

Measurable sets with excluded distances*

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March 31, 2007

Abstract

For a set of distances $D = \{d_1, \dots, d_k\}$ a set A is called D -avoiding if no pair of points of A is at distance d_i for some i . We show that the density of A is exponentially small in k provided the ratios $d_1/d_2, d_2/d_3, \dots, d_{k-1}/d_k$ are all small enough. This resolves a question of Székely, and generalizes a theorem of Furstenberg-Katznelson-Weiss, Falconer-Marstrand, and Bourgain. Several more results on D -avoiding sets are presented.

1 Introduction

The problem of determining the least number of colors required to color the points of the plane \mathbb{R}^d so that no pair of points at distance 1 is colored in the same color was first investigated by Nelson and Hadwiger in 1940s. This number, which we denote by $\chi_{\mathbb{R}^d}(\{1\})$, is called the chromatic number of \mathbb{R}^d because it is the chromatic number of the graph whose vertices are the points of \mathbb{R}^d and the edges are pairs of points that are distance 1 apart. We denote this graph by $G_{\mathbb{R}^d}(\{1\})$.

In the dimension two, there has been no improvement on the bounds $4 \leq \chi_{\mathbb{R}^2}(\{1\}) \leq 7$ in the past forty-five years [Had61, MM61]. In higher dimensions, however, Frankl and Wilson [FW81] showed that the chromatic number grows exponentially in the dimension, $\chi_{\mathbb{R}^d}(\{1\}) \geq (1.207\dots + o(1))^d$, confirming an earlier conjecture of Erdős. The paper of Frankl and Wilson in conjunction with the earlier work of Ray-Chaudhuri and Wilson [RCW75] laid down the theory of set families with restricted intersection, which led to many other results including the disproof of Borsuk's conjecture by Kahn and Kalai [KK93].

It was first shown by Erdős and de Bruijn [dBE51] that the chromatic number of any infinite graph, and $G_{\mathbb{R}^d}(\{1\})$ in particular, is the maximum of the chromatic numbers of its finite subgraphs, provided the maximum is finite. The proof relied on the axiom of choice, which suggested that the chromatic number might depend on the underlying axiom system. This was partially confirmed by Falconer [Fal81] who showed that there is no coloring of \mathbb{R}^2 into four colors such that each color class is a Lebesgue measurable set. Since as shown by Solovay [Sol70] the axiom that all subsets of \mathbb{R} are Lebesgue measurable is consistent with the usual Zermelo-Fraenkel set theory without the axiom of choice, $\chi_{\mathbb{R}^2}(\{1\}) = 4$ is unprovable in the set theory without the axiom of choice.

Thus, we denote by $\chi_{\mathbb{R}^d}^m(\{1\})$ the least number of colors required to color \mathbb{R}^d so that no points at distance 1 are assigned the same color, and each color class is a measurable set. A set with no pairs of points at distance 1 is going to be called $\{1\}$ -avoiding. The most natural way to show that $\chi_{\mathbb{R}^d}^m(\{1\})$ is large is by showing that no color class can be large. Denote by $\bar{d}(A)$ the upper limit density of A , and by $m_{\mathbb{R}^d}(\{1\}) = \sup \bar{d}(A)$, where the supremum is over all measurable $\{1\}$ -avoiding sets. Then $\chi_{\mathbb{R}^d}^m(\{1\}) \leq 1/m_{\mathbb{R}^d}(\{1\})$. Unfortunately, Falconer's proof that $\chi_{\mathbb{R}^2}^m(\{1\}) \geq 5$ does not show that $m_{\mathbb{R}^2}(\{1\}) < 1/4$. The best known bounds are $0.229365 \leq m_{\mathbb{R}^2}(\{1\}) \leq 12/43$

*This work is a part of a Ph. D. thesis under the supervision of Benjamin Sudakov.

(see [SU97, p. 61] and [Szé85] respectively), and it is a conjecture of Erdős that $m_{\mathbb{R}^2}(\{1\}) < 1/4$ [Szé02].

The problem of forbidding more than one distance was first studied by Székely in his thesis [Szé85]. There he established the first bounds on $\chi_{\mathbb{R}^d}^m(D)$ and $m_{\mathbb{R}^d}(D)$ which denote the analogues of $\chi_{\mathbb{R}^d}^m(\{1\})$ and $m_{\mathbb{R}^d}(\{1\})$, respectively, where a finite set of distances $D = \{d_1, \dots, d_k\}$ is forbidden. Székely conjectured that in dimension $d \geq 2$ for any set A with $\bar{d}(A) > 0$ there is a d_0 such that all the distances greater than d_0 occur among the points of A . The conjecture was proved by Furstenberg, Katznelson and Weiss [FKW90]. Their proof was ergodic-theoretic. Later Bourgain found a harmonic-analytic proof [Bou86], and Falconer and Marstrand gave a direct geometric proof [FM86]. Székely also conjectured that if d_1, d_2, \dots is a sequence converging to 0, then $m_{\mathbb{R}^d}(\{d_1, \dots, d_k\}) \rightarrow 0$ as $k \rightarrow \infty$. This was proved by Falconer [Fal86].

It is not known how large $\chi_{\mathbb{R}^d}(D)$ can be for a set D of given size. It has been long known that $\sup_{|D|=k} \chi_{\mathbb{R}^2}(D) \geq ck\sqrt{\log k}$ [CFG94, p. 180]. The only known upper bound $\sup_{|D|=k} \chi_{\mathbb{R}^d}(D) \leq \chi_{\mathbb{R}^d}(\{1\})^k$ comes from the observation that the coloring, which is a product of colorings that avoid D_1 and D_2 , avoids both D_1 and D_2 . Croft, Falconer and Guy asked whether $\sup_{|D|=k} \chi_{\mathbb{R}^d}(D)$ is exponential in k [CFG94, Prob. G11]. Erdős conjectured that $\sup_{|D|=k} \chi_{\mathbb{R}^d}(D)$ is polynomial in k [Erd81].

In this paper we answer the question of Croft, Falconer and Guy in the measurable setting by showing that in the dimension $d \geq 2$ as the ratios $d_1/d_2, d_2/d_3, \dots, d_{k-1}/d_k$ all tend to infinity $m_{\mathbb{R}^d}(D)$ tends to $m(\{1\})^k$, and thus $\sup_{|D|=k} \chi_{\mathbb{R}^d}(D) \geq 1/m(\{1\})^k$. This also generalizes the above-mentioned theorems of Furstenberg-Katznelson-Weiss and Falconer. Indeed, to deduce Furstenberg-Katznelson-Weiss theorem suppose there is a set A with $\bar{d}(A) > 0$ and a sequence d_1, d_2, \dots going to infinity such that the distance d_i does not occur between points of A . Then there is a subsequence such that $d_{i_2}/d_{i_1}, d_{i_3}/d_{i_2}, \dots$ tends to infinity, implying $\bar{d}(A) \leq m(\{d_{i_1}, \dots\}) \leq m(\{1\})^k$ for any positive integer k . In fact our result is stronger:

Theorem 1. *Suppose $d \geq 2$ and let $D_1, \dots, D_k \subset \mathbb{R}^+$ be arbitrary finite sets. If the ratios $t_1/t_2, t_2/t_3, \dots, t_{k-1}/t_k$ tend to infinity, then*

$$m_{\mathbb{R}^d}(t_1 \cdot D_1 \cup \dots \cup t_k \cdot D_k) \rightarrow \prod_{i=1}^k m_{\mathbb{R}^d}(D_i).$$

It is conceivable that there might be denser and denser D -avoiding sets whose density approaches $m_{\mathbb{R}^d}(D)$ without there being a D -avoiding set of density $m_{\mathbb{R}^d}(D)$. However, that is not the case. We show that there is a set which not just achieves this density, but which measure cannot be increased by an alteration on a bounded subset. Moreover, we show that the constants $m_{\mathbb{R}^d}(D)$ can in principle be computed for any finite set D . However, the high time complexity of our algorithm prohibits us from settling the question whether $m_{\mathbb{R}^2}(\{1\}) < 1/4$.

The principal tool of the paper is the so-called zooming-out lemma stating that under the appropriate conditions we can ignore the small-scale details of the measurable sets in question. In this sense, it is similar to the celebrated Szemerédi regularity lemma. The Szemerédi regularity lemma implies that for the purpose of counting subgraph every graph can be replaced by a much smaller “reduced graph” [KS96]. The zooming-out lemma states that every measurable set can be replaced by a “zoomed-out set” which captures some of information about counting (by an appropriate integral) pairs of points that are at a given distance away.

2 The 1-dimensional case and the main idea

Before delving into the proof of the results in \mathbb{R}^d it is instructive to examine the situation in \mathbb{Z} , for it is much simpler, of interest on its own right, and illustrates some of the ideas used in the main results.

Throughout the paper we identify sets with their characteristic functions, i.e., for a set A we define $A(x) = 1$ if $x \in A$ and $A(x) = 0$ if $x \notin A$. In this section we use the notation $[a..b]$ to denote the interval of the integers from a to b , i.e., $[a..b] = \mathbb{Z} \cap [a, b]$.

For a set $A \subset \mathbb{Z}$ define upper and lower densities by

$$\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A \cap [-n..n]|}{2n+1}, \quad \underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{|A \cap [-n..n]|}{2n+1}.$$

The set A is D -avoiding if $(A - A) \cap D = \emptyset$, where $A - A = \{a_1 - a_2 : a_1, a_2 \in A\}$ is the difference set of A . Define $m(D) = \sup \bar{d}(A)$ where the supremum is over all D -avoiding sets.

The simple-minded analogue of theorem 1 is false. If A is $\{1\}$ -avoiding set, then $A(x-1) + A(x) \leq 1$ and thus $2|A \cap [-n..n]| \leq \sum_{k=-n}^{n+1} (A(x-1) + A(x)) \leq 2n+2$ showing that $m(\{1\}) \leq 1/2$. On the other hand, the set of even integers shows that $m(\{1\}) = 1/2$. However, for every odd integer t the set of even integers shows that $m(\{1\} \cup t \cdot \{1\}) = m(\{1, t\}) = 1/2$. This example also shows why the theorem 1 is itself false in \mathbb{R}^1 . In \mathbb{R}^1 the integration of the inequality $A(x) + A(x+1) \leq 1$ yields $m(\{1\}) \leq 1/2$. The set $\bigcup_{k \in \mathbb{Z}} [2k, 2k+1]$ shows that $m(\{1\}) = 1/2$, and the same set shows that $m(\{1, t\}) = 1/2$ for every odd integer t .

The version of theorem 1 that works in one dimension involves excluding thickened sets, in order to avoid this kind of congruential obstacles. For a set $D \subset \mathbb{Z}$ we denote by D^k the k -neighborhood of D , i.e., $D^k = \{x \in \mathbb{Z} : |x - y| \leq k \text{ for some } y \in D\}$.

Theorem 2. *For every finite set $D_1 \subset \mathbb{Z}$ there is a k such that for every finite non-empty set $D_2 \subset \mathbb{Z}$ we have*

$$m(D_1 \cup (t \cdot D_2)^k) < m(D_1)m(D_2),$$

no matter how large the integer t is.

