Conflict-Free Coloring for Rectangle Ranges Using $O(n^{.382})$ Colors

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ABSTRACT

Given a set of points $P \subseteq \mathbb{R}^2$, a *conflict-free coloring* of P w.r.t. rectangle ranges is an assignment of colors to points of P, such that each non-empty axis-parallel rectangle T in the plane contains a point whose color is distinct from all other points in $P \cap T$. This notion has been the subject of recent interest, and is motivated by frequency assignment in wireless cellular networks: one naturally would like to minimize the number of frequencies (colors) assigned to bases stations (points), such that within any range (for instance, rectangle), there is no interference. We show that any set of n points in $\mathbb{R}^2$ can be conflict-free colored with $\tilde{O}(n^{\beta+\epsilon})$ colors in expected polynomial time, for any arbitrarily small $\epsilon > 0$ and $\beta = \frac{3-\sqrt{5}}{2} < 0.382$. This improves upon the previously known bound of $O(\sqrt{n \log\log n / \log n})$.

Categories and Subject Descriptors

G.2 [**Combinatorics**]: Combinatorial algorithms

General Terms

Algorithms, Theory

Keywords

Frequency assignment in wireless networks, conflict-free coloring, axis-parallel rectangles, dominating sets, monotone sequences

1. INTRODUCTION

The study of conflict-free coloring is motivated by the frequency assignment problem in wireless networks. A wireless network is a heterogeneous network consisting of *base stations* and *agents*. The base stations have a fixed location, and are interlinked via a fixed backbone network, while the agents are typically mobile and can connect to the base stations via radio links. The base stations are assigned fixed frequencies to enable links to agents. The agents can connect to any base station, provided that the radio link to that particular station has good reception. Good reception is only possible if i) the base station is located within range, and ii) no other base station within range of the agent has the same frequency assignment (to avoid interference). Thus the fundamental problem of frequency-assignment in cellular networks is to assign frequencies to base stations, such that an agent can always find a base station with unique frequency among the base stations in its range. Naturally, due to cost, flexibility and other restrictions, one would like to minimize the total number of assigned frequencies.

The study of the above problem was initiated in [8], and continued in a series of recent papers [2, 3, 4, 5, 7, 9, 10, 12, 13]. It can be formally described as follows. Let $P \subseteq \mathbb{R}^2$ be a set of points and $\mathcal{R}$ be a set of ranges (e.g. the set of all discs or rectangles in the plane). A *conflict-free* coloring (CF-coloring in short) of P w.r.t. the range $\mathcal{R}$ is an assignment of a color to each point $p \in P$ such that for any range $T \in \mathcal{R}$ with $T \cap P \neq 0$, the set $T \cap P$ contains a point of unique color. Naturally, the goal is to assign a conflict-free coloring to the points of P with the *smallest* number of colors possible.

The work in [8] presented a general framework for computing a conflict-free coloring for several types of ranges. In particular, for the case where the ranges are discs in the plane, they present a polynomial time coloring algorithm that uses $O(\log n)$ colors for conflict-free coloring and this bound is shown to be tight. This result was then extended in [10] by considering the case where the ranges are axis-parallel rectangles in the plane. This seems much harder than the disc case, and the work in [10] presented a simple algorithm that uses $O(\sqrt{n})$ colors. As mentioned in [10], this can be further improved to $O(\sqrt{n \log\log n / \log n})$ using the sparse neighborhood property of the conflict-free graph, as independently observed by Noga Alon, Timothy Chan, and János Pach and Geza Tóth [1, 12]. Currently, this is the best known upper bound for CF-coloring axis-parallel rectangles. A lower bound of $\Omega(\log n)$ trivially follows from the lower bound for intervals. Very recently, Chen et al. [6] showed that there exists a set of n points for which the maximum size of an independent set in the conflict-free graph is $O(n \log^2 \log n / \log n)$, suggesting that the number of colors in any conflict-free coloring is likely to be at least superlinear in $\log n$.

Recent works have shown that one can obtain better upper bounds for special cases of this problem. In [10], it was shown that for the case of random points in a unit square, $O(\log^4 n)$ colors suffice, and for points lying in an *exact* uniform $\sqrt{n} \times \sqrt{n}$ grid, $O(\log n)$ colors are sufficient. Chen [4] showed that polylogarithmic number of colors suffice for the case of *nearly equal* rectangle ranges. Elbassioni and Mustafa [7] asked the following question: Given a set of points P in the plane, can we insert new points Q such that the conflict free coloring of $P \cup Q$ requires fewer colors? They showed that by inserting $O(n^{1-\epsilon})$ points, $P \cup Q$ can be conflict free colored using $\tilde{O}(n^{3(1+\epsilon)/8})$ colors.

While the CF-coloring problem is closed for disc ranges, the upper bounds are very far from the currently known lower bounds for axis-parallel rectangular ranges. It remains very interesting to reduce this gap between upper and lower bounds, and this is, in fact, the main open problem posed in [10]. In this paper, we improve the upper bound significantly.

THEOREM 1.1. *Any set of n points in $\mathbb{R}^2$ can be conflict-free colored with respect to rectangle ranges using $\tilde{O}(n^{\beta+\epsilon})$ colors, in expected polynomial time, for any arbitrarily small $\epsilon > 0$ and $\beta = \frac{3-\sqrt{5}}{2} < 0.382$.*

Our main tool for proving this theorem is a probabilistic coloring technique, introduced in [7], that can be used to get a coloring with weaker properties, which we call *quasi-conflict-free* coloring. This will be combined with dominating sets, monotone sequences, and careful griding of the point set, in a recursive way, to obtain the claimed result. We start with some definitions and preliminaries in Section 2. To illustrate our ideas, we sketch a simple $\tilde{O}(n^{6/13})$ conflict free coloring algorithm in Section 3. The main algorithm will be given in Section 4. We describe the quasi-conflict-free coloring technique in a slightly more general form in Section 5. Section 6 contains the analysis of the main algorithm.

2. PRELIMINARIES

By $\mathcal{R} \subseteq 2^{\mathbb{R}^2}$, we denote the set of all *axis-parallel* rectangles. Let P be a set of points in $\mathbb{R}^2$.

