippable edges, and $k - l > 0$ flippable edges not in F. Let $f_1, f_2, \ldots, f_j$ be the non-triangular faces of the convex subdivision $T_P \setminus F$. (Notice that $T_P \setminus F$ is a convex subdivision of a polygon, and not of the convex hull of a point set.) Each f_i is a convex polygon with $m_i \geqslant 4$ sides and $m_i - 3$ diagonals, and thus, $l = \sum_{i=1}^{j}(m_i - 3)$.

A flippable edge not in F must be on the boundary of some f_i, since otherwise F is not maximal. We say that such a flippable edge $e \notin F$ *is covered by a face* f_i if (i) e is on the boundary of f_i, and (ii) the triangle Δ in T_P that is contained in f_i and incident to e has two edges in F (so e is the only edge of Δ on the boundary of f_i). Ears are incident to two edges of their containing f_i, and thus, edges of an ear cannot be covered by the face of the ear. Since any triangulated non-triangular polygon contains at least two ears, with four distinct boundary edges, a polygon with $m \geqslant 4$ sides (and $m - 3$ diagonals) can cover at most $m - 4$ edges. Therefore, if all flippable edges not in F are covered, then $k - l \leqslant \sum_{i=1}^{j}(m_i - 4)$. By multiplying the number of triangulations of the f_i's and applying (12), we get that the number of triangulations of P that contain $T_P \setminus F$ is at least

$$\prod_{i=1}^{j} 2^{m_i-3}\left(\frac{5}{4}\right)^{m_i-4} = 2^{\sum_{i=1}^{j}(m_i-3)}\left(\frac{5}{4}\right)^{\sum_{i=1}^{j}(m_i-4)} \geqslant 2^l\left(\frac{5}{4}\right)^{k-l} \geqslant 2^l \alpha^{k-l}.$$

We are left with the case where there is a flippable edge $e \notin F$ that is not covered by any f_i. Let e' be the edge obtained by flipping e. We now derive a lower bound on the number of triangulations of P that contain e and on the number of triangulations of P that contain e'. To bound the number of triangulations that contain e, we partition P into two interior disjoint simple polygons P', P'' by "cutting" P at e. More precisely, we consider the two triangulated polygons $T_{P'}, T_{P''} \subset T_P$. Together, these two triangulated polygons contain $k - 1$ diagonals that are flippable. Moreover, the set F remains a set of a total of l ps-flippable edges, some being diagonals of P' and some of P''. Thus, the induction hypothesis implies that there are at least $2^l \alpha^{k-1-l}$ triangulations of P that contain e. To obtain a similar bound for the number of triangulations containing e', we produce at most four triangulated sub-polygons of T_P by removing the two triangles incident to e (whose union forms the same quadrilateral as the union of the two triangles incident to e' in the new triangulation). Such a case is illustrated in Fig. 8(a), where the triangles incident to e' are shaded. This partitioning may cancel the flippability of at most five edges (those incident to the two triangles adjacent to e). At most two out of the five edges may belong to F, since e can only be incident to ears of polygons of $T_P \setminus F$. Using the induction hypothesis again, we get that there are at least $2^{l-2}\alpha^{(k-5)-(l-2)} = 2^{l-2}\alpha^{k-l-3}$ triangulations that contain e' (and the four edges around it); in the last estimate we have used the fact that $2^{l-2}\alpha^{k-l-3}$ is increasing in l. Therefore, P has at least

$$2^l\alpha^{k-1-l} + 2^{l-2}\alpha^{k-l-3} = 2^l\alpha^{k-l}\left(\frac{1}{\alpha} + \frac{1}{4\alpha^3}\right) = 2^l\alpha^{k-l} \cdot \frac{4\alpha^2+1}{4\alpha^3} = 2^l\alpha^{k-l}$$

triangulations, as asserted (recall that α is the root of the polynomial $1 + 4x^2 - 4x^3$). $\square$

Next, we show how to use c_{gon} (or rather, its lower bound α) and $\mathsf{flip}(T)$ to upper bound $\sum_{C\in\mathcal{C}(T)} \frac{1}{\mathsf{supp}(C)}$.