Proof. Denote $\text{diam } D = \max_{d \in D} |d|$. Let k be any even integer so that $\text{diam } D_1 - km(D_1) \leq -1$.

Suppose A is $D_1 \cup (t \cdot D_2)^k$ -avoiding. Then the set $A^{k/2}$ is $t \cdot D_2$ -avoiding. To see that suppose $x_1, x_2 \in A^{k/2}$ is a pair of elements such that $x_1 - x_2 \in t \cdot D_2$. By the definition of $A^{k/2}$ there are $y_1, y_2 \in A$ with $|x_1 - y_1| \leq k/2$ and $|x_2 - y_2| \leq k/2$. By the triangle inequality $y_1 - y_2 \in (t \cdot D_2)^k$, which is a contradiction.

Write the set $A^{k/2}$ as a union of disjoint intervals $A^{k/2} = [a_1..b_1] \cup [a_2..b_2] \cup \dots$ where for no i, j we have $b_i + 1 = a_j$. Each of these intervals has length at least k . If q is the smallest element of D_2 , then none of these intervals has length exceeding tq , for $A^{k/2}$ is $\{tq\}$ -avoiding. The density of $A^{k/2}$ does not exceed $m(t \cdot D_2) = m(D_2)$. The set A is contained $A^{k/2}$, so it suffices to bound the density of A on each of the intervals $[a_i..b_i]$. By translating the interval $[a_i..b_i]$ it suffices to consider the case $[a_i..b_i] = [0..n-1]$.

So, suppose $A' \subset [0..n-1]$ is D_1 -avoiding and $|A'| = s$. Then $\tilde{A} = A' + (n + \text{diam } D_1)\mathbb{Z} = \bigcup_{t \in \mathbb{Z}} (A' + (n + \text{diam } D_1)t)$ is D_1 -avoiding because the copies of a D_1 -avoiding set A' are too far from each other for there to be elements x, y in different copies such that $x - y \in D_1$. Since \tilde{A} has density $s/(n + \text{diam } D_1) \leq m(D_1)$, we infer $s \leq m(D_1)n + \text{diam } D_1$.

Now let us turn back to the proof of the theorem. For each interval $[a_i..b_i]$ the subintervals $[a_i..a_i + k/2 - 1]$ and $[b_i - k/2 + 1..b_i]$ do not meet A . Thus each interval in $A^{k/2}$ of length n contains no more than $m(D_1)(n - k) + \text{diam } D_1 \leq m(D_1)n - 1 \leq (m(D_1) - 1/tq)n$ elements of A .

Similarly no more than $m(t \cdot D_2)r + \text{diam}(t \cdot D_2) = m(D_2)r + t \text{diam } D_2$ elements belong to $A^{k/2}$ in any interval of length r . Let n be an arbitrary positive integer. Consider $B = \bigcup_{[a_i..b_i] \subset [-n..n]} [a_i..b_i]$. Since at most two intervals contain elements in $[-n..n]$, but not contained in $[-n..n]$, we have $|B| \geq |[-n..n] \cap A| - 2tq$. Hence,

$$|[-n..n] \cap A| \leq |B| + 2tq \leq (m(D_2)(2n+1) + t \text{diam } D_2)(m(D_1) - 1/tq) + 2tq.$$

Letting $n \rightarrow \infty$ we conclude that $\bar{d}(A) \leq m(D_2)(m(D_1) - 1/tq) < m(D_1)m(D_2)$. \square

As remarked above the reason why theorem 1 fails in the dimension one is because the largest D -avoiding sets can be periodic (in fact there is always a set of density $m(D)$ which is periodic as shown by Cantor and Gordon [CG73]), and thus avoid many more distances than required of it. By the theorem of Furstenberg, Katznelson and Weiss [FKW90] this cannot happen in higher dimensions because any periodic set has positive density, and all sufficiently large distances occur in sets of positive density. So, it is not surprising that in the higher dimensions it becomes possible to carry out a proof very similar in spirit to the proof of theorem 2 above, but technically more complicated.

The approach employed in this paper is rooted in the proof of Bourgain [Bou86] of the Furstenberg-Katznelson-Weiss theorem.

3 Notation

Throughout the rest of the paper the dimension $d \geq 2$ is going to be fixed, so we will often omit the dependency on d from our notation.

For a measurable set $A \subset \mathbb{R}^d$ the notation $|A|$ denotes the measure of A . The notation $Q(x, r)$ denotes the open axis-parallel box of side length r centered at the point x .

For a set $A \subset \mathbb{R}^d$ and a bounded domain $\Omega \subset \mathbb{R}^d$ the density of A on Ω is

$$d_\Omega(A) = \frac{|A \cap \Omega|}{|\Omega|}.$$

The upper and lower limit densities of A are

$$\bar{d}(A) = \limsup_{R \rightarrow \infty} d_{Q(0,R)}(A), \quad \underline{d}(A) = \liminf_{R \rightarrow \infty} d_{Q(0,R)}(A).$$

Whenever $\bar{d}(A) = \underline{d}(A)$ we write $d(A) = \bar{d}(A) = \underline{d}(A)$. Note that we measure the densities with respect to cubes, and not balls as it is usually done. Whereas, in general these densities are different, corollary 12 below implies that our results do not depend on the kind of density chosen, and the proofs are more transparent if one employs the density with respect to the cubes.

Being interested in the largest D -avoiding sets, we define

$$m(D) = \sup_{A \text{ is } D\text{-avoiding}} \bar{d}(A).$$

More generally, we will be looking at the properties of sets that are more general than the property of being D -avoiding. So, we let $\mathfrak{M}(\mathbb{R}^d)$ denote the family of all the measurable subsets of \mathbb{R}^d and call a function $P: \mathfrak{M}(\mathbb{R}^d) \rightarrow \{0, 1\}$ a *property*. If $P(A) = 1$, we say that A has the property P , and if $P(A) = 0$, we say that A does not have it. We define

$$m(P) = \sup_{P(A)=1} \bar{d}(A), \quad m_\Omega(P) = \sup_{P(A)=1} d_\Omega(A).$$

For a property P and a real number $t > 0$ the property $t \cdot P$ is the property that holds for A precisely when the property P holds for $(1/t) \cdot A$. This is in agreement with the definition of $t \cdot D$ -avoiding set as a set A such that $(1/t) \cdot A$ is D -avoiding. Note that the function m is scale-invariant: for every $t > 0$ we have $m(t \cdot P) = m(P)$.

If P_1 and P_2 are two properties, then P_1 AND P_2 denotes the property asserting that both P_1 and P_2 hold, i.e., $(P_1 \text{ AND } P_2)(A) = P_1(A)P_2(A)$. In particular, if P_1 and P_2 are the properties of being D_1 - and D_2 -avoiding respectively, then P_1 AND P_2 is the property of being $D_1 \cup D_2$ -avoiding.

4 Supersaturable properties

In this section we prove basic theorems about a class of properties for which the analogue of theorem 1 holds.

As explained in the introduction, the crucial tool is the ability to ignore the fine details of the sets. The intuition here is that given a set A and a large real number t in order to understand whether the set has points which are at distance t apart we should zoom-out away from the set A and look at the scale where t stays constant. If we think of the set A as colored black on the otherwise white background, then the very fine details of A will blur into some shade of gray. The zooming-out lemma says that for our purposes if the shade is not too light, then we can treat gray points as if they were black.

More formally, for each $\delta > 0$ and $\epsilon > 0$ we define a zooming-out operator $\mathcal{Z}_\delta(\epsilon)$ acting on $\mathfrak{M}(\mathbb{R}^d)$ by

$$\mathcal{Z}_\delta(\epsilon)A = \{x \in \mathbb{R}^d : |Q(x, \delta) \cap A| > \epsilon|Q(x, \delta)|\}.$$

One can think of the zooming-out operator as the replacement for the operation of thickening sets $A \mapsto A^{k/2}$ in the integers. In the sequel we use the following easy properties of the zooming-out operator which we now state.

Lemma 3. *a. $\mathcal{Z}_\delta(\epsilon)A + Q(0, (t-1)\delta) \subset \mathcal{Z}_{t\delta}(t^{-d}\epsilon)A$ for any $t \geq 1$.*

$$b. \mathcal{Z}_{\delta_1}(\epsilon_1)\mathcal{Z}_{\delta_2}(\epsilon_2)A \subset \mathcal{Z}(\delta_1 + \delta_2) \left(\epsilon_1 \epsilon_2 \frac{\delta_1^d \delta_2^d}{(\delta_1 + \delta_2)^d \min(\delta_1, \delta_2)^d} \right) A.$$

Proof. The claim a is clear, so we show b. If $x \in \mathcal{Z}_{\delta_1}(\epsilon_1)\mathcal{Z}_{\delta_2}(\epsilon_2)A$, then

$$\epsilon_1 \delta_1^d \leq \int \mathcal{Z}_{\delta_2}(\epsilon_2)A(y)Q(x, \delta_1)(y) dy.$$

Since for all y we have

$$\epsilon_2 \delta_2^d \mathcal{Z}_{\delta_2}(\epsilon_2)A(y) \leq \int A(z)Q(y, \delta_2)(z) dz,$$

it follows that

$$\epsilon_1 \epsilon_2 \delta_1^d \delta_2^d \leq \int A(z)|Q(z, \delta_2) \cap Q(x, \delta_1)| dz \leq \min(\delta_1, \delta_2)^d |A \cap Q(x, \delta_1 + \delta_2)|.$$

Therefore $x \in \mathcal{Z}(\delta_1 + \delta_2) \left(\epsilon_1 \epsilon_2 \frac{\delta_1^d \delta_2^d}{(\delta_1 + \delta_2)^d \min(\delta_1, \delta_2)^d} \right) A$ as desired. \square

We say that a property P is *supersaturable* if there is a function $I_P: \mathfrak{M}(\mathbb{R}^d) \rightarrow [0, \infty]$ such that the following seven conditions are satisfied:

- I. $0 < m(P)$.
- II. $I_P(A)$ is monotone nondecreasing and P is monotone, i.e., $I_P(A) \geq I_P(B)$ and $P(A) \leq P(B)$ if $A \supset B$.
- III. $I_P(A) > 0$ implies that A does not have the property P .
- IV. Both P and I_P are translation-invariant: $P(A) = P(A+x)$ and $I_P(A) = I_P(A+x)$ for every $x \in \mathbb{R}^d$.
- V. There is a real number, which we denote by $\text{diam}(P)$, such that if A_1 and A_2 are sets which are at distance at least $\text{diam}(P)$ away from each other, then $I_P(A_1 \cup A_2) \geq I_P(A_1) + I_P(A_2)$ and $A_1 \cup A_2$ has the property P iff both A_1 and A_2 have the property P .

VI. There is $\epsilon > 0$ such that if $\mathcal{Z}_\delta(1 - \epsilon)A$ does not have the property P , then $I_P(A) \geq f(\delta)$ for some strictly positive function f .

VII. (Zooming-out lemma) If $A \subset Q(0, R)$ then $I_P(A) \geq g_P(\epsilon)I_P(\mathcal{Z}_\delta(\epsilon)A) - h_P(\epsilon, \delta)R^d$, where g_P is non-negative and $h_P(\epsilon, \delta) \rightarrow 0$ as $\delta \rightarrow 0$ for any fixed $\epsilon > 0$.