DEFINITION 2.1. (Conflict-free coloring) *A coloring of P is a function $\chi : P \mapsto \mathcal{N}$ from P to some finite set $\mathcal{N}$. A rectangle $T \in \mathcal{R}$ is said to be* conflict-free *with respect to a coloring χ if either $T \cap P = \emptyset$, or there exists a point $p \in P \cap T$ such that $\chi(p) \neq \chi(p')$ for all $p \neq p' \in P \cap T$. A coloring χ is said to be* conflict-free *(w.r.t. $\mathcal{R}$) if every rectangle $T \in \mathcal{R}$ is conflict-free w.r.t. χ.*

DEFINITION 2.2. (Dominating sets) *For a point $p = (p^x, p^y) \in \mathbb{R}^2$, define $W_1(p) = \{q \in \mathbb{R}^2 | q^x \geq p^x, q^y \geq p^y\}$ to be the upper right quadrant defined by p. Similarly, let $W_2(p)$, $W_3(p)$ and $W_4(p)$ be the upper left, lower right and lower left quadrants respectively. Define the* dominating set *of type i for P, denoted by $D_i(P)$, $1 \leq i \leq 4$, as follows:*

$$D_i(P) = \{p \in P | W_i(p) \cap P = \{p\}\}$$

DEFINITION 2.3. (Monotonic sets) *Let $P = \{p_1, p_2, \ldots, p_k\}$ be a set of points that is sorted by x coordinate. P is monotonic non-decreasing (resp. monotonic non-increasing) if $p_j^y \geq p_i^y$ (resp. $p_j^y \leq p_i^y$) $\forall 1 \leq i, j \leq k, j > i$.*

It is easy to see that the dominating set of type 2 and 3 (resp. type 1 and 4) are monotonic non-decreasing (resp. non-increasing); see Figure 1.

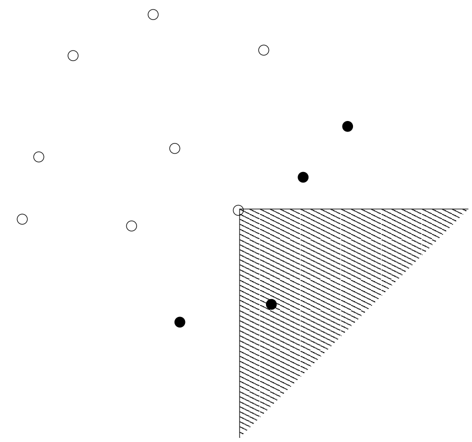

Figure 1: Dominating sets: the shaded region represents the lower right quadrant, and the solid black points represent the dominating set $D_3(P)$ of type 3.

DEFINITION 2.4. (r-Grid) *Let $r \in \mathbb{Z}_+$ be a positive integer. An r-grid on P (see Figure 2), denoted by $G_r = G_r(P)$, is an $r \times r$ axis-parallel grid containing all points of P. For $i = 1, \ldots, r$, denote by R_i and C_i the subsets of P lying in the ith row and column of G_r, respectively. Denote by $B(G_r)$, the maximum number of points of P in a row or a column of G_r. For $1 \leq h \leq 2r-1$, let M_h^1 (resp. M_h^2) be the set of grid cells lying along a diagonal h of positive slope (resp. negative slope) in G_r. For $l = 2, 3$ (resp. $l = 1, 4$), let $\mathcal{D}_l^h = \cup_{(i,j) \in M_h^1} D_l(R_i \cap C_j)$ (resp. $\mathcal{D}_l^h = \cup_{(i,j) \in M_h^2} D_l(R_i \cap C_j)$) be the union of dominating sets of type l over grid cells in M_h^1 (resp. M_h^2). Let $\mathcal{D}_l = \cup_{(i,j) \in G_r} D_l(R_i \cap C_j)$ be the union of dominating sets of type l over all the grid cells in G_r.*

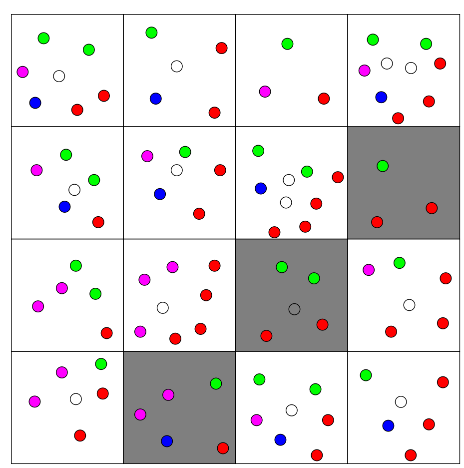

Figure 2: r-grid $G_r(P)$: $r = 4$, $B(G_r) = 24$, the four types of dominating sets are shown as solid circles in four different colors, and the remaining points are shown as hollow circles. The shaded cells represent the set M_h^1.

Note that, for $l = 2, 3$ and $1 \le h \le 2r - 1$, $\mathcal{D}_l^h$ is monotonic non-decreasing, since the grid cells in M_h^1, which lie along the diagonal of positive slope, are horizontally and vertically separated and hence the union of $D_l(R_i \cap C_j)$ (which are monotonic non-decreasing), is also monotonic non-decreasing. By similar argument, for $l = 1, 4$ with M_h^2 and $1 \le h \le 2r - 1$, $\mathcal{D}_l^h$ is monotonic non-increasing.

DEFINITION 2.5. (Quasi-conflict-free coloring) *Given a grid $G_r = G_r(P)$ on P, we call a coloring $\chi : P \mapsto \mathcal{N}$* quasi-conflict-free *with respect to G_r, if every axis-parallel rectangle which contains points only from the same row or the same column of G_r is conflict-free.*

Let G_r be an r-grid on a point set P such that $B(G_r) = B$. It is shown in [7] that there exists a quasi-conflict-free coloring of $G_r(P)$ requiring $\tilde{O}(B^{3/4})$ colors.

3. A SIMPLE $\tilde{O}(n^{6/13})$ CONFLICT-FREE COLORING ALGORITHM

In this section, we sketch a simple algorithm for CF-coloring P in order to illustrate the main ideas. This algorithm CF-colors P using $\tilde{O}(n^{6/13})$ colors. We can assume w.l.o.g. that P has no monotone sequences of size $\Omega(n^{7/13})$. If there is one, we pick every other point in the sequence (this is a set I of size $\Omega(n^{7/13})$), color them all with one color, and recurse on the rest of the points with a different set of colors. It is easy to show that this gives an $O(n^{6/13})$ CF-coloring if such a monotone sequence always exists (see [10] for more details).

Let $\mathcal{A}$ be an $O(n^{1/2})$ conflict-free coloring algorithm [10]. Our algorithm can be described as follows. Let $r = n^{\frac{5}{13}}$. Grid the point set P using G_r such that each row and column has $B = n^{\frac{8}{13}}$ points of P. Compute the dominating sets $\mathcal{D}_l(P), 1 \le l \le 4$ and let $D = \cup_{l=1}^{4} D_l(P)$ and $P' = P \setminus D$. We quasi-CF color P' with $\tilde{O}(B^{3/4})$ colors using the algorithm of [7] (which uses $\mathcal{A}$ as subroutine). Then, we CF-color D using $\mathcal{A}$ with a different set of colors.

LEMMA 3.1. *The above coloring of P is conflict-free.*

PROOF. Let $T \in \mathcal{R}$ be a rectangle such that $T \cap P \neq \emptyset$. We show that T contains a point of unique color among the points in $T \cap P$.