Lemma 5.3. *Consider a triangulation T with $\mathsf{flip}(T) = N/2 - 3 + \kappa N$, for some $\kappa \geqslant 0$, and let α be the constant in Lemma 5.2. Then*

$$\sum_{C\in\mathcal{C}(T)} \frac{1}{\mathsf{supp}(C)} < 8\left(\frac{(3+(\gamma^2-1)(\kappa+1/2))(4+(\alpha^2-1)\kappa)}{\alpha^{4\kappa}}\right)^{N/4},$$

where

$$\gamma = \alpha \cdot e^{-\frac{\alpha^2-1}{4(4+(\alpha^2-1)\kappa)}}.$$

Proof. Once again, we treat every spanning cycle as the union of a pair of edge-disjoint perfect matchings $M_1, M_2 \in \mathcal{M}(T)$, and use Kasteleyn's technique (as presented in Section 3) to bound the number of such pairs. We start by fixing some perfect matching $M_1 \in \mathcal{M}(T)$ and denote the number of flippable edges of T that are in M_1 as $\mathsf{flip}_T(M_1)$. As shown in the proof of Theorem 3.1, there is a set of at least $N/2 - 3$ ps-flippable edges in $T \setminus M_1$. We restrict our attention to a set F of *exactly* $N/2 - 3$ ps-flippable edges in $T \setminus M_1$.

For analyzing the complementary matchings M_2, we define a weight function $\mu(\cdot)$ on the edges of $T \setminus M_1$, such that

$$\mu(e) = \begin{cases} 2, & \text{if } e \in F, \\ \alpha, & \text{if } e \notin F \text{ is flippable}, \\ 1, & \text{if } e \text{ is not flippable}. \end{cases}$$

Notice that any spanning cycle partitions the convex hull of its point set into interior-disjoint simple polygons. The support of the spanning cycle is the product of the number of triangulations of each of these polygons. For a fixed choice of M_2 (and of M_1), denote by $P_1, \ldots, P_m$ the polygons in the partition produced by $M_1 \cup M_2$ (assuming that $M_1 \cup M_2$ is indeed a spanning cycle). For each i, let k_i be the number of flippable diagonals of P_i, and let l_i be the number of those diagonals (among the k_i flippable ones) that belong to F. If M_2 uses $\mathsf{flip}_T(M_2)$ flippable edges of $T \setminus M_1$, l of which are in F, then $\sum_{i=1}^m k_i = \mathsf{flip}(T) - \mathsf{flip}_T(M_1) - \mathsf{flip}_T(M_2)$ and $\sum_{i=1}^m l_i = |F| - l = N/2 - 3 - l$. Applying Lemma 5.2 to each P_i and multiplying the resulting bounds, we obtain a total of at least $2^{\sum l_i}\alpha^{\sum k_i - \sum l_i}$ triangulations. Hence,

$$\mathsf{supp}(M_1 \cup M_2) \geqslant (2/\alpha)^{\sum l_i}\alpha^{\sum k_i} = (2/\alpha)^{N/2-3-l}\alpha^{\mathsf{flip}(T)-\mathsf{flip}_T(M_1)-\mathsf{flip}_T(M_2)}.$$

Next, notice that $\mu(M_2) = 2^l\alpha^{\mathsf{flip}_T(M_2)-l}$, so we have

$$\mathsf{supp}(M_1 \cup M_2) \geqslant \frac{2^{N/2-3}\alpha^{\mathsf{flip}(T)-\mathsf{flip}_T(M_1)-(N/2-3)}}{\mu(M_2)} = \frac{2^{N/2-3}\alpha^{\kappa N-\mathsf{flip}_T(M_1)}}{\mu(M_2)}. \tag{13}$$