We call I_P a *saturation function* for the property P . An example of supersaturable property to keep in mind is the property of being $\{1\}$ -avoiding, for which the saturation function can be chosen to be $I(A) = \iint A(x)A(x+y) d\sigma(y) dx$ where σ is the uniform measure on the unit circle, and here for the second time we use the convention that a set A is identified with the characteristic function of A . In this example, with the exception of the zooming-out lemma all the conditions are not hard to check, and the zooming-out lemma will be proved in section 5. We will also see that the property of being D -avoiding is an example of a supersaturable property.

The motivation for the definition of the supersaturable properties is that not only $I_P(A) > 0$ implies that A does not have the property P , but also $\bar{d}(A) > m(P)$ implies that $I_P(A) > 0$. The latter statement is the content of the following lemma.

Supersaturation lemma. *Let P be a supersaturable property. For every $\epsilon_1, \epsilon_2 > 0$ there is a constant $c = c(\epsilon_1, \epsilon_2) > 0$ such that for any $R > 0$ there is $\delta_0 = \delta_0(\epsilon_1, \epsilon_2, R)$ such that the following holds. For any $\delta \leq \delta_0$ and any measurable set $A \subset \mathbb{R}^d$ if*

$$d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_1)A) > m_{Q(0,R)}(P)(1 + \epsilon_2),$$

then

$$I_P(A) \geq cR^d.$$

In particular, A does not have the property P .

Moreover, $\delta_0(\epsilon_1, \epsilon_2, R)$ is a monotone non-decreasing function of R for any fixed ϵ_1, ϵ_2 .

Before proving the supersaturation lemma, we need to show that the rate of convergence in the definition of $m(P)$ cannot be too slow:

Lemma 4. *Let P be a property satisfying the conditions IV and V. If A has the property P , then*

$$m(P) \geq d_{Q(0,R)}(A) / \left(1 + \frac{\text{diam } P}{R}\right)^d.$$

Proof. Set $A' = A \cap Q(0, R)$. Then the tiling $T = A' + (R + \text{diam } P)\mathbb{Z}^d$ has the property P because the distance between the translates of A' is $\text{diam } P$ and A' has the property P . Since $m(P) \geq d(T)$, the fact follows. \square

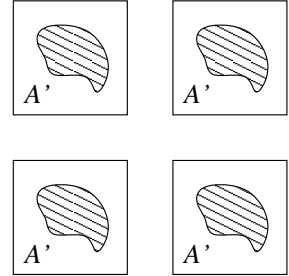


Figure 1: Tiling T

Proof of supersaturation lemma. Since the condition of the lemma refers only to the set $Q(0, R) \cap \mathcal{Z}_\delta(\epsilon_1)A$ we can assume without any loss of generality that $A \subset Q(0, R + 2\delta)$. Denote by $\mathbf{Lemma}(\epsilon_1, \epsilon_2)$ the statement that the lemma holds for some specific ϵ_1 and ϵ_2 . Denote by $\mathbf{Lemma}'(\epsilon_2)$ the statement that if $A \subset Q(0, R)$ with $d_{Q(0,R)}(A) \geq m_{Q(0,R)}(P)(1 + \epsilon_2)$, then the inequality $I_P(A) \geq c'R^d$ holds with $c' = c'(\epsilon_2) > 0$. First, for every $\epsilon_2 > 0$ and some specific $\epsilon_T > 0$ we will show a weaker version of $\mathbf{Lemma}(1 - \epsilon_T, \epsilon_2)$ where under the same conditions as in the supersaturation lemma we deduce the conclusion that is weakened by allowing the constant c to depend not only on ϵ_2 but also on δ . Then we will show that $\mathbf{Lemma}'(\epsilon_2)$ implies $\mathbf{Lemma}(\epsilon_1, \epsilon_2)$ for any $\epsilon_1 > 0$. Finally, we will show that $\mathbf{Lemma}(\epsilon_2 \epsilon_T m_{Q(0,R)}(P)/4, (1 + \epsilon_T/8)\epsilon_2)$ and the weaker version of $\mathbf{Lemma}(1 - \epsilon_T, \epsilon_2/2)$ together imply $\mathbf{Lemma}'(\epsilon_2)$. Since for $\epsilon_2 \geq 1/m_{Q(0,R)}(P)$ the $\mathbf{Lemma}'(\epsilon_2)$ is

vacuously true, all of these imply $\text{Lemma}'(\epsilon_2)$ for all $\epsilon_2 > 0$ by induction on $\lceil \log_{1+\epsilon_T/8} \frac{1}{\epsilon_2} \rceil$. Then the proof will be complete.

Weaker version of Lemma($1 - \epsilon_T, \epsilon_2$): We let ϵ_T to be the ϵ whose existence is postulated in the condition VI. We set $\delta_0 = \text{diam } P$. Choose R' so large that

$$\left(\frac{R'}{R' - 3 \text{diam } P} \right)^d \leq \min \left(1 + \frac{m_{Q(0,R)}(P)\epsilon_2}{4}, \frac{1 + \epsilon_2/2}{1 + \epsilon_2/3} \right).$$

If $R \leq \frac{8d}{m_{Q(0,R)}(P)\epsilon_2} R'$ then since $\mathcal{Z}_\delta(\epsilon_T)(A)$ does not have the property P , the condition VI tells us $I_P(A) \geq f(\delta) \geq c(\epsilon_2, \delta)R^d$ provided c is chosen small enough. So, assume $R > \frac{8d}{m_{Q(0,R)}(P)\epsilon_2} R'$. Let $k = \lfloor R/R' \rfloor$. Let \mathcal{C}_1 be a collection of k^d disjoint cubes inside $Q(0, R)$ of side length R' each. Let $\mathcal{C}_2 = \{Q(x, R' - 3 \text{diam } P) : Q(x, R') \in \mathcal{C}_1\}$. Then

$$\begin{aligned} d_{\bigcup \mathcal{C}_2}(\mathcal{Z}_\delta(\epsilon_T)A) &\geq \frac{|Q(0, R)|d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_T)A) - |Q(0, R) \setminus \bigcup \mathcal{C}_2|}{|\bigcup \mathcal{C}_2|} \\ &= 1 - (1 - d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_T)A)) \frac{R^d}{|\bigcup \mathcal{C}_1|} \cdot \frac{|\bigcup \mathcal{C}_1|}{|\bigcup \mathcal{C}_2|} \\ &\geq 1 - (1 - d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_T)A)) \frac{R^d}{(R - R')^d} \cdot \left(\frac{R'}{R' - 3 \text{diam } P} \right)^d \\ &\geq 1 - (1 - d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_T)A)) \frac{1}{1 - \frac{m_{Q(0,R)}(P)\epsilon_2}{8}} \cdot \left(1 + \frac{m_{Q(0,R)}(P)\epsilon_2}{4} \right) \\ &\geq 1 - (1 - d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_T)A)) \left(1 + \frac{m_{Q(0,R)}(P)\epsilon_2}{2} \right) \end{aligned}$$

where in the last inequality we used that $m_{Q(0,R)}(P)\epsilon_2 < 1$. Thus

$$\begin{aligned} d_{\bigcup \mathcal{C}_2}(\mathcal{Z}_\delta(\epsilon_T)A) &\geq d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_T)A) - \frac{m_{Q(0,R)}(P)\epsilon_2}{2} \\ &\geq m_{Q(0,R)}(P)(1 + \epsilon_2/2) \\ &\geq m(P)(1 + \epsilon_2/2) \end{aligned}$$

From lemma 4, and the choice of R' we get

$$d_{\bigcup \mathcal{C}_2}(\mathcal{Z}_\delta(\epsilon_T)A) \geq m_{Q(0,R'-3 \text{diam } P)}(P)(1 + \epsilon_2/3)$$

Let $\mathcal{C}_3 = \{Q(x, R' - 3 \text{diam } P) : d_{Q(x, R' - 3 \text{diam } P)}(\mathcal{Z}_\delta(\epsilon_T)A) > m_{Q(0,R'-3 \text{diam } P)}(P)(1 + \epsilon_2/6)\}$. Set $n = |\mathcal{C}_3|$. Then

$$k^d m_{Q(0,R'-3 \text{diam } P)}(P)(1 + \epsilon_2/3) \leq n + (k^d - n) m_{Q(0,R'-3 \text{diam } P)}(P)(1 + \epsilon_2/6). \quad (1)$$

Since by lemma 4

$$\begin{aligned} m_{Q(0,R'-3 \text{diam } P)}(P)(1 + \epsilon_2/6) &\leq m(P) \left(1 + \frac{3 \text{diam } P}{R' - 3 \text{diam } P} \right)^d (1 + \epsilon_2/6) \\ &\leq m_{Q(0,R)}(P)(1 + \epsilon_2/2) < 1, \end{aligned}$$

the inequality (1) implies

$$n \geq k^d \frac{m_{Q(0,R'-3 \text{diam } P)}(P)\epsilon_2/6}{1 - m_{Q(0,R'-3 \text{diam } P)}(P)(1 + \epsilon_2/6)}.$$

Since $\delta \leq \delta_0 = \text{diam } P$ we have $Q(x, R' - 3 \text{diam } P) \cap \mathcal{Z}_\delta(\epsilon_T)A \subset \mathcal{Z}_\delta(\epsilon_T)(Q(x, R' - \text{diam } P) \cap A)$. Therefore if $Q(x, R' - 3 \text{diam } P) \in \mathcal{C}_3$, then the condition VI implies $I_P(Q(x, R' - \text{diam } P) \cap A) \geq f(\delta)$. Since $Q(x_1, R' - \text{diam } P)$ and $Q(x_2, R' - \text{diam } P)$ are at distance at least $\text{diam } P$ for distinct $Q(x_1, R' - 3 \text{diam } P), Q(x_2, R' - 3 \text{diam } P) \in \mathcal{C}_3$, we can apply the condition V to deduce

$$I_P \left(A \cap \bigcup_{Q(x, R' - 3 \text{diam } P) \in \mathcal{C}_3} Q(x, R' - \text{diam } P) \right) \geq c(\delta, \epsilon_2)R^d.$$

The monotonicity condition II allows us to conclude that $I_P(A) \geq c(\delta, \epsilon_2)R^d$.

Lemma'(ϵ_2) implies Lemma(ϵ_1, ϵ_2): Suppose a set A satisfies conditions of Lemma(ϵ_1, ϵ_2). Then the zooming-out lemma and Lemma'(ϵ_2) tell us that

$$I_P(A) \geq g(\epsilon_1)I_P(\mathcal{Z}_\delta(\epsilon_1)A) - c_6 h(\epsilon_1, \delta)(R + \delta)^d \geq g(\epsilon_1)c'(\epsilon_2)R^d - c_6 h(\epsilon_1, \delta)(R + \delta)^d.$$

If δ small enough, we obtain that $I_P(A) \geq c(\epsilon_1, \epsilon_2)R^d$.