We consider 4 cases:

Case 1. A monotone sequence of size $\Omega(n^{7/13})$ is found and we colored every other point in the sequence (set I) with one color: if $(T \cap P) \setminus I \neq \emptyset$, then by induction and the fact that I and $P \setminus I$ are colored with distinct sets of colors, we know that $T \cap P$ contains a point of a unique color. If $T \cap P \subseteq I$, then $|T \cap P| = 1$ (since I consists of every other point in a monotone sequence) and the statement trivially holds.

We assume thus that case 1 does not hold.

Case 2. $T \cap D \neq \emptyset$: The CF-coloring of D guarantees that there is a point p of unique color among points in $T \cap D$. Since D and $P' = P \setminus D$ are colored with distinct sets of colors, p is a point of unique color among points in $T \cap P$ also.

Case 3. T spans at least 2 rows and 2 columns of G_r: Let (i, j) be a grid cell of G_r such that $T \cap (R_i \cap C_j) \neq \emptyset$. Since T contains at least one corner of grid cell (i, j), $T \cap D_l(R_i \cap C_j) \neq \emptyset$ for some $l \in \{1, 4\}$, i.e., T contains at least one point of the dominating set of points in grid cell (i, j). This implies that $T \cap D \neq \emptyset$ and we are back to Case 2.

We may assume now that cases 1, 2 and 3 do not hold.

Case 4. T lies completely within one row or one column of G_r: Since $T \cap P \neq \emptyset$ and $T \cap D = \emptyset$, we have $T \cap P' \neq \emptyset$. The quasi-CF coloring of P' guarantees that there is a point p of unique color among the points in $T \cap P'$. p is also a point of unique color among points in $T \cap P$. ☐

We now bound the total number of colors used by our algorithm. Quasi-CF-coloring of P' requires $\tilde{O}(n^{\frac{8}{13} \times \frac{3}{4}}) = \tilde{O}(n^{6/13})$ colors. To bound the number of colors used in CF-coloring D, we first bound the size of D: $|\mathcal{D}_l^k| = O(n^{7/13})$ for all k, because $\mathcal{D}_l^k$ is a monotonic sequence. Since $D = \cup_{l,h} \mathcal{D}_l^h$ over $1 \le h \le 2n^{5/13} - 1$, and $1 \le l \le 4$, we have $|D| = O(n^{12/13})$. Thus, the CF-coloring of D (using the $O(n^{1/2})$-coloring algorithm $\mathcal{A}$) requires $O(n^{6/13})$ colors. The total number of colors used by our algorithm is thus $\tilde{O}(n^{6/13})$.

THEOREM 3.1. *Any set of n points $P \subseteq R^2$, can be CF-colored with $\tilde{O}(n^{6/13})$ colors.*

4. GENERALIZED ALGORITHM

In this section, we generalize the algorithm described in Section 3. Recall that, in our coloring algorithm, we used an $O(n^{1/2})$ "black-box" $\mathcal{A}$ for CF-coloring the dominating set D and the quasi-CF-coloring of P'. As a result we obtained an $\tilde{O}(n^{6/13})$ CF-coloring algorithm. We can improve this coloring further by using now this $\tilde{O}(n^{6/13})$ as a new black-box for CF-coloring the dominating set D and quasi-CF-coloring of P'. An easy calculation shows that the number of colors used is asymptotically smaller than $\tilde{O}(n^{6/13})$.

We can now take this approach (almost) to the limit. This results in a sequence of strictly improved algorithms, $\mathcal{A} = \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ For $k = 1, 2, \ldots$, the structure of $\mathcal{A}_k$ is similar to the algorithm described in Section 3: Grid the point set P using G_r, where $r = n^{1-\alpha_k}$, for some α_k; Partition P into dominating set D and $P' = P \setminus D$ and use algorithm $\mathcal{A}_{k-1}$ for CF-coloring D and quasi-CF-coloring P'. We choose the parameter α_k such that both the CF-coloring of D and quasi-CF-coloring of P' balance-out into using an $\tilde{O}(n^{\beta_k})$ colors, for some β_k as small as possible. Ideally, one would like to always recursively apply algorithm $\mathcal{A}_\infty$ to get a bound of $\tilde{O}(n^{\beta_\infty})$ on the number of colors. However, there is a technical problem with such a recursion: the sublinearity of the bound on the number of colors implies that the power of the logarithmic factor increases exponentially with k. To solve this problem, we stop the recursion at a level of $O(\log \frac{1}{\epsilon})$, settling at a bound of $\tilde{O}(n^{\beta_\infty + \epsilon})$, for any arbitrarily small constant $\epsilon > 0$.

To be more precise and describe our coloring procedure formally, we need a few more definitions. Given a coloring $\chi : P \mapsto \mathcal{N}$, we denote by $\mathrm{range}(\chi) = \{\chi(p) : p \in P\}$, the set of distinct colors used to color P. Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a *monotone sublinear* function on the positive reals. An $f(\cdot)$-*conflict-free coloring algorithm* $\mathcal{A}$ takes as an input a point set $P \subseteq \mathbb{R}^2$, and a set of colors $N \subseteq \mathcal{N}$ such that $|N| \ge f(|P|)$, and returns a conflict-free coloring $\chi : P \mapsto N$ of P such that $|\mathrm{range}(\chi)| \le f(|P|)$.

REMARK 4.1. *It will simplify the analysis to assume, without loss of generality, that there exists a polylogarithmic factor (or even a constant) $\delta = \mathrm{polylog}(|P|)$ such that the size of each color class (that is $\max_\ell |\{p \in P : \chi(p) = \ell\}$) in the coloring returned by $\mathcal{A}$ is at most $\delta|P|/f(|P|)$. (This can be justified as follows. Let $P' \subseteq P$ be the largest monochromatic set returned by $\mathcal{A}$ when applied to P. If $|P'| \ge \delta n/f(n)$, where $n = |P|$, then let P'' be a subset of P' of size exactly $\delta n/f(n)$. Color all points of P'' with the same color ℓ, then the points $P - P''$ recursively with colors different from ℓ. Since $f(n)$ is sublinear, we get the required bound.)*

It will be convenient to think of the set of colors $\mathcal{N}$, which we use to color the points, as a subset of the sequences of natural numbers $\mathbb{N}^* \stackrel{\text{def}}{=} \mathbb{N} \cup \mathbb{N}^2 \cup \ldots$ This allows us to take unions and products of colors. More precisely, for disjoint subsets $P', P'' \subseteq P$ and colorings $\chi' : P' \mapsto \mathbb{N}^*$ and $\chi'' : P'' \mapsto \mathbb{N}^*$, we let $\chi' + \chi''$ denote the coloring $\chi : P' \cup P'' \mapsto \mathbb{N}^*$ defined by $\chi(p) = \chi'(p)$ if $p \in P'$ and $\chi(p) = \chi''(p)$ if $p \in P''$. For two colorings $\chi', \chi'' : P \mapsto \mathbb{N}^*$, we denote by $\chi' \times \chi''$ the coloring $\chi : P \mapsto \mathbb{N}^*$ given by $\chi(p) = (\chi'(p), \chi''(p))$ for $p \in P$.