By combining (13) with Kasteleyn's method, we obtain

$$\sum_{M_2 \in \mathcal{M}(T \setminus M_1)} \frac{1}{\mathsf{supp}(M_1 \cup M_2)} \leqslant \sum_{M_2 \in \mathcal{M}(T \setminus M_1)} \frac{\mu(M_2)}{2^{N/2-3}\alpha^{\kappa N - \mathsf{flip}_T(M_1)}}$$
$$\leqslant \frac{1}{2^{N/2-3}\alpha^{\kappa N - \mathsf{flip}_T(M_1)}} \cdot \left(\frac{2}{N}\sum_{e \in T \setminus M_1} \mu(e)^2\right)^{N/4}. \tag{14}$$

To bound the sum in the parentheses, we notice that $T \setminus M_1$ contains exactly $N/2 - 3$ edges of F, exactly $\kappa N - \mathsf{flip}_T(M_1)$ flippable edges not in F, and fewer than $2N - (\kappa N - \mathsf{flip}_T(M_1))$ non-flippable edges. Therefore,

$$\frac{2}{N}\sum_{e \in T \setminus M_1} \mu(e)^2 < \frac{2}{N}\left(1^2 \cdot \left(2N - \left(\kappa N - \mathsf{flip}_T(M_1)\right)\right) + \alpha^2 \cdot \left(\kappa N - \mathsf{flip}_T(M_1)\right) + 2^2 \cdot N/2\right)$$
$$= 8 + 2\left(\alpha^2 - 1\right)\kappa - 2\left(\alpha^2 - 1\right) \cdot \mathsf{flip}_T(M_1)/N$$
$$= \left(8 + 2\left(\alpha^2 - 1\right)\kappa\right)\left(1 - \frac{2(\alpha^2 - 1)}{8 + 2(\alpha^2 - 1)\kappa} \cdot \frac{\mathsf{flip}_T(M_1)}{N}\right)$$
$$\leqslant \left(8 + 2\left(\alpha^2 - 1\right)\kappa\right) \cdot e^{-\frac{\alpha^2 - 1}{4 + (\alpha^2 - 1)\kappa} \cdot \frac{\mathsf{flip}_T(M_1)}{N}} \quad \left(\text{using } 1 - u \leqslant e^{-u} \text{ for } u \geqslant 0\right)$$
$$= \left(8 + 2\left(\alpha^2 - 1\right)\kappa\right) \cdot (\gamma/\alpha)^{4 \cdot \mathsf{flip}_T(M_1)/N}. \tag{15}$$

Combining (14) and (15), we get

$$\sum_{C \in \mathcal{C}(T)} \frac{1}{\mathsf{supp}(C)} \leqslant \sum_{\substack{M_1, M_2 \in \mathcal{M}(T) \\ M_1, M_2 \text{ edge-disjoint}}} \frac{1}{\mathsf{supp}(M_1 \cup M_2)}$$
$$\leqslant \sum_{M_1 \in \mathcal{M}(T)} \frac{1}{2^{N/2-3}\alpha^{\kappa N - \mathsf{flip}_T(M_1)}}\left(\left(8 + 2\left(\alpha^2 - 1\right)\kappa\right) \cdot (\gamma/\alpha)^{4 \cdot \mathsf{flip}_T(M_1)/N}\right)^{N/4}$$
$$= 8\left(\frac{8 + 2(\alpha^2 - 1)\kappa}{4\alpha^{4\kappa}}\right)^{N/4} \sum_{M_1 \in \mathcal{M}(T)} \gamma^{\mathsf{flip}_T(M_1)}. \tag{16}$$