Lemma($\epsilon_2 \epsilon_T m_{Q(0,R)}(P)/4, (1 + \epsilon_T/8)\epsilon_2$) and the weaker version of Lemma($1 - \epsilon_T, \epsilon_2/2$) imply Lemma'(ϵ_2): With hindsight we set $\epsilon_1 = \epsilon_2 \epsilon_T m_{Q(0,R)}(P)/4$. The condition I asserts that $m(P) > 0$ ensuring that $\epsilon_1 > 0$. Let $\delta = \min(\text{diam } P, R\epsilon_1/25d, \delta_0(\epsilon_1, (1 + \epsilon_T/8)\epsilon_2, R))$. Suppose we have a set A satisfying the conditions of Lemma'(ϵ_2). If A also satisfies the conditions of the weaker version of Lemma($1 - \epsilon_T, \epsilon_2/2$), then $I_P(A)$ is as large as it should be, and we are done. Hence, the conditions of Lemma($1 - \epsilon_T, \epsilon_2/2$) do not hold. Since $\delta \leq \text{diam } P$, the only way in which the conditions of Lemma($1 - \epsilon_T, \epsilon_2/2$) can fail is

$$d_{Q(0,R)}(\mathcal{Z}_\delta(1 - \epsilon_T)A) \leq m_{Q(0,R)}(P)(1 + \epsilon_2/2).$$

Since the average density of A is at least $m_{Q(0,R)}(P)(1 + \epsilon_2)$ and the inequality above says that the density of points that are centers of cubes of large density is no more than $m_{Q(0,R)}(P)(1 + \epsilon_2/2)$, there should be many points that are centers of cubes with medium density ϵ_1 . For this we need to first relate $|A|$ to $|\mathcal{Z}_\delta(\epsilon_1)A|$. For that we need to allow for the edge effects due to averaging over the cube of edge length $R + 2\delta$ rather than R . Since $A \subset Q(0, R + 2\delta)$,

$$\begin{aligned} |A| &\leq |A \cap Q(0, R - 2\delta)| + 4d\delta(R + 2\delta)^{d-1} \\ &\leq (\epsilon_1 + (1 - \epsilon_T)d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_1)A) + \epsilon_T d_{Q(0,R)}(\mathcal{Z}_\delta(1 - \epsilon_T)A)) R^d + 5d\delta R^{d-1}. \end{aligned}$$

The definition of A gives us

$$|A| \geq m_{Q(0,R)}(P)(1 + \epsilon_2)R^d.$$

The two inequalities together yield

$$d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_1)A) \geq \frac{m_{Q(0,R)}(P)(1 - \epsilon_T + \epsilon_2(1 - \epsilon_T/2)) - \epsilon_1 - 5d\delta/R}{1 - \epsilon_T}.$$

Our choice of ϵ_1 and δ , made in the beginning of the proof, assures us that the left side is at least $m_{Q(0,R)}(P)(1 + (1 + \epsilon_T/8)\epsilon_2)$. Thus, we can apply Lemma($\epsilon_1, (1 + \epsilon_T/8)\epsilon_2$), and get the desired bound on $I_P(A)$. This completes the proof of the final implication, and thus the supersaturation lemma is proved. \square

One can combine the supersaturation lemma with lemma 4 to obtain a weak form of supersaturation lemma which is easier to apply:

Weak supersaturation lemma. *Let P be a supersaturable property. For every $\epsilon_1, \epsilon_2 > 0$ there are $\delta_0 = \delta_0(\epsilon_1, \epsilon_2) > 0$ and $R_0 = R_0(\epsilon_2) > 0$ such that for any $\delta < \delta_0$ and $R \geq R_0$ and any measurable set $A \subset \mathbb{R}^d$ if*

$$d_{Q(0,R)}(\mathcal{Z}_\delta(\epsilon_1)A) > m(P)(1 + \epsilon_2),$$

then

$$I_P(A) \geq cR^d$$

for some constant $c = c(\epsilon_1, \epsilon_2) > 0$ independent of δ and A .

Proof. Choose R_0 to be large enough so that $m_{Q(0,R)}(P) \leq m(P)(1 + \epsilon_2/2)$ for $R \geq R_0$. Set $\delta_0 = \delta_0(\epsilon_1, \epsilon_2, R_0)$. The monotonicity of $\delta_0(\epsilon_1, \epsilon_2, R)$ in the supersaturation lemma then insures that any choice of $\delta \leq \delta_0$ and $R \geq R_0$ satisfies the conditions of the supersaturation lemma. \square

Lemma 5. *If P_1 and P_2 are properties satisfying the conditions IV and V, then $m(P_1 \text{ AND } P_2) \geq m(P_1)m(P_2)$.*

Proof. Fix $\epsilon > 0$. Take R to be a large enough function of ϵ . Then pick a set A_1 with property P_1 such that $\bar{d}(A_1) \geq m(P)(1 - \epsilon)$. By averaging there is a cube $Q(x, R)$ such that $d_{Q(x,R)}(A_1) \geq m(P)(1 - 2\epsilon)$. Then the proof of lemma 4 shows existence of a periodic set A'_1 with property P_1 of period R with $d(A'_1) \geq m(P) \geq (1 - 3\epsilon)$. Similarly, we can construct a periodic set A'_2 with property P_2 with period R and $m(A'_2) \geq m(P_2)(1 - 3\epsilon)$. Then averaging $d((A'_1 + x) \cap A'_2)$ over $x \in Q(0, R)$ yields existence of a x_0 such that $d((A'_1 + x_0) \cap A'_2) \geq m(P_1)m(P_2)(1 - 3\epsilon)^2$. Since ϵ was arbitrary, the lemma follows. \square

Lemma 6. *If P_1 and P_2 are any two supersaturable properties, then so is $P_1 \text{ AND } P_2$.*

Proof. Let I_{P_1} and I_{P_2} be the saturation functions for P_1 and P_2 . Then $I_{P_1} + I_{P_2}$ is a saturation function for $P_1 \text{ AND } P_2$. The condition I follows from the lemma above. The conditions II, III, IV and VII follow from the corresponding conditions for P_1 and P_2 . For the conditions V and VI we can take $\text{diam}(P_1 \text{ AND } P_2) = \max(\text{diam } P_1, \text{diam } P_2)$ and $\epsilon(P_1 \text{ AND } P_2) = \min(\epsilon(P_1), \epsilon(P_2))$ respectively. \square

Now we are ready to derive a generalization of theorem 1:

Theorem 7. *Suppose P_1, \dots, P_n are supersaturable properties. Then*

$$m(t_1 \cdot P_1 \text{ AND } \dots \text{ AND } t_n P_n) \rightarrow \prod_{i=1}^n m(P_i)$$

if for all $i \neq j$ the limit of t_i/t_j is either 0 or ∞ .

Proof. The inequality $m(t_1 \cdot P_1 \text{ AND } \dots \text{ AND } t_n P_n) \geq \prod_{i=1}^n m(P_i)$ follows from lemma 5 and scale-invariance of m by induction on n .

For the proof of the opposite inequality we permute P_1, \dots, P_n and the corresponding variables t_1, \dots, t_n so that $t_{i+1}/t_i \rightarrow 0$ for all i . Furthermore, we scale t 's so that $t_1 = 1$. Fix an arbitrary $\epsilon > 0$. Let δ be the minimum of $\delta_0(\epsilon, \epsilon)$ over all the properties P_1, \dots, P_{n-1} , where δ_0 is as in the statement of the weak supersaturation lemma. Consider any set A with the property $t_1 \cdot P_1 \text{ AND } \dots \text{ AND } t_n \cdot P_n$. Write $A_1 = A$. The weak supersaturation lemma applied to this set and the property P_1 asserts that

$$\bar{d}(\mathcal{Z}_\delta(\epsilon)A_1) \leq m(P_1)(1 + \epsilon).$$

For each point $x \in \mathcal{Z}_\delta(\epsilon)A_1$ the set $Q(x, \delta) \cap A_1$ has the property $t_2 \cdot P_2 \text{ AND } \dots \text{ AND } t_n \cdot P_n$. Therefore, the set $A_2 = (1/t_2) \cdot ((A_1 - x) \cap Q(0, \delta))$ has the property $P_2 \text{ AND } (t_3/t_2) \cdot P_3 \dots \text{ AND } (t_n/t_2) \cdot P_n$. The set A_2 is contained in the cube $Q(0, \delta/t_2)$. Since $t_2 \rightarrow 0$ we can assume that t_2 is small enough so that we can apply the weak supersaturation lemma to the set A_2 and property P_2 to get

$$d_{Q(0,\delta/t_2)}(\mathcal{Z}_\delta(\epsilon)A_2) \leq m(P_2)(1 + \epsilon).$$

Repeating the argument, we eventually arrive at the inequalities

$$d_{Q(0,\delta/t_{n-1})}(\mathcal{Z}_\delta(\epsilon)A_{n-1}) \leq m(P_{n-1})(1 + \epsilon)$$

and

$$d_{Q(0,\delta/t_n)}(A_n) \leq m(P_n)(1 + \epsilon).$$

These two inequalities mean that the density of A_{n-1} on cubes of size δ is no more than ϵ except a set of density no more than $m(P_{n-1})(1 + \epsilon)$ on which the density is no more than $m(P_n)(1 + \epsilon)$. Hence, averaging implies that

$$d_{Q(0,\delta/t_{n-1})}(A_{n-1}) \leq m(P_{n-1})m(P_n)(1 + \epsilon)^2 + \epsilon.$$

Then by similarly unfolding the recursion, one arrives at the inequality

$$\bar{d}(A_1) \leq \prod_{i=1}^n m(P_i)(1 + \epsilon)^n + \mathcal{O}(\epsilon).$$

Since ϵ is arbitrary, this implies that $m(t_1 \cdot P_1 \text{ AND } \dots \text{ AND } t_n \cdot P_n) = \prod_{i=1}^n m(P_i)$. \square

The definition of $m(P)$ leaves unclear whether there is “a largest” set with property P or there are larger and larger sets. If the property in question is the property of not containing a copy of a finite subset in a given family, then a largest set exists in a very strong sense.

Definition 8. A property P is said to be finite if there is a family \mathcal{P} of finite sets such that A has the property P iff no set in \mathcal{P} is a subset of A . If in addition the diameter of sets in \mathcal{P} is bounded, then the property P is said to be boundedly finite.

Definition 9. We call a measurable set $A \subset \Omega$ having property P locally optimal for the property P with respect to a measurable set Ω if the following condition holds for every bounded measurable set S : there is no measurable set $A' \subset \Omega$ with property P such that $A \cap (\mathbb{R}^d \setminus S) = A' \cap (\mathbb{R}^d \setminus S)$ such that $|A' \cap S| > |A \cap S|$. If $\Omega = \mathbb{R}^d$, then we simply say that A is locally optimal for P .

Theorem 10. If P is any boundedly finite supersaturable property and Ω is a measurable set, then there is a locally optimal set for P with respect to Ω .

The proof of theorem 10 requires an appropriate compactness result. A characteristic function of any set lies in $L^\infty(\mathbb{R}^d)$, which is a dual of $L^1(\mathbb{R}^d)$. The space $L^1(\mathbb{R}^d)$ induces a weak* topology on $L^\infty(\mathbb{R}^d)$ which is the topology in which $f_1, f_2, \dots \rightarrow f$ when $\int f_k g \rightarrow \int f g$ as $k \rightarrow \infty$ for all $g \in L^1(\mathbb{R}^d)$.