The generalized coloring algorithm is given in Figure 3. We set the values of α_k, β_k, n_0 for $k \geq 1$, by the following recurrence relations and formulas:

$$\beta_0 = 1/2, \quad \beta_k = \frac{2\beta_{k-1}(2-\beta_{k-1})}{3+\beta_{k-1}-\beta_{k-1}^2}, \quad \beta_\infty = (3-\sqrt{5})/2 \qquad (1)$$

$$\alpha_k = \frac{2-\beta_k}{3-\beta_{k-1}} \qquad (2)$$

$$\gamma_0 = 4, \quad \gamma_k = 2^{k+2}-1 \qquad (3)$$

$$n_0 = 2^{10} \qquad (4)$$

Procedure $\mathcal{A}_k(P, S)$**:**
Input: A point set $P \subseteq \mathbb{R}^2, |P| = n$, a set of colors S
Output: A CF-coloring $\chi : P \mapsto S$ with $|\operatorname{range}(\chi)| \leq f_k(n)$

if $k = 0$ or $n \leq n_0$ **then**
  return a coloring of P using the $O(\sqrt{n})$-coloring algorithm
else
  Compute α_k and β_k using (1)-(2); Set $r \leftarrow n^{1-\alpha_k}$
  if $\exists$ a monotonic sequence L of size $2n^{1-\beta_k}$ **then**
    Let I be the set consisting of every other point of L
    Color every point of I with the same color $i \in S$,
      i.e. set $\chi'(p) \leftarrow i$ for all $p \in I$
    $\chi'' \leftarrow \mathcal{A}_k(P \setminus I, S \setminus \{i\})$
    return $\chi' + \chi''$
  else
    Grid P using G_r
    Compute the dominating set D w.r.t. $G_r(P)$
    $\chi' \leftarrow$QCFC$(P \setminus D, G_r, \mathcal{A}_{k-1}, S)$
    $\chi'' \leftarrow \mathcal{A}_{k-1}(D, S \setminus \operatorname{range}(\chi'))$
    return $\chi' + \chi''$

Figure 3: Conflict-free coloring

The structure of the generalized coloring algorithm is the same as the algorithm described in Section 3. Hence, by Lemma 3.1, the coloring returned by the algorithm is conflict-free.

5. GENERALIZED QUASI CONFLICT FREE COLORING

In this section we describe the quasi-CF coloring algorithm. Given an r-grid $G_r(P)$ on point set P, we start by coloring the points of each column, using a CF-coloring algorithm $\mathcal{A}$ as a black-box. We use the same set of colors for all columns. Then randomly and independently for each column, we redistribute the colors on the different color classes of the column. Finally, a recoloring step is applied on each monochromatic set of points in each row, again using algorithm $\mathcal{A}$ as the CF-coloring procedure. When we do the recoloring, we color all sets assigned a color ℓ in the first step using the same set of colors S_ℓ. A formal description of this procedure is given in Figure 4.

Procedure QCFC$(P, \mathcal{A}, G_r, S)$**:**
Input: A point set $P \subseteq \mathbb{R}^2$, an $f(\cdot)$-CF-coloring algorithm $\mathcal{A}$
  an r-grid G_r on P, and a set of possible colors S
Output: A quasi-CF-coloring $\chi : P \mapsto S$ of P w.r.t. G_r

Let $h = f(B(G_r))$; $N = \{1, \ldots, h\}$
for $i = 1, \ldots, r$ **do**
  $\chi_i \leftarrow \mathcal{A}(C_i, N)$
  Let $\pi \in \mathbb{S}_h$ be a random permutation
  foreach $p \in C_i$ **do**
    $\chi_i'(p) \leftarrow \pi(\chi_i(p))$
$\chi' \leftarrow \sum_{i=1}^{r} \chi_i'$
Let $S_1, \ldots, S_h$ be disjoint subsets of $\mathbb{N}$ of size $B(G_r)$
for $i = 1, \ldots, r$ **do**
  for $\ell = 1, \ldots, h$ **do**
    $P_i^\ell \leftarrow \{p \in R_i : \chi'(p) = \ell\}$
    $\chi_{i,\ell}'' \leftarrow \mathcal{A}(P_i^\ell, S_\ell)$
$\chi'' \leftarrow \sum_{i=1}^{r} \sum_{\ell=1}^{h} \chi_{i,\ell}''$
return $\chi' \times \chi''$ (mapped to S)

Figure 4: Quasi-conflict-free coloring of a grid

The following is a straightforward generalization of Theorem 3 in [7]. We include the proof in the Appendix A for completeness.

THEOREM 5.1. *Given any point set $P \subseteq \mathbb{R}^2$, a grid $G_r = G_r(P)$ with $B(G_R) = B$ on P, and a conflict-free coloring algorithm $\mathcal{A}$ with guarantee $f(\cdot)$ such that*

$$r \cdot f(B)(\log(\delta B / f(B)) + 1)(\delta \log B)^{(-\log B)/4} \leq \frac{1}{2}, \qquad (5)$$

procedure QCFC returns a quasi-conflict-free coloring of $G_r(P)$ using

$$q(B) = f(B) f\left(\frac{2\delta B \log B \log((\delta B / f(B)) + 1)}{f(B)}\right) \qquad (6)$$

colors, in expected polynomial time, where δ is the parameter given in Remark 4.1.

6. ANALYSIS

We denote by $f_k(\cdot)$ an upper bound on the number of colors required by the algorithm at the k^{th} level. If $n \leq n_0$ or $k = 0$, we use a $4\sqrt{n}$ coloring algorithm. Thus, $f_k(n) \geq 4\sqrt{n}$ for $4 \leq n \leq n_0$ or $k = 0$. Otherwise, if any of the monotonic sets $\mathcal{D}_l^h$ (for $1 \leq l \leq 4$ and $1 \leq h \leq 2\, n^{1-\alpha_k} - 1$) is larger than $2n^{1-\beta_k}$, we color every alternate node in the monotonically increasing dominating set with a single color and recurse on the rest. Thus, $f_k(n) \geq 1 + f_k(n - n^{1-\beta_k})$. If there is no such monotonic set, we grid the point set such that all rows and columns contain approximately (but not more than) n^{α_k} points, recursively color the dominating set D and quasi color $P \setminus D$. Let $\delta_k(n) = \log^{\gamma_k} n$. We can conclude from Theorem 5.1 that if for all $n > n_0$ and $k > 1$,