To bound the sum in (16), we once again use Kasteleyn's technique. This time, we define a weight function $\nu(\cdot)$ over the edges of T, such that every flippable edge gets a weight of γ, and every other edge a weight of 1. Notice that, in this manner, $\nu(M_1) = \gamma^{\mathsf{flip}_T(M_1)}$ for every $M_1 \in \mathcal{M}(T)$. We thus have

$$\sum_{M_1 \in \mathcal{M}(T)} \gamma^{\mathsf{flip}_T(M_1)} \leqslant \left(\frac{2}{N}\sum_{e \in T} \nu(e)^2\right)^{N/4}$$
$$< \left(\frac{2}{N}\left(\gamma^2 \cdot \mathsf{flip}(T) + 1 \cdot \left(3N - \mathsf{flip}(T)\right)\right)\right)^{N/4}$$
$$< \left(6 + \frac{2}{N}\left(\gamma^2 - 1\right)(N/2 + \kappa N)\right)^{N/4}$$
$$< \left(6 + 2\left(\gamma^2 - 1\right)(\kappa + 1/2)\right)^{N/4}. \tag{17}$$

Finally, combining (16) and (17) implies the assertion of the lemma. □

Note that in the worst case, when $\kappa = 0$, the bound becomes $O((10 + 2\gamma^2)^{N/4})$. For $\kappa = 0$, we have $\gamma = \alpha \cdot e^{-(\alpha^2 - 1)/16}$, and it is easy to verify that $\gamma > 1$ for $1 < \alpha \leqslant 5/4$. So the bound is actually asymptotically *worse* than our initial bound of $12^{N/4}$, and it continues to be worse when κ is sufficiently small. As the next subsection shows, in this case the v_3-dependent bound from Section 5.1 becomes small and can be used instead.

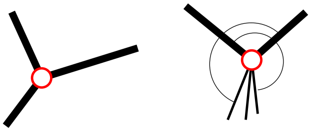


Fig. 9. Separable edges.

5.3. Integration

In this subsection we combine the results from the two previous subsections to obtain an improved bound for $\mathsf{sc}(N)$. This is done by deriving a connection between $v_3(T)$ and $\mathsf{flip}(T)$. We start by presenting a generalization of Lemma 2.4.

Lemma 5.4. *Let $c > 1$ be a constant such that every set S of an even number of points in the plane and a triangular convex hull satisfies $\mathsf{sc}(S) = O(c^{|S|})$. Then $\mathsf{sc}(S) = O(c^{|S|})$ also holds for every other finite point set S in the plane.*

Proof. Consider a point set S. If S has an even number of points, we pick a new point p outside the convex hull of S, and put $S' = S \cup \{p\}$. As mentioned in the proof of Lemma 2.4, inserting an additional vertex outside the convex hull of the point set can only increase the number of spanning cycles. If S has an odd number of points, we put $S' = S$. Notice that, either way, S' has an odd number of points. Let $\triangle abc$ be a large triangle containing S' in its interior, and let $S'' = S' \cup \{a, b, c\}$. Again, since inserting an additional vertex outside the convex hull of the point set can only increase the number of spanning cycles, we have $\mathsf{sc}(S') \leqslant \mathsf{sc}(S'')$. Since S'' has an even number of points and a triangular convex hull, $\mathsf{sc}(S) \leqslant \mathsf{sc}(S') \leqslant \mathsf{sc}(S'') = O(c^{N+4}) = O(c^{|S|})$. $\square$

We also require the notion of *separable edges*, as presented in [23]. Consider a point set S, a triangulation $T \in \mathcal{T}(S)$, and an interior point $p \in S$. We call an edge e incident to p in T a *separable edge at p* if it can be separated from the other edges incident to p by a line through p. An equivalent condition is that the two angles between e and its clockwise and counterclockwise neighboring edges (around p) sum up to more than π. We observe the following easy properties (see Fig. 9 for an illustration).

(S0) No edge is separable at both vertices induced by its endpoints.
(S1) If p has degree 3 in T, every edge incident to it is separable (recall that p is an interior point).
(S2) If p has degree at least 4 in T, at most two incident edges can be separable at w.
(S3) If p has degree at least 4 in T and there are two edges separable at p, then they must be consecutive in the order around it.

We are now ready for the main theorem of the section.