Lemma 11. If P is a finite supersaturable property, and A_1, A_2, \dots is a sequence of sets with property P whose characteristic functions converge in the weak* topology of $L^\infty(\mathbb{R}^d)$. Then there is a nonnegative function $A \in L^\infty(\mathbb{R}^d)$ such that $A_1, A_2, \dots \rightarrow A$ in the weak* topology, and $\text{supp } A = \{x : A(x) > 0\}$ has the property P .

Proof. Since A_1, A_2, \dots converge, they converge to some function, which we will call A . The Lebesgue differentiation theorem states that

$$\lim_{\delta \rightarrow 0} \frac{1}{|Q(x, \delta)|} \int_{Q(x, \delta)} |A(y) - A(x)| dy = 0 \quad \text{for almost every } x. \quad (2)$$

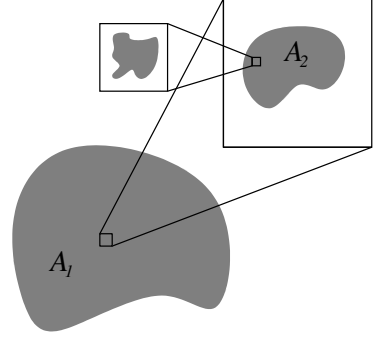


Figure 2: Recursive zooming-in

By setting A to 0 on a set of measure zero if necessary, we can assume that this holds whenever $A(x) > 0$ and A is nonnegative. We will show that this modified function A satisfies the conclusion of the lemma. Suppose that on the contrary that the set $\text{supp } A$ lacked the property P . Then by finiteness of P there would be a finite set $X = \{x_1, \dots, x_n\} \subset \text{supp } A$ such that every set containing X lacks P . Let $\epsilon = \min_{1 \leq j \leq n} A(x_j)$. Let ϵ_T be the ϵ whose existence is postulated in the condition VI. By (2) there is δ such that for every $1 \leq j \leq n$ the set $\{y \in Q(x_j, \delta) : |A(y) - A(x_j)| > \epsilon/4\}$ is of measure not exceeding $\frac{1}{4}\epsilon_T\epsilon\delta^d/24$. Let $\delta' > 0$ be any number small enough so that $(1 - \delta'/\delta)^d > 2/3$. Let $Y_j = \{z \in Q(x_j, \delta - \delta') : \int_{Q(z, \delta')} |A(y) - A(x_j)| dy > \epsilon\delta'^d/4\}$. Since

$$\begin{aligned} \int_{Q(x_j, \delta)} |A(y) - A(x_j)| dy &\geq \int_{Q(x, \delta - 2\delta')} \frac{1}{\delta'^d} \int_{Q(z, \delta')} |A(y) - A(x_j)| dy dz \\ &\geq |Y_j| \frac{\epsilon}{4}, \end{aligned}$$

it follows that $|Y_j| \leq \epsilon_T \delta^d / 6$. Let

$$W_j = \{y \in Q(x, \delta) : |d_{Q(y, \delta')} (A_k) - \frac{1}{\delta'^d} \int_{Q(y, \delta')} A(z) dz| > \epsilon/4\}.$$

Choose R to be so large that $Q(x_j, 2\delta) \subset Q(0, R)$ for all $1 \leq j \leq n$. By the definition of weak* convergence for every x we have $|A_k \cap Q(x, \delta')| = \int_{Q(x, \delta')} A_k(y) dy \rightarrow \int_{Q(x, \delta')} A(y) dy$ as $k \rightarrow \infty$. So choose k so large that $|W_j| \leq \epsilon_T \delta^d / 6$. Thus for $y \in Q(x_j, \delta) \setminus (Y_j \cup W_j)$ we have $d_{Q(y, \delta')} (A_k) \geq \epsilon/2$. Since $|Y_j \cup W_j| \leq \epsilon_T (\delta - \delta')^d / 2$, we can also write this as $x_j \in \mathcal{Z}_\delta (1 - \epsilon_T / 2) \mathcal{Z}_{\delta'} (\epsilon/2) (A_k \cap Q(0, R))$.

Let $t = \sqrt{\frac{1 - \epsilon_T / 2}{1 - \epsilon_T}}$. By lemma 3

$$Q(x_j, (t - 1)\delta) \subset \mathcal{Z}_{t\delta} (1 - \epsilon_T) \mathcal{Z}_{\delta'} (\epsilon/2) (A_k \cap Q(0, R)).$$

Since $x_j \in Q(x_j, (t - 1)\delta)$, from the condition VI we infer $I_P(\mathcal{Z}_{\delta'} (\epsilon/2) (A_k \cap Q(0, R))) > f(t\delta)$. By the zooming-out lemma VII we have

$$I_P(A_k \cap Q(0, R)) \geq g_P(\epsilon/2) f(t\delta) - h_P(\epsilon/2, \delta') R^d.$$

Since δ' is independent of both ϵ and δ , for sufficiently small δ' we would have that $I_P(A_k) \geq I_P(A_k \cap Q(0, R)) > 0$ contradicting the assumption that A_k had the property P . The contradiction shows that $\text{supp } A$ has the property P . \square

Proof of theorem 10. Let $A_1, A_2 \dots \subset \Omega$ be a sequence of sets, each having the property P , such that

$$d_{\Omega \cap Q(0, i)}(A_i) \geq m_{\Omega \cap Q(0, i)}(P) - 2^{-i}.$$

We can and will assume that $A_i \subset \Omega \cap Q(0, i)$. The Banach-Alaoglu theorem states that the closed ball in the dual of a Banach space is compact in weak* topology [Rud73, theorem 3.15]. Thus there is a subsequence A_{i_1}, A_{i_2}, \dots which converges in weak* topology. By lemma above there is a limit A of the subsequence such that the set $\text{supp } A$ has the property P . The set $\text{supp } A$ is the desired locally optimal set. Indeed, suppose that is not so, and there are R , and $\epsilon > 0$, and a set $A' \subset \Omega$ such that $|A' \cap Q(0, R)| \geq |\text{supp } A \cap Q(0, R)| + \epsilon$ and $A' \setminus Q(0, R) = \text{supp } A \setminus Q(0, R)$. Since P is boundedly finite, there is a R' and a family of sets \mathcal{P} of diameter at most R' each such that a set does not have the property P precisely when the set contains a member of \mathcal{P} . Let f be the characteristic function of $Q(0, R) \cap \text{supp } A$. By the definition of weak* convergence there are arbitrarily large k so that

$$\left| \int f(x) (A_{i_k}(x) - A(x)) dx \right| \leq \epsilon/4,$$

and

$$\left| \int_{Q(0,R)} (A_{i_k}(x) - A(x)) dx \right| \leq \epsilon/4.$$

Let $\tilde{A} = (A' \cap Q(0, R)) \cup \left(\text{supp } A \cap A_{i_k} \cap (Q(0, R + R') \setminus Q(0, R)) \right) \cup (A_{i_k} \setminus Q(0, R + R'))$. Note that $\tilde{A} \cap Q(0, R + R') \subset A' \cap Q(0, R + R')$.

If \tilde{A} did not have the property P , then there would be a finite set $X \subset \tilde{A}$ such that every set containing X does not have the property P . By the definition of R' , we would have that either X is a subset of either $(A' \cap Q(0, R)) \cup \left(\text{supp } A \cap A_{i_k} \cap (Q(0, R + R') \setminus Q(0, R)) \right)$ or $\left(\text{supp } A \cap A_{i_k} \cap (Q(0, R + R') \setminus Q(0, R)) \right) \cup (A_{i_k} \setminus Q(0, R + R'))$. Since the former is a subset of A' and the latter is a subset of A_{i_k} , we would reach a contradiction with the assumption that A' and A_{i_k} both have the property P . Thus, \tilde{A} has the property P .

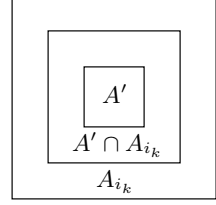


Figure 3: Set \tilde{A}

On the other hand,

$$\begin{aligned} |\tilde{A}| &= |A' \cap Q(0, R)| + \left| \text{supp } A \cap A_{i_k} \cap (Q(0, R + R') \setminus Q(0, R)) \right| + |A_{i_k} \setminus Q(0, R + R')| \\ &\geq \epsilon + |\text{supp } A \cap Q(0, R + R') \cap A_{i_k}| + |A_{i_k} \setminus Q(0, R + R')| \\ &= \epsilon + \int f(x)(A_{i_k}(x) - A(x)) dx + \int_{Q(0, R + R')} (A(x) - A_{i_k}(x)) dx \\ &\quad + |A_{i_k} \cap Q(0, R + R')| + |A_{i_k} \setminus Q(0, R + R')| \\ &\geq \epsilon/2 + |A_{i_k}| \end{aligned}$$

If k was chosen large enough, we obtain $d_{\Omega \cap Q(0, i_k)}(\tilde{A}) \geq m_{\Omega \cap Q(0, i_k)}(P) - 2^{-i_k} + i_k^{-d} \epsilon/2 > m_{\Omega \cap Q(0, i_k)}(P)$. The contradiction implies that $\text{supp } A$ is locally optimal. \square

Corollary 12. *If P is any boundedly finite supersaturable property, then there is a set A with property P such that for any open bounded set Ω*

$$\lim_{t \rightarrow \infty} d_{t \cdot \Omega}(A) = m(P).$$

Proof. It follows from Whitney decomposition, for example, that we can write Ω as a union of countably many disjoint open cubes and a set of measure zero, i.e., $\Omega = Z \cup \bigcup_{i \geq 0} Q(x_i, r_i)$ where Z is of measure zero. Let $\epsilon > 0$ be arbitrary and let A be a locally optimal set for the property P . Choose n to be large enough so that $|\bigcup_{i > n} Q(x_i, r_i)| < \epsilon$.

By lemma 4 the measure of $(t \cdot Q(x_i, r_i)) \cap A = Q(tx_i, tr_i) \cap A$ cannot exceed $|Q(tx_i, tr_i + \text{diam } P)|m(P)$. By the local optimality of A the measure of $Q(tx_i, tr_i) \cap A$ cannot be any less than $|Q(tx_i, tr_i - 2 \text{diam } P)|m(P)$. Hence

$$|Q(tx_i, tr_i) \cap A| = |Q(tx_i, tr_i)|m(P)(1 + \mathcal{O}(1/rit)).$$

Summing over i with $i \leq n$ we obtain

$$|(t \cdot \Omega) \cap A| - |t \cdot \Omega|m(P)| < \epsilon + \mathcal{O}\left(\frac{1}{t} \sum_{i \leq n} \frac{1}{r_i}\right).$$

Since t goes to infinity and ϵ is arbitrary, the corollary follows. \square

5 Zooming-out lemma

In this section we establish that several properties including the property of being D -avoiding are supersaturable.