$$n^{1-\alpha_k} \cdot f_{k-1}(n^{\alpha_k}) \left(\log\left(\frac{\delta_{k-1}(n^{\alpha_k}) n^{\alpha_k}}{f_{k-1}(n^{\alpha_k})}\right) + 1\right) \cdot (\delta_{k-1}(n^{\alpha_k}) \log n^{\alpha_k})^{(-\log n^{\alpha_k})/4} \leq 1/2 \qquad (*)$$

then the upper bound on the number of colors required by our algorithm will be

$$f_k(n) \geq f_{k-1}(16 \cdot n^{(2-\alpha_k-\beta_k)}) + f_{k-1}(n^{\alpha_k})$$
$$\cdot\ f_{k-1}\left(\frac{2\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} \log n^{\alpha_k} \left(\log\left(\delta_{k-1}(n^{\alpha_k}) \cdot n^{\alpha_k} / f_{k-1}(n^{\alpha_k})\right) + 1\right)}{f_{k-1}(n^{\alpha_k})}\right)$$

Our main theorem in this Section is that $f_k(n)$ defined as

$$f_k(n) = \begin{cases} n & \text{for } n < 4 \\ \min\{4\sqrt{n},\ \delta_{k-1}(n)\ n^{\beta_k}\} & \text{for } n \geq 4 \end{cases}$$

satisfies all the recursions mentioned above. Coupled with Lemma 6.1, where we prove that for $k = O(\log 1/\epsilon)$, $\beta_\infty < \beta_k \leq \beta_\infty + \epsilon$, this shows that the number of colors required by our algorithm is bounded above by $n^{\beta_\infty+\epsilon} \log^{O(1/\epsilon)} n = \tilde{O}(n^{\beta+\epsilon})$ for any $\epsilon > 0$.

LEMMA 6.1. *$\beta_\infty < \beta_k \leq \beta_\infty + \epsilon$ for $k = O(\log 1/\epsilon)$*

PROOF. The first inequality follows from the monotonicity of the sequence β_k and its convergence to β_∞. For the second inequality, let us define $\epsilon_k = \beta_k - \beta_\infty$. We show that $\epsilon_k \leq \epsilon$ for $k = \log_{\frac{2-\beta_\infty}{2-2\beta_\infty}} \frac{0.5-\beta_\infty}{\epsilon}$. By definition,

$$\begin{aligned}
\beta_k &= \frac{2(\beta_\infty + \epsilon_{k-1})(2 - \beta_\infty - \epsilon_{k-1})}{3 + (\beta_\infty + \epsilon_{k-1}) - (\beta_\infty + \epsilon_{k-1})^2} \\
&= \frac{2\beta_\infty(2-\beta_\infty) + 4\epsilon_{k-1}(1-\beta_\infty) - 2\epsilon_{k-1}^2}{(3 + \beta_\infty - \beta_\infty^2) + (\epsilon_{k-1} - 2\beta_\infty\epsilon_{k-1} - \epsilon_{k-1}^2)} \\
&\leq \frac{2\beta_\infty(2-\beta_\infty) + 4\epsilon_{k-1}(1-\beta_\infty)}{(3 + \beta_\infty - \beta_\infty^2) + (\epsilon_{k-1} - 2\beta_\infty\epsilon_{k-1})} \\
&\leq \frac{2\beta_\infty(2-\beta_\infty) + 4\epsilon_{k-1}(1-\beta_\infty)}{(3 + \beta_\infty - \beta_\infty^2)} \quad \text{(assuming } 2\beta_\infty \leq 1)
\end{aligned}$$

Since $3 + \beta_\infty - \beta_\infty^2 = 2(2 - \beta_\infty)$, we get

$$\beta_k \leq \beta_\infty + \frac{4\epsilon_{k-1}(1-\beta_\infty)}{2(2-\beta_\infty)}$$

Therefore,

$$\epsilon_k \leq \frac{\epsilon_{k-1}(2 - 2\beta_\infty)}{(2-\beta_\infty)}$$

Since this is true for any $k > 0$, we get

$$\begin{aligned}
\epsilon_k &\leq \left(\frac{2-2\beta_\infty}{2-\beta_\infty}\right)^k \epsilon_0 \\
&= \left(\frac{2-2\beta_\infty}{2-\beta_\infty}\right)^k (0.5 - \beta_\infty)
\end{aligned}$$

Thus for $k = \log_{\frac{2-\beta_\infty}{2-2\beta_\infty}} \frac{0.5-\beta_\infty}{\epsilon}$, $\epsilon_k \leq \epsilon$. $\square$

The following claims can be easily verified:

CLAIM 6.1. *For $k \geq 1$, $0.61 < \alpha_1 \leq \alpha_k \leq \alpha_\infty < 0.62$.*

CLAIM 6.2. *For $k \geq 0$, $0.38 < \beta_\infty \leq \beta_k \leq \beta_0 \leq 0.50$.*

CLAIM 6.3. *For $k \geq 1$, $\alpha_k + \beta_k \geq 1$.*

LEMMA 6.2. *For all k and $4 \leq n \leq n_0$, $f_k(n) \geq 4\, n^{0.5}$.*

PROOF. $f_k(n) = n^{\beta_k} \log^{\gamma_k} n \geq n^{\beta_\infty} \log^{\gamma_0} n \geq 4\, n^{0.5}$
The last inequality holds because $4 \leq n \leq n_0$. $\square$

LEMMA 6.3. *$f_k(n) \geq 1 + f_k(n - n^{1-\beta_k})$ for $k > 0$ and $n > n_0$.*

PROOF.

$$f_k(n) - f_k(n - n^{1-\beta_k}) \geq (n_k^\beta - (n - n^{1-\beta_k})^{\beta_k}) \log^{\gamma_k} n$$

Since $n_k^\beta - (n - n^{1-\beta_k})^{\beta_k}) \geq \beta_k \geq \beta_\infty$ for $n > n_0$ and $\log^{\gamma_k} n \geq \log^{\gamma_0} n_0 > 1/\beta_\infty$,

$$(n_k^\beta - (n - n^{1-\beta_k})^{\beta_k}) \log^{\gamma_k} n \geq 1$$

Thus, $f_k(n) - f_k(n - n^{1-\beta_k}) \geq 1$. $\square$

LEMMA 6.4. *For $k > 0$ and $n > n_0$, $(*)$ holds.*

PROOF. Since $f_{k-1}(n^{\alpha_k}) \leq n^{\alpha_k}$, the left hand side of $(*)$ is at most

$$\begin{aligned}
& n^{1-\alpha_k} n^{\alpha_k} (\log n^{\alpha_k(1-\beta_{k-1})} + 1) \left(\delta_{k-1}(n^{\alpha_k}) \log n^{\alpha_k}\right)^{(-\log n^{\alpha_k})/4} \\
&\leq n \left(\log n^{\alpha_k(1-\beta_{k-1})} + 1\right) \left(\log^{\gamma_{k-1}}(n^{\alpha_k}) \log n^{\alpha_k}\right)^{(-\log n^{\alpha_k})/4} \\
&\leq n \left(\log n^{\alpha_\infty(1-\beta_\infty)} + 1\right) \left(\log^{\gamma_0}(n^{\alpha_1}) \log n^{\alpha_1}\right)^{(-\log n^{\alpha_1})/4} \\
&\qquad \text{(using Claim 6.1, 6.2 and the definition of } \gamma_k) \\
&= h(n)
\end{aligned}$$