Theorem 5.5. *For any set S of N points in the plane,*

$$\mathsf{sc}(S) = O\left(10.9247^{N/4}\right) \cdot \mathsf{tr}(S) = O\left(1.8181^N\right) \cdot \mathsf{tr}(S) = O\left(54.543^N\right).$$

Proof. By Lemma 5.4, we may assume that N is even and that S has a triangular convex hull. Recall that

$$\mathsf{sc}(S) = \sum_{T \in \mathcal{T}(S)} \sum_{C \in \mathcal{C}(T)} \frac{1}{\mathsf{supp}(C)}.$$

We sort the triangulations in the first sum according to the number of interior vertices of degree 3 that they contain, and get

$$\mathrm{sc}(S)=\sum_{i=0}^{(2N+1)/3}\sum_{\substack{T\in\mathcal{T}(S)\\ v_3(T)=i}}\sum_{C\in\mathcal{C}(T)}\frac{1}{\mathrm{supp}(C)}. \tag{18}$$

(The fact that $v_3(T)\leqslant(2N+1)/3$ for every triangulation T is established, e.g., in [25].) Given a triangulation T with $v_3(T)=i$, we can use Lemma 5.1 to bound $\sum_{C\in\mathcal{C}(T)}\frac{1}{\mathrm{supp}(C)}$. However, when $v_3(T)$ is small, the improvement in Lemma 5.1 is not significant. In this case we will use instead the bound in Lemma 5.3 which, as we now proceed to show, becomes significant when $v_3(T)$ is small.

Consider a triangulation $T\in\mathcal{T}(S)$. Since S has a triangular convex hull, T contains $3N-9$ interior edges. Notice that an interior edge e is flippable if and only if e is not separable at either of its endpoints (this property is equivalent to e being a diagonal of a convex quadrilateral). From the above properties of separable edges, we have

$$\mathrm{flip}(T)\geqslant\overbrace{3N-9}^{\text{Interior edges}}-3\cdot\overbrace{v_3(T)}^{\text{Interior vertices of degree 3}}-2\cdot\overbrace{\big(N-v_3(T)-3\big)}^{\text{Other interior vertices}}=N-3-v_3(T).$$

To find for which values of i it is better to use Lemma 5.1, and for which values it is better to use Lemma 5.3, we define $t=v_3(T)/N$ and

$$\kappa=\frac{\mathrm{flip}(T)-(N/2-3)}{N}\geqslant\frac{(N-tN-3)-(N/2-3)}{N}=1/2-t,$$

and solve the equation

$$8\left(\frac{3}{2^t}\left(\frac{(2-t)(2-t/2)}{(1-t)^2}\right)^{1-t}\right)^{N/4}=8\left(\frac{(3+(\gamma^2-1)(\kappa+1/2))(4+(\alpha^2-1)\kappa)}{\alpha^{4\kappa}}\right)^{N/4},$$

with $\kappa=1/2-t$, where $\alpha\approx1.17965$ and $\gamma=\alpha\cdot e^{-\frac{\alpha^2-1}{4(4+(\alpha^2-1)\kappa)}}$; this will determine the threshold where the two bounds coincide. That is, we need to solve the equation (again, with $\kappa=1/2-t$)

$$\frac{3}{2^t}\left(\frac{(2-t)(2-t/2)}{(1-t)^2}\right)^{1-t}=\frac{(3+(\gamma^2-1)(\kappa+1/2))(4+(\alpha^2-1)\kappa)}{\alpha^{4\kappa}}.$$

For this, we use the Wolfram Mathematica software [27], and obtain the solution $t\approx0.1072$. Moreover, it is easily shown that for $i\geqslant0.1072N$ the bound from Lemma 5.1 is smaller, and for $i\leqslant0.1072N$ the bound from Lemma 5.3 is smaller. In fact, these bounds, in their appropriate ranges, are all dominated by the common bound for $t\approx0.1072$, which is $10.9247^{N/4}$. This, together with (18), Lemma 5.1, and Lemma 5.3 imply the asserted bound. □

By combining Theorem 5.5 with the bound $\mathrm{tr}(N)<30^N$ [22], we obtain:

Corollary 5.6. $\mathrm{sc}(N)=O(54.543^N)$.

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