We shall use Fourier transform on \mathbb{R}^d which is defined via

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad \hat{\sigma}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle x, \xi \rangle} d\sigma(x)$$

for a function f or a Borel measure σ , respectively. For functions $f, g \in L^1 \cap L^\infty$ and a measure $\sigma \in \mathcal{M}(\mathbb{R}^d)$ the convolutions are defined by $(f * g)(y) = \int f(y-x)g(x) dx$ and $(f * \sigma) = \int f(y-x) d\sigma(x)$, which satisfy the following well-known identities

$$\begin{aligned} \widehat{f * g}(\xi) &= \hat{f}(\xi)\hat{g}(\xi), & \int f(x)g(x) dx &= \int \hat{f}(\xi)\hat{g}(-\xi) d\xi, \\ \widehat{f * \sigma}(\xi) &= \hat{f}(\xi)\hat{\sigma}(\xi), & \int f(x)d\sigma(x) &= \int \hat{f}(-\xi)\hat{\sigma}(\xi) d\xi. \end{aligned} \tag{3}$$

Definition 13. A probability measure $\sigma \in \mathcal{M}(\mathbb{R}^d)$ with support $\text{supp } \sigma$ is admissible if σ is symmetric around 0, has compact support, $0 \notin \text{supp } \sigma$, and $\hat{\sigma}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$.

We say that a set A is σ -avoiding if there are no points $x, y \in A$ such that $x - y \in \text{supp } \sigma$. For example, the property of being $\{1\}$ -avoiding in Euclidean distance is the same as being σ -avoiding for σ being the surface measure on the unit sphere. We can assume without loss of generality that σ is symmetric around 0. Indeed if we let $\sigma'(A) = (\sigma(A) + \sigma(-A))/2$, then being σ' -avoiding is same as being σ -avoiding. Define the saturation function for the property of being σ -avoiding by $I_\sigma(A) = \int A(x)A(x+y) d\sigma(y) dx$. The saturation function is well-defined by Tonelli's theorem.

Write Q_δ for the function $Q_\delta(x) = \delta^{-d}Q(0, \delta)(x)$.

Lemma 14. There is an absolute constant c_1 such that if $\sigma \in \mathcal{M}(\mathbb{R}^n)$ is a probability measure, then for every $T > 0$

$$\left| \iint f(x)g(x+y) d\sigma(y) dx - \iint f(x)(g * Q_\delta)(x+y) d\sigma(y) dx \right| \leq \left(c_1 \delta^2 T^2 + \sup_{|\xi| > T} 2|\hat{\sigma}(\xi)| \right) \|f\|_{L^2} \|g\|_{L^2}.$$

Proof. Applying (3) we obtain

$$\begin{aligned} \iint f(x)g(x+y) d\sigma(y) dx &= \int f(x) \int g(x+y) d\sigma(y) dx = \iint f(x) e^{-2\pi i \langle -x, \xi \rangle} \hat{g}(\xi) \hat{\sigma}(-\xi) dx d\xi \\ &= \int \hat{f}(-\xi) \hat{g}(\xi) \hat{\sigma}(-\xi) d\xi \end{aligned}$$

Since $|\widehat{Q_\delta}(\xi) - 1| \leq c_1 \delta^2 |\xi|^2$ and $|\widehat{Q_\delta}(\xi)| \leq |\widehat{Q_\delta}(0)| = 1$, we obtain

$$\begin{aligned} \left| \iint f(x)(g(x+y) - (g * Q_\delta)(x+y)) d\sigma(y) dx \right| &= \left| \int \hat{f}(-\xi) \hat{g}(\xi) (1 - \widehat{Q_\delta}(\xi)) \hat{\sigma}(-\xi) d\xi \right| \\ &\leq c_1 \delta^2 T^2 \|\sigma\| \int_{|\xi| < T} |\hat{f}(-\xi) \hat{g}(\xi)| d\xi \\ &\quad + \sup_{|\xi| > T} 2|\hat{\sigma}(\xi)| \int_{|\xi| \geq T} |\hat{f}(-\xi) \hat{g}(\xi)| d\xi \end{aligned}$$

Cauchy-Schwarz and Parseval imply that $\int |\hat{f}(\xi) \hat{g}(\xi)| \leq \|\hat{f}\|_{L^2} \|\hat{g}\|_{L^2} = \|f\|_{L^2} \|g\|_{L^2}$, completing the proof. \square

Corollary 15. *If a probability measure $\sigma \in \mathcal{M}(\mathbb{R}^n)$ is admissible, then the property of being σ -avoiding satisfies the condition VII.*

Proof. Suppose $A \subset Q(0, R)$. By the definition of $\mathcal{Z}_\delta(\epsilon)A$ we have

$$\epsilon \mathcal{Z}_\delta(\epsilon)A(x) \leq (A * Q_\delta)(x)$$

which implies

$$\epsilon^2 I_\sigma(\mathcal{Z}_\delta(\epsilon)A(x)) \leq \iint (A * Q_\delta)(x)(A * Q_\delta)(x + y) d\sigma(y) dx.$$

Since σ is symmetric around 0, we have

$$\iint (A * Q_\delta)(x)A(x + y) d\sigma(y) dx = \iint A(x)(A * Q_\delta)(x + y) d\sigma(y) dx$$

and the lemma 14 applied twice yields

$$\begin{aligned} & \left| \iint (A * Q_\delta)(x)(A * Q_\delta)(x + y) d\sigma(y) dx - \iint A(x)A(x + y) d\sigma(y) dx \right| \\ & \leq \left| \iint (A * Q_\delta)(x)(A * Q_\delta)(x + y) d\sigma(y) dx - \iint (A * Q_\delta)(x)A(x + y) d\sigma(y) dx \right| \\ & \quad + \left| \iint (A * Q_\delta)(x)A(x + y) d\sigma(y) dx - \iint A(x)A(x + y) d\sigma(y) dx \right| \\ & \leq \left(c_1 \delta^2 T^2 + \sup_{|\xi| > T} 2|\hat{\sigma}(\xi)| \right) (\|A\|_{L^2} \|A * Q_\delta\|_{L^2} + \|A\|_{L^2}^2) \end{aligned}$$

Since $\|A\|_{L^2}^2 = |A| \leq R^d$ and $\|A * Q_\delta\|_{L^2} = \|\hat{A}\hat{Q}_\delta\|_{L^2} \leq \|\hat{A}\|_{L^2} = \|A\|_{L^2}$, it follows that

$$I_\sigma(A) \geq \epsilon^2 I_\sigma(\mathcal{Z}_\delta(\epsilon)A) - \left(c_1 \delta^2 T^2 + \sup_{|\xi| > T} 2|\hat{\sigma}(\xi)| \right) R^d.$$

If we let $T = \delta^{-1/2}$, the condition $\hat{\sigma}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$ implies the condition VII. \square

With the zooming-out lemma in place we are ready to show supersaturability:

Theorem 16. *If $\sigma \in \mathcal{M}(\mathbb{R}^n)$ is admissible, then the property of being σ -avoiding is supersaturable.*

Proof. The conditions II, III, IV are obvious. The compact support of σ implies the condition V. Since $0 \notin \text{supp } \sigma$ there is an $\epsilon > 0$ such that $Q(0, \epsilon) \cap \text{supp } \sigma = \emptyset$. Then the set $Q(0, \epsilon/2) + \text{diam}(\text{supp } \sigma)\mathbb{Z}^d$ has positive density and is σ -avoiding. Thus the condition I is fulfilled.

Finally to verify the condition VI let $\epsilon = 1/4$ and suppose $\mathcal{Z}_\delta(1 - \epsilon)A$ is not σ -avoiding. Then there are $x_0, y_0 \in \mathcal{Z}_\delta(1 - \epsilon)A$ such that $x_0 - y_0 \in \text{supp } A$. Then for every $z \in Q(0, \delta/8d)$ the set $(A - x_0 - z) \cap (A - y_0) \cap Q(0, \delta)$ has measure at least $\delta^d(1 - 2\epsilon - 2d/8d) = \delta^d/4$. Therefore

$$\begin{aligned} I_\sigma(A) &= \int A(x)A(x + y) d\sigma(y) dx \\ &\geq \iint_{x \in Q(x_0, \delta/8d)} A(x)A(x + y) dx d\sigma(y) \\ &\geq \frac{\delta^d}{4} \sigma(Q(y_0 - x_0, \delta/8d)) \end{aligned}$$

which is positive since $y_0 - x_0 \in \text{supp } \sigma$. \square

In particular since the surface measure on the unit sphere in \mathbb{R}^d with L^p norm for $1 < p < \infty$ and $d \geq 2$ satisfies the condition of the theorem, the property of being $\{1\}$ -avoiding in L^p is supersaturable property not only for the usual Euclidean distance, but also in L^p for $1 < p < \infty$. By lemma 6 the property of being D -avoiding for a finite set $D \subset \mathbb{R}^+$ is also of this form.

To avoid the false impression that the property of being σ -avoiding is the only supersaturable property, we demonstrate another class of natural supersaturable properties. For symmetric probability measures $\sigma_1, \sigma_2 \in \mathcal{M}(\mathbb{R}^d)$ say that a set A is σ_1 OR σ_2 -avoiding if for every point $x \in A$ either there is no point $y \in A$ such that $x - y \in \text{supp } \sigma_1$ or there is no point $y \in A$ such that $x - y \in \sigma_2$.

Theorem 17. *If σ_1, σ_2 are two admissible measures, then the property of being σ_1 OR σ_2 -avoiding is supersaturable with the saturation function $I_{\sigma_1 \text{ OR } \sigma_2}(A) = \iiint A(x)A(x+y_1)A(x+y_2) d\sigma_1(y_1) d\sigma_2(y_2) dx$.*

Proof. The conditions I through VI are checked in the same way as in the theorem 16. We will show that VII is satisfied. Since σ_1 is a probability measure, we have $\int A(x+y_1) d\sigma_1(y) \leq 1$ for every x . Therefore, lemma 14 implies the inequality

$$\begin{aligned} & \left| \iiint A(x)A(x+y_1)(A(x+y_2) - (A * Q_{2\delta})(x+y_2)) d\sigma_1(y_1) d\sigma_2(y_2) dx \right| \\ & \leq \left(4c_1\delta^2T^2 + \sup_{|\xi|>T} 2|\hat{\sigma}_2(\xi)| \right) \|A\|_{L^2}^2. \end{aligned} \quad (4a)$$

Similarly,

$$\begin{aligned} & \left| \iiint A(x)(A(x+y_1) - (A * Q_{2\delta})(x+y_1))(A * Q_{2\delta})(x+y_2) d\sigma_1(y_1) d\sigma_2(y_2) dx \right| \\ & \leq \left(4c_1\delta^2T^2 + \sup_{|\xi|>T} 2|\hat{\sigma}_1(\xi)| \right) \|A\|_{L^2}^2. \end{aligned} \quad (4b)$$

Define translation operator by $(\mathcal{T}_x f)(z) = f(z-x)$. Set $I'(f) = \iint f(y_1)f(y_2) d\sigma_1(y_1) d\sigma_2(y_2)$. Then the inequalities (4a) and (4b) imply that

$$I_{\sigma_1 \text{ OR } \sigma_2}(A) \geq \int A(x)I'(\mathcal{T}_x A * Q_{2\delta}) dx - \left(8c_1\delta^2T^2 + \sup_{|\xi|>T} 2|\hat{\sigma}_1(\xi)| + \sup_{|\xi|>T} 2|\hat{\sigma}_2(\xi)| \right) R^d. \quad (5)$$

Since for every $y \in Q(x, \delta)$ we have $(\mathcal{T}_x A * Q_\delta)(z) \leq (2^d \mathcal{T}_y A * Q_{2\delta})(z)$ and I' is monotone, it follows that

$$\begin{aligned} \int A(x)I'(\mathcal{T}_x A * Q_{2\delta}) dx & \geq 4^{-d}\delta^{-d} \iint_{y \in Q(x, \delta)} A(x)I'(\mathcal{T}_y A * Q_\delta) dx dy \\ & = 4^{-d}\delta^{-d} \iint_{y \in Q(0, \delta)} (\mathcal{T}_y A)(x)I'(\mathcal{T}_x A * Q_\delta) dx dy \\ & = 4^{-d} \int (A * Q_\delta)(x)I'(\mathcal{T}_x A * Q_\delta) dx \\ & \geq 4^{-d}\epsilon^3 I_{\sigma_1 \text{ OR } \sigma_2}(\mathcal{Z}_\delta(\epsilon)A) \end{aligned} \quad (6)$$

If we set $T = \delta^{-1/2}$, the inequalities (5) and (6) together imply the condition VII. \square

6 Applications

In this section is devoted to two applications of the general results proved above.