The function $h(n)$ is monotonically decreasing for $n \geq n_0$ and $h(n_0) < 1/2$. We therefore conclude that for $k > 0$ and $n > n_0$, $(*)$ holds. $\square$

LEMMA 6.5. *For $k > 0$ and $n > n_0$,*

$$f_k(n) \geq f_{k-1}(16 \cdot n^{2-\alpha_k-\beta_k}) + f_{k-1}(n^{\alpha_k})$$
$$\cdot\ f_{k-1}\left(\frac{2\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} \log n^{\alpha_k} \left(\log\left(\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} / f_{k-1}(n_k^{\alpha})\right) + 1\right)}{f_{k-1}(n)}\right)$$

PROOF. We first show that

$$\begin{aligned}
& f_{k-1}\left(\frac{2\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} \log n^{\alpha_k} \left(\log\left(\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} / f_{k-1}(n_k^{\alpha})\right) + 1\right)}{f_{k-1}(n^{\alpha_k})}\right) \\
&\leq n^{\alpha_k\beta_{k-1}(1-\beta_{k-1})} \left(2 \log n^{\alpha_k} (\log(n^{\alpha_k(1-\beta_{k-1})}) + 1)\right)^{\beta_{k-1}} \delta_{k-1}(n)
\end{aligned}$$

Note that

$$\begin{aligned}
& f_{k-1}\left(\frac{2\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} \log n^{\alpha_k} \left(\log\left(\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} / f_{k-1}(n_k^{\alpha})\right) + 1\right)}{f_{k-1}(n^{\alpha_k})}\right) \\
&= f_{k-1}\left(\frac{2\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} \log n^{\alpha_k} \left(\log\left(\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} / f_{k-1}(n_k^{\alpha})\right) + 1\right)}{\delta_{k-1}(n^{\alpha_k}) n^{\alpha_k\beta_{k-1}}}\right) \\
&= f_{k-1}\left(2n^{\alpha_k(1-\beta_{k-1})} \log n^{\alpha_k} \left(\log n^{\alpha_k(1-\beta_{k-1})} + 1\right)\right) \\
&= n^{\alpha_k\beta_{k-1}(1-\beta_{k-1})} \left(2 \log n^{\alpha_k} \left(\log n^{\alpha_k(1-\beta_{k-1})} + 1\right)\right)^{\beta_{k-1}} \\
&\quad \cdot \delta_{k-1}\left(2n^{\alpha_k(1-\beta_{k-1})} \log n^{\alpha_k} (\log n^{\alpha_k(1-\beta_{k-1})} + 1)\right)
\end{aligned}$$

Now,

$$\begin{aligned}
& 2n^{\alpha_k(1-\beta_{k-1})} \log n^{\alpha_k} (\log n^{\alpha_k(1-\beta_{k-1})} + 1) \\
&\leq 2n^{\alpha_\infty(1-\beta_\infty)} \log n^{\alpha_\infty} (\log n^{\alpha_\infty(1-\beta_\infty)} + 1) \\
&\qquad \text{(using Claim 6.1 and 6.2)} \\
&\leq n,
\end{aligned}$$

where the last inequality follows from the fact that n is asymptotically larger and the inequality holds for $n = n_0$. So, we have

$$\begin{aligned}
& f_{k-1}\left(\frac{2\delta_{k-1}(n^{\alpha_k})n^{\alpha_k} \log n^{\alpha_k} (\log(\delta_{k-1}(n^{\alpha_k})n^{\alpha_k}) + 1)}{f_{k-1}(n^{\alpha_k})}\right) \\
&\leq n^{\alpha_k\beta_{k-1}(1-\beta_{k-1})} \left(2 \log n^{\alpha_k} (\log n^{\alpha_k(1-\beta_{k-1})} + 1)\right)^{\beta_{k-1}} \delta_{k-1}(n)
\end{aligned}$$