Theorem 18. Let σ_1, σ_2 be a pair of admissible measures. Let $P_1, P_2, P_1 \text{ OR } t \cdot P_2$ denote the properties of being σ_1 -avoiding, σ_2 -avoiding and $\sigma_1 \text{ OR } t \cdot \sigma_2$ -avoiding, respectively. Then

$$\lim_{t \rightarrow \infty} m(P_1 \text{ OR } t \cdot P_2) = \max(m(P_1), m(P_2)).$$

Theorem 19. There is an algorithm that given as input $\epsilon > 0$ and a finite set D of distances outputs $m(D)$ with absolute error at most ϵ .

Before proving the theorem 18 we need some notation and a lemma. If a set A is σ -avoiding for admissible measure σ , we set $F(A) = \{x \in \mathbb{R}^d : x - y \in \text{supp } \sigma \text{ for some } y \in A\}$ and $S(A) = F(A) \cup A$. Intuitively, if we try to enlarge A to another σ -avoiding set, then $F(A)$ is the sets which is forbidden by A and $S(A)$ is the set which is already ‘‘occupied’’ by A . Write $\text{diam } \sigma = \max_{x \in \text{supp } \sigma} |x|$.

Lemma 20. Let P be the property of being σ -avoiding. For every σ -avoiding set A we have $d_{Q(0,R)}(A) < m(P) \left(d_{Q(0,R)}(S(A)) + (1 + \text{diam } \sigma/R)^d - 1 \right)$.

Proof. We use the same trick that was used in the proof of lemma 4. The set $A_1 = (A \cap Q(0, R)) + (R + \text{diam } \sigma)\mathbb{Z}^d$ is σ -avoiding and $S(A_1) \subset \left((S(A) \cap Q(0, R)) \cup (Q(0, R + \text{diam } \sigma) \setminus Q(0, R)) \right) + (R + \text{diam } \sigma)\mathbb{Z}^d$. Let $\alpha = d(A_1)/d(S(A_1))$. Since $d(S(A_1)) \leq d_{Q(0,R)}(S(A))/(1 + \text{diam } \sigma/R)^d + (1 - 1/(1 + \text{diam } \sigma/R)^d)$ and $d(A_1) \geq d_{Q(0,R)}(A)/(1 + \text{diam } \sigma/R)^d$, it suffices to show that $\alpha \leq m(P)$.

Let $\gamma = 1 - d(S(A_1))$ be the proportion of \mathbb{R}^d which is not occupied yet. Choose a vector x uniformly at random from $Q(0, R + \text{diam } \sigma)$. For any set X periodic with fundamental region $Q(0, R + \text{diam } \sigma)$ we have that $E[d((X + x) \setminus S(A_1))] = \gamma d(X)$ where E denotes the expectation. Let $A_2 = A_1 \cup ((A_1 + x) \setminus S(A_1))$. Then $E[d(A_2)] = d(A_1) + \gamma d(A_1)$, and $E[d(S(A_2))] = d(S(A_1)) + \gamma d(S(A_1))$. Hence $E[\alpha d(S(A_2)) - d(A_2)] = 0$. It follows that the set $\{x \in Q(0, R + \text{diam } \sigma) : \alpha d(S(A_2)) - d(A_2) \leq 0\}$ has non-zero measure. In particular, it contains an element which is not a period of the set A_1 . Thus, we can ensure $d(A_2) > d(A_1)$.

Similarly we can build an increasing sequence A_1, A_2, A_3, \dots of σ -avoiding sets such that $d(A_k)/d(S(A_k)) \geq \alpha$. If the set $S(\bigcup_k A_k)$ had density 1, then we would be done, but that need not be the case. We use compactness lemma 11 to circumvent this.

So, suppose $\alpha > m(P)$. Let \mathcal{A} be the family of all σ -avoiding sets $A \subset \mathbb{R}^d$ which are periodic with the fundamental region $Q(0, R + \text{diam } \sigma)$. Let \mathcal{A}' be those of them that satisfy $d(A)/d(S(A)) \geq \alpha$. Let $\beta = \sup_{A \in \mathcal{A}'} d(A)$. Note that by the argument above the supremum is not achieved. Let $A_1, A_2 \dots \in \mathcal{A}'$ be a sequence such that $d(A_i) \rightarrow \beta$.

By passing to a subsequence if needed, assume that the sequence A_1, A_2, \dots , converges in the weak* topology of $L^\infty(\mathbb{R}^d)$. By lemma 11 there is a weak* limit A of the sequence such that $\text{supp } A$ is σ -avoiding. Let $Y_i = A_i \setminus \text{supp } A$. We claim that $d(Y_i) \rightarrow 0$ as $i \rightarrow \infty$. Suppose that there was a subsequence Y_{i_1}, Y_{i_2}, \dots on which $d(Y_i) > \epsilon > 0$. Banach-Alaoglu tells us that, by passing to a subsequence again if needed, we can assume that Y_{i_1}, Y_{i_2}, \dots converges to some function Y in weak* topology. Since $|\text{supp } Y \cap \text{supp } A| = 0$, we conclude that $\lim \int_{\text{supp } Y} (A(x) - A_i(x)) dx < -\epsilon < 0$ which contradicts the definition of the weak* convergence. Thus, $d(Y_i) \rightarrow 0$. Therefore, the sequence $A_i \cap \text{supp } A$ converges to A in the weak* topology.

Next we show that

$$d(F(\text{supp } A) \setminus F(A_i)) \rightarrow 0 \quad \text{as } i \rightarrow \infty. \quad (7)$$

Pick an $\epsilon > 0$. We will first cover almost all of the set $F(\text{supp } A)$ by cubes on which $F(\text{supp } A)$ has density at least $1 - \epsilon$. Then we will show that $F(A_i)$ has density at least $1 - 3\epsilon$ on each of the cubes provided i is large.

Let $\mathcal{Q} = \{Q(x, r) : d_{Q(x,r)}(F(\text{supp } A)) > 1 - \epsilon\}$, and let \mathcal{C} be a family of all collections of cubes from \mathcal{Q} which are pairwise disjoint. By Hausdorff maximum principle there is a maximal collection

\mathcal{M} in \mathcal{C} . Then $W = F(\text{supp } A) \setminus \bigcup \mathcal{M}$ is of measure null. Indeed, if $|W| > 0$ then by Lebesgue density for almost every $x \in W$ we would have $\lim_{\delta \rightarrow 0} d_{Q(x, \delta)}(W) = 1$, which implies that there is x and δ such that $d_{Q(x, \delta)}(W) > 1 - \epsilon$. That contradiction shows that the desired covering exists.

Now let $Q(x_0, r)$ be any cube in the covering. Let $f(x) = \int A(x+y) d\sigma(y)$. The function f is defined almost everywhere by Tonelli's theorem. Let $Z = \{x \in Q(x_0, r) \cap F(\text{supp } A) : f(x) = 0\}$. The set Z is of measure null. Indeed, if $|Z| > 0$, then by Lebesgue density theorem there would exist an $x \in F(\text{supp } A)$ such that $x \in \mathcal{Z}_\delta(2/3)Z$ for all sufficiently small δ . Let $y \in A$ be such that $|x-y| \in \text{supp } \sigma$. Then since every point of $\text{supp } A$ is a point of density, there are arbitrarily small δ such that $y \in \mathcal{Z}_\delta(2/3)(\text{supp } A)$. Thus, $\iint_{Q(x, \delta)} f(x) dx > 0$. This contradicts the definition of Z and so $|Z| = 0$. Therefore, there is a ϵ' such that the measure of $\{x \in Q(x_0, r) \cap F(\text{supp } A) : f(x) < \epsilon'\}$ does not exceed ϵr^d . Therefore, if $d_{Q(x_0, r)}(Y) \geq 3\epsilon$, then $\iint Y(x)A(x+y) d\sigma(y) dx > \epsilon \epsilon' r^d$.

Suppose there are arbitrarily large i 's such that $F(A_i)$ has density less than $1 - 3\epsilon$ on $Q(x_0, r)$. Let $Y_i = Q(x_0, r) \setminus F(A_i)$. Then $\iint Y_i(x)A(x+y) d\sigma(y) dx > \epsilon \epsilon' r^d$. Let $W_i = A_i \cap Q(x_0, r + 2 \text{diam } P + 1)$. Clearly, $\iint Y_i(x)W_i(x+y) d\sigma(y) dx = 0$. For small enough δ lemma 14 implies that

$$\begin{aligned} \iint Y_i(x)(W_i * Q_\delta)(x+y) d\sigma(y) dx &\leq \epsilon \epsilon' r^d / 4 \\ \iint Y_i(x)(A * Q_\delta)(x+y) d\sigma(y) dx &\geq \epsilon \epsilon' r^d / 2. \end{aligned} \tag{8}$$

For any $\delta < 1$ and for large enough i we have $|(W_i * Q_\delta)(x) - (A_i * Q_\delta)(x)| \leq \frac{1}{9} \epsilon \epsilon' r^d (R + 2 \text{diam } P + 1)^{-d}$ for all $x \in Q(x, r + 2 \text{diam } P)$ except a set of measure $\frac{1}{9} \epsilon \epsilon' r^d$. Then

$$\left| \iint Y_i(x+y)(A_i * Q_\delta - W_i * Q_\delta)(x) d\sigma(y) dx \right| \leq \frac{2}{9} \epsilon \epsilon' r^d,$$

which contradicts 8. Therefore, $F(W_i)$ has density at least $1 - 3\epsilon$ on $Q(x_0, r)$ for all sufficiently large i .