Therefore,

$$f_{k-1}(16\cdot n^{(2-\alpha_k-\beta_k)}) + f_{k-1}(n^{\alpha_k})$$
$$\cdot\ f_{k-1}\left(\frac{2\delta_{k-1}(n^{\alpha_k})n^{\alpha_k}\log n^{\alpha_k}(\log(\delta_{k-1}(n^{\alpha_k})n^{\alpha_k})+1)}{f_{k-1}(n)}\right)$$
$$\leq f_{k-1}(16\, n^{2-\alpha_k-\beta_k}) + f_{k-1}(n^{\alpha_k})$$
$$\cdot\, n^{\alpha_k\beta_{k-1}(1-\beta_{k-1})}\left(2\log n^{\alpha_k}(\log n^{\alpha_k(1-\beta_{k-1})}+1)\right)^{\beta_{k-1}}\delta_{k-1}(n)$$
$$\leq 16^{\beta_{k-1}}\, n^{(2-\alpha_k-\beta_k)\beta_{k-1}}\delta_{k-1}(16\, n^{2-\alpha_k-\beta_k}) + n^{\alpha_k\beta_{k-1}}\delta_{k-1}(n^{\alpha_k})$$
$$\cdot\, n^{\alpha_k\beta_{k-1}(1-\beta_{k-1})}(2\log n^{\alpha_k}(\log n^{\alpha_k(1-\beta_{k-1})}+1))^{\beta_{k-1}}\delta_{k-1}(n)$$
$$= n^{\beta_k}\ (16^{\beta_{k-1}}\ \delta_{k-1}(16\, n^{2-\alpha_k-\beta_k}) + \delta_{k-1}(n^{\alpha_k})$$
$$\cdot\left(2\log n^{\alpha_k}(\log n^{\alpha_k(1-\beta_{k-1})}+1)\right)^{\beta_{k-1}}\ \delta_{k-1}(n))$$
$$\text{(since } (2-\alpha_k-\beta_k)\beta_{k-1} = \alpha_k\beta_{k-1} + \alpha_k(1-\beta_{k-1})\beta_{k-1} = \beta_k\text{)}$$
$$\leq n^{\beta_k}\ (16^{\beta_{k-1}}\ \delta_{k-1}(16\, n) + \delta_{k-1}(n^{\alpha_k})\delta_{k-1}(n)$$
$$\cdot\left(2\log n^{\alpha_k}(\log n^{\alpha_k(1-\beta_{k-1})}+1)\right)^{\beta_{k-1}})\qquad\text{(using Claim 6.3)}$$
$$\leq n^{\beta_k}\ (16^{\beta_{k-1}}\ \delta_{k-1}(n^2) + \delta_{k-1}(n^{\alpha_k})\delta_{k-1}(n)$$
$$\cdot\left(2\log n^{\alpha_k}(\log n^{\alpha_k(1-\beta_{k-1})}+1)\right)^{\beta_{k-1}})\qquad\text{(since } n \geq n_0 > 16\text{)}$$
$$\leq \left(n^{\beta_k}\ \log^{2\gamma_{k-1}+1} n\right)$$
$$\cdot\left(\frac{16^{\beta_{k-1}}\ \log^{\gamma_{k-1}} n^2}{\log^{2\gamma_{k-1}+1} n} + \frac{\alpha_k^{\gamma_{k-1}}}{\log n}\left(2\log n^{\alpha_k}(\log n^{\alpha_k(1-\beta_{k-1})}+1)\right)^{\beta_{k-1}}\right)$$
$$= f_k(n)\left(\frac{16^{\beta_{k-1}}\log^{\gamma_{k-1}} n^2}{\log^{2\gamma_{k-1}+1} n} + \frac{\alpha_k^{\gamma_{k-1}}}{\log n}\left(2\log n^{\alpha_k}(\log n^{\alpha_k(1-\beta_{k-1})}+1)\right)^{\beta_{k-1}}\right)$$
$$\text{(since } \gamma_k = 2\gamma_{k-1}+1\text{)}$$
$$= f_k(n)\left(\frac{16^{\beta_{k-1}}\ 2^{\gamma_{k-1}}}{\log^{\gamma_{k-1}+1} n} + \frac{\alpha_k^{\gamma_{k-1}}}{\log n}\left(2\log n^{\alpha_k}(\log n^{\alpha_k(1-\beta_{k-1})}+1)\right)^{\beta_{k-1}}\right)$$
$$= f_k(n)\left(\frac{0.5\cdot 16^{\beta_{k-1}}}{(0.5\log n)^{\gamma_{k-1}+1}} + \frac{\alpha_k^{\gamma_{k-1}}}{\log n}\left(2\log n^{\alpha_k}(\log n^{\alpha_k(1-\beta_{k-1})}+1)\right)^{\beta_{k-1}}\right)$$
$$\leq f_k(n)\left(\frac{0.5\cdot 16^{\beta_0}}{(0.5\log n)^{\gamma_0+1}} + \frac{\alpha_\infty^{\gamma_0}}{\log n}\left(2\log n^{\alpha_\infty}(\log n^{\alpha_\infty(1-\beta_\infty)}+1)\right)^{\beta_0}\right)$$
$$\text{(using Claim 6.1, 6.2 and the definition of } \gamma_k\text{)}$$
$$\leq f_k(n)$$

It can be verified that the last inequality holds for $n > n_0$. □

Combining Lemmas 6.2, 6.3, 6.4 and 6.5, we get the following theorem:

THEOREM 6.1. *$f_k(n) = n^{\beta_k}\log^{\gamma_k} n$ satisfies all of the following recursions:*

(i) $f_k(n) \geq 4\sqrt{n}$ *(for all $k > 0$, $4 \leq n \leq n_0$ or $k = 0$)*

(ii) $f_k(n) \geq 1 + f_k(n - n^{1-\beta_k})$ *(for $k > 0$ and $n > n_0$)*

(iii) $$n^{1-\alpha_k} f_{k-1}(n^{\alpha_k})\log\left(\frac{\delta_{k-1}(n^{\alpha_k})n^{\alpha_k}}{f_{k-1}(n^{\alpha_k})}+1\right)\cdot(\delta_{k-1}(n^{\alpha_k})\log n^{\alpha_k})^{(-\log n^{\alpha_k})/4} \leq 0.5$$
(for $k > 0$ and $n > n_0$)

(iv) $$f_k(n) \geq f_{k-1}(16\cdot n^{2-\alpha_k-\beta_k}) + f_{k-1}(n^{\alpha_k})\cdot f_{k-1}\left(\frac{2\delta_{k-1}(n^{\alpha_k})n^{\alpha_k}\log n^{\alpha_k}(\log(\delta_{k-1}(n^{\alpha_k})n^{\alpha_k})/f_{k-1}(n_k^{\alpha})+1)}{f_{k-1}(n)}\right)$$
(for $k > 0$ and $n > n_0$)

Using Theorem 6.1 and Lemma 6.1, we get the following:

THEOREM 6.2. *Let P be a set of n points in $\mathbb{R}^2$. Algorithm $\mathcal{A}_k(P,S)$ conflict-free colors P with respect to rectangle ranges using $n^{\beta+\epsilon}\log^{O(1/\epsilon)} n = \tilde{O}(n^{\beta+\epsilon})$ colors, for any arbitrarily small $\epsilon > 0$.*

7. DISCUSSION

Note that the quasi-CF-coloring Algorithm *QCFC* when fed with an $\tilde{O}(n^{\beta_{k-1}})$-CF-coloring algorithm $\mathcal{A}_{k-1}$ as an input, returns a $\tilde{O}(n^{\beta_k}) = \tilde{O}(B(G_r)^{\beta_{k-1}(2-\beta_{k-1})})$-quasi-CF-coloring of P', $P' \subseteq P$, where $r = n^{1-\alpha_k}$. This is actually how the successive improvements are made, since $\beta_k < \beta_{k-1}$. Suppose there exists a quasi-CF-coloring algorithm that uses $B(G_r)^c$ colors for some $c > 0$. Then an easy calculation shows that our algorithm $\mathcal{A}_k$ returns a CF-coloring using $n^{c/(c+1)}$ colors. Clearly any improvement on the quasi-CF-coloring algorithm will translate to an improvement on the general case. By setting $c = \epsilon$, for any $\epsilon > 0$, we obtain the following:

COROLLARY 7.1. *Let P be a set of points of size n, and $r = n^{1-\alpha}$ for some $\alpha \in (0,1)$. If there exists a quasi-CF-coloring algorithm of the grid $G_r(P)$ that requires $O(B(G_r)^\epsilon)$ colors, for any $\epsilon > 0$, then we can obtain a CF-coloring algorithm of P that requires $O(n^{\epsilon'})$ colors, where $\epsilon' = \epsilon/(\epsilon+1)$.*

Similarly, any improved coloring of the dominating set D also leads to an improved CF-coloring algorithm.

One might think that, if an improved CF-coloring algorithm (that uses $O(n^c)$ colors, $c < \beta_\infty$) is obtained, it could be further improved using our iterated improvement scheme given in Section 4. However, this is not possible since it can be easily seen that, if $\beta_{k-1} < \beta_\infty$, then $\beta_k > \beta_{k-1}$. Thus, one cannot hope to improve a CF-coloring algorithm that uses fewer than $O(n^{\beta_\infty})$ colors by directly using our iterated improvement scheme.

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APPENDIX

A. PROOF OF THEOREM 5.1

Let $\chi_i, \chi', \chi'', h, P_i^\ell$ be as defined in the procedure, and $\chi = \chi' \times \chi''$ be the coloring returned in Step 14. The theorem follows from the following two claims.

CLAIM A.1. ([7]) *χ is quasi-conflict-free.*

PROOF. Let $T \in \mathcal{R}$ be any rectangle that lies completely inside a row or a column of G_r, such that $T \cap P \neq \emptyset$. If T contains only points belonging to a single column C_j of G_r, then the fact that algorithm $\mathcal{A}$ returns a conflict-free coloring of C_j and the definition of χ'_j imply that T contains a point $p \in T \cap C_j$ such that $\chi'_j(p) \neq \chi'_j(p')$ for all $p' \in T \cap P$, $p' \neq p$. Then $\chi'(p)$ and hence $\chi(p)$ is different in the first coordinate from $\chi(p')$ for every $p' \in T \cap P$, $p' \neq p$. Now assume that T contains only points belonging to a single row i of G_r. Since $T \cap P \neq \emptyset$, there is an $\ell \in [h]$ such that $T \cap P_i^\ell \neq \emptyset$. Since $\mathcal{A}$ returns a conflict-free coloring $\chi''_{i,\ell}$ of P_i^ℓ, there is a point $p \in T \cap P_i^\ell$, such that $\chi''_{i,\ell}(p) \neq \chi''_{i,\ell}(p')$ for all $p' \in T \cap P_i^\ell$, $p' \neq p$. Thus if $p' \in T \cap R_i$, then either $p' \in P_i^{\ell'}$ for $\ell' \neq \ell$ in which case $\chi'(p') \neq \chi'(p)$, or $p' \in P_i^\ell$ but $\chi''(p') \neq \chi''(p)$. In both cases $\chi(p') \neq \chi(p)$. □

CLAIM A.2. *With probability at least* $1/2$, $|\operatorname{range}(\chi)| \leq q(B)$ *given by (6).*

PROOF. Fix $i \in [r]$ and $\ell \in [h]$. Define $t = \delta B/h$. For $j \in [r]$, let $A_{i,j}^\ell = \{p \in R_i \cap C_j : \chi_j(p) = \ell\}$ and note that $|A_{i,j}^\ell| \leq \delta B/f(B) = t$ (by Remark 4.1 and the sub-linearity of $f(\cdot)$). For $m = 1, 2, \ldots, \lceil \log t \rceil$, let

$$\mathcal{A}_{i,j}^m = \{A_{i,j}^\ell : \ 2^{m-1} \leq |A_{i,j}^\ell| \leq 2^m, \ \ell = 1, \ldots, h\},$$

and note that

$$\sum_{j=1}^{r} |\mathcal{A}_{i,j}^m| \leq \frac{B}{2^{m-1}}, \tag{7}$$

since the total number of points in row i of G_r is at most B, and each set in $\mathcal{A}_{i,j}^m$ has at least 2^{m-1} points.

Note that, for any $j \in [r]$, every point $p \in A_{i,j}^\ell$ gets the same color $\chi'(p)$ in Step 6. Thus we can think of the coloring in Step 6 as of permuting randomly the colors to the sets $A_{i,j}^\ell$, $\ell = 1, \ldots, h$, and may use $\chi'(A_{i,j}^\ell)$ to denote the color assigned in Step 6 to all points in $A_{i,j}^\ell$. Let $Y_{i,j}^{m,\ell}$ be the indicator random variable that takes value 1 if and only if there exists a set $S \in \mathcal{A}_{i,j}^m$ with $\chi'(S) = \ell$ (if $\mathcal{A}_{i,j}^m$ is empty, then the corresponding random variable is 0 with probability 1). Let $Y_i^{m,\ell} = \sum_{j=1}^r Y_{i,j}^{m,\ell}$. Then,

$$\mathbb{E}[Y_{i,j}^{m,\ell}] = \Pr[Y_{i,j}^{m,\ell} = 1] = \frac{|\mathcal{A}_{i,j}^m|}{h}$$

$$\mathbb{E}[Y_i^{m,\ell}] = \sum_{j=1}^{r} \frac{|\mathcal{A}_{i,j}^m|}{h} \leq \frac{B}{h2^{m-1}} = \frac{t}{\delta 2^{m-1}},$$

where the last inequality follows from (7).

Note that the variable $Y_i^{m,\ell}$ is the sum of independent Bernoulli trials, and thus applying the Chernoff bound[1], we get

$$\Pr\left[Y_i^{m,\ell} > \frac{t \log B}{2^{m-1}}\right] \leq e^{-\frac{t \log B}{4 \cdot 2^{m-1}} \ln\left(\frac{t \log B}{\mathbb{E}[Y_i^{m,\ell}] \cdot 2^{m-1}}\right)}. \tag{8}$$

Using $\mathbb{E}[Y_m^{i,\ell}] \leq t/(\delta 2^{m-1})$ and $2^m \leq 2t$, we deduce from (8) that

$$\Pr\left[Y_i^{m,\ell} > \frac{t \log B}{2^{m-1}}\right] \leq (\delta \log B)^{-(\log B)/4}.$$

Thus, the probability that there exist i, ℓ, and m such that $Y_i^{m,\ell} > t \log B/2^{m-1}$ is at most

$$rh(\log t + 1)(\delta \log B)^{-(\log B)/4} \leq \frac{1}{2},$$

by (5). Therefore with probability at least $1/2$, $Y_i^{m,\ell} \leq t \log B/2^{m-1}$ for all i, ℓ, and m. In particular, with constant probability, for all i and ℓ, we have

$$|P_i^\ell| \leq \sum_{m=1}^{\lceil \log t \rceil} Y_i^{m,\ell} \cdot 2^m \leq 2t \log B(\log t + 1).$$

Since algorithm $\mathcal{A}$ has guarantee $f(\cdot)$, with constant probability, the total number of colors needed, by the sublinearity of $f(\cdot)$, is

$$|\operatorname{range}(\chi)| \leq \sum_{\ell=1}^{h} f(|P_i^\ell|) \leq h \cdot f(2t \log B(\log t + 1)) \leq q(B),$$

as claimed. □

[1] In particular, the following version [11]: $\Pr[X \geq (1+\theta)\mu] \leq e^{-(1+\theta)\ln(1+\theta)\mu/4}$, for $\theta > 1$ and $\mu = \mathbb{E}[X]$.