By (7) we get

$$d(F(\text{supp } A)) \leq \liminf_{i \rightarrow \infty} d(F(A_i)) \leq \liminf_{i \rightarrow \infty} \left(\frac{1}{\alpha} - 1 \right) d(A_i) = \left(\frac{1}{\alpha} - 1 \right) \beta.$$

Since

$$d(\text{supp } A) \geq \frac{1}{|Q(0, R + \text{diam } \sigma)|} \int_{Q(0, R + \text{diam } \sigma)} A(x) dx = \beta,$$

we conclude that

$$\alpha d(F(\text{supp } A)) \leq (1 - \alpha)\beta + \alpha d(\text{supp } A) \leq d(\text{supp } A)$$

implying that $\text{supp } A \in \mathcal{A}'$, which contradicts the assumption that the supremum in the definition of β is not achieved. \square

Proof of theorem 18. The inequality $m(P_1 \text{ OR } P_2) \geq \max(m(P_1), m(P_2))$ is obvious. Let $\epsilon > 0$ be arbitrary. Let $R = \frac{5d}{\epsilon} \text{diam } P_2$. We will show that $\limsup_{t \rightarrow 0} m_{Q(0, R)}((1/t) \cdot P_1 \text{ OR } P_2) \leq \max(m(P_1), m(P_2)) + \epsilon$.

Suppose the contrary. Let t_1, t_2, \dots be a sequence of t 's going to infinity for which $m_{Q(0, R)}((1/t_i) \cdot P_1 \text{ OR } P_2) \geq \max(m(P_1), m(P_2)) + \epsilon$. Let A_i be a locally optimal set for the property $(1/t_i) \cdot P_1 \text{ OR } P_2$. Let $A_i^1 = \{x \in A_i : \forall y \in A_i |x-y| \notin \frac{1}{t_i} \cdot \text{supp } \sigma_1\}$, and $A_i^2 = A_i \setminus A_i^1$. Note that A_i^1 is σ_1 -avoiding. By passing to a subsequence if necessary we can assume that the sequences $\{A_i^1\}_{i=1}^\infty$ and $\{A_i^2\}_{i=1}^\infty$ converge in weak*. By lemma 11 there is a limit A^2 of the sequence $\{A_i^2\}_{i=1}^\infty$ such that $\text{supp } A^2$ is σ_2 -avoiding, and every point of $\text{supp } A^2$ is a density point as in the Lebesgue

density theorem. Let A^1 be a limit of $\{A_i^1\}_{i=1}^\infty$. Moreover we can set A^1 to zero wherever the conclusion of Lebesgue differentiation theorem (2) fails. We claim that $\text{supp } A^1 \cap F(\text{supp } A^2) = \emptyset$.

Suppose that is not the case. Then there are points $a_2 \in \text{supp } A_2$ and $a_1 \in \text{supp } A_1$ such that $|a_2 - a_1| \in \text{supp } \sigma_2$. Pick a small enough δ so that $Q(0, \delta) \cap \text{supp } \sigma_2 = \emptyset$. Then $(Q(0, \delta) \cap A_i^1) \cup A_i^2$ is σ_2 -avoiding for every i . Since the conclusion of Lebesgue differentiation theorem holds for every point of $(Q(0, \delta) \cap \text{supp } A^1) \cup \text{supp } A^2$ by the argument of theorem 11 the set $(Q(0, \delta) \cap \text{supp } A^1) \cup \text{supp } A^2$ is σ_2 -avoiding. This proves the claim.

Furthermore, $A^1(x) \leq m(P_1)$ for all x . Indeed, suppose $A^1(x_0) \geq m(P_1)(1 + \epsilon)$. Since A^1 satisfies the conclusion of Lebesgue differentiation theorem (2) at x_0 , we can choose δ small enough so that $\delta^{-d} \int_{Q(x_0, \delta)} A^1(x) dx \geq m(P_1)(1 + \epsilon/2)$. Then for all sufficiently large i by lemma 4 we have

$$d_{Q(x_0, \delta)}(A_i^1) \geq m(P_1) \left(1 + \frac{\epsilon}{3}\right) \geq m_{Q(x_0, \delta)}\left(\frac{1}{t_i} \cdot P_1\right) \left(1 + \frac{\epsilon}{3}\right) / \left(1 + \frac{\text{diam } P_1}{t_i \delta}\right)^d > m_{Q(x_0, \delta)}(P_1)$$

which is in contradiction with the fact that A_i^1 is σ_1 -avoiding.

Let $\alpha = d_{Q(0, R)}(F(\text{supp } A^2))$. Therefore by the lemma above

$$\begin{aligned} \lim_{i \rightarrow \infty} d_{Q(0, R)}(A_i) &= \frac{1}{|Q(0, R)|} \int (A^1(x) + A^2(x)) dx \\ &\leq \alpha m_{Q(0, R)}(P_2) \left(1 + \frac{\text{diam } P_2}{R}\right)^d + (1 + \text{diam } P_2/R)^d - 1 + (1 - \alpha)m(P_1) \\ &\leq \max(m(P_1), m(P_2)) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, the proof is complete. \square

Proof of theorem 19. Note that the proofs of zooming-out lemma and supersaturation lemma are effective: the dependencies between all the constants are effectively computable.

For integer k partition $Q(0, 1)$ into k^d cubes in the natural way. Say a set A is k -granular if A is union of some of these cubes. Let \mathcal{G}_k be the collection of k -granular sets. Let $\mathcal{P}_k = \{A + \mathbb{Z}^d : Z \in \mathcal{G}_k\}$. The following simple algorithm outputs $m(D)$ within absolute error 8ϵ .

1. If $\epsilon > 1/10$, set $\epsilon = 1/10$.
2. Make the following assignments:

$$\begin{aligned} r &= \min D, \\ \tilde{m} &= \left(\frac{rd^{-1/2}}{rd^{-1/2} + \text{diam } D}\right)^d, \\ \epsilon_2 &= \epsilon^{2d}, \\ \epsilon_1 &= \tilde{m}\epsilon^{2d}. \end{aligned}$$

3. Set δ_0 and R_0 to the values whose existence is asserted by the weak supersaturation lemma with ϵ_1 and ϵ_2 as above.
4. Make the following assignments

$$\begin{aligned} R &= \max\left(R_0, \frac{d \text{diam } D}{\epsilon^{2d}}\right), \\ k &= \max(\lceil 1/\epsilon \rceil, \lceil 1/\delta_0 \rceil). \end{aligned}$$

5. Let P be the property of being $(1/R) \cdot D$ -avoiding. By checking each set in \mathcal{P}_{k^2} compute

$$m' = \max_{\substack{A \in \mathcal{P}_{k^2} \\ P(A)=1}} d(A).$$

6. Output m' .

The first step of the algorithm allows us to assume that $\epsilon \leq 1/10$ in our analysis. Note that since $Q(0, rd^{-1/2}) + (rd^{-1/2} + \text{diam } D)\mathbb{Z}^d$ is D -avoiding, we have $\tilde{m} \leq m(D)$. Clearly, $m' \leq m(t \cdot D) = m(D)$. We will show that $m' \geq m(D)(1 - 8\epsilon)$.

By theorems 10 and 12 there is a D -avoiding set of density $m(D)$. Thus, by the proof of lemma 4 there is a periodic D -avoiding set A with the period R and density $d(A) \geq m(D)/(1 + \epsilon^{2d}/d)^d \geq m(D)(1 - \epsilon^{2d})$. Let $A' = \mathcal{Z}_{1/k}(1 - \epsilon^d)A$. If $d(A') \leq m(D)(1 - 3\epsilon^d)$ then

$$m(D)(1 - \epsilon^{2d}) \leq d(A) \leq \epsilon^d d(A') + d(\mathcal{Z}_{1/k}(\epsilon_1)A)(1 - \epsilon^d) + \epsilon_1$$

implies that

$$d(\mathcal{Z}_{1/k}(\epsilon_1)A) \geq \frac{m(D)(1 - \epsilon^d + \epsilon^{2d})}{1 - \epsilon} \geq m(D)(1 + \epsilon^{2d})$$

and weak supersaturation lemma tells us that A is not D -avoiding. Thus $d(A') \geq m(D)(1 - 3\epsilon^d)$.

Consider the set $(A' + x) \cap (R/k)\mathbb{Z}^d$ for a vector x chosen uniformly at random from $Q(0, R/k)$. By averaging there is a choice of x for which $|(A' + x) \cap (R/k)\mathbb{Z}^d| \geq d(A')k^d$. Let x_0 be such a choice. Set $A'' = ((A' + x_0) \cap (R/k)\mathbb{Z}^d) + Q(0, (1 - 3^{1/d}\epsilon)R/k)$. The set A'' is D -avoiding. Indeed, suppose for some $x, y \in A''$ we have $|x - y| \in D$. Then $Q(x, 3^{1/d}\epsilon R/k)$ is contained in a cube of side length $1/k$ on which $A + x_0$ has density at least $1 - \epsilon^d$. Let $\tau = d_{Q(x, 3^{1/d}\epsilon R/k)}(A + x_0)$. Then $1 - \epsilon^d \leq 1 - (1 - \tau)(3^{1/d}\epsilon)^d$ implying $\tau \geq 2/3$. Similarly, for $d_{Q(y, 3^{1/d}\epsilon R/k)}(A + x_0) \geq 2/3$. Therefore A is not D -avoiding. This contradiction proves that A'' is D -avoiding.

The set $(1/R) \cdot A''$ is $(1/R) \cdot D$ -avoiding and periodic with the fundamental region $Q(0, 1)$. It is also a union of cubes of side length $(1 - 3^{1/d}\epsilon)/k$. Each such cube contains k^2 -granular set of measure at least $[(1 - 3^{1/d}\epsilon)/k - 2/k^2]^d$. Therefore A'' contains a k^2 -granular set of density at least $d(A'') \left(1 - \frac{2}{k(1 - 3^{1/d}\epsilon)}\right) \geq d(A'')(1 - 3/k)$ since $\epsilon \leq 1/10$. Thus there is a k^2 -granular set of density at least $m(D)(1 - 3\epsilon)(1 - 3^{1/d}\epsilon)(1 - 3\epsilon^d) \geq m(D)(1 - 8\epsilon)$. \square

7 Concluding remarks

Let G be a finite graph, and suppose that for every edge $e \in E(G)$ there is an admissible measure $\sigma_e \in \mathcal{M}(\mathbb{R}^d)$. Then we say that a copy of the graph G occurs in a set $A \subset \mathbb{R}^d$ if there is a map $f: V(G) \rightarrow A$ such that for every edge $xy \in E(G)$ we have $f(x) - f(y) \in \text{supp } \sigma_{xy}$. The theorems 16 and 17 show that if G is either a single edge or a path of length 2, then the property of avoiding G is supersaturable. The proof of theorem 17 can be easily modified to the case when G is a star. I conjecture that the property of avoiding G is supersaturable whenever G is a tree.

An example of Bourgain [Bou86] shows that the property of avoiding a triangle K_3 fails to be supersaturable. However, in his example the points of the triangle are forced to lie on the same line. Perhaps with an appropriate non-degeneracy condition the property of avoiding K_3 is supersaturable.

Suppose G_1 and G_2 are two graphs as above. Let r_1, r_2 be two distinguished vertices in G_1 and G_2 respectively. Then $G_1 \text{ OR } G_2$ is a graph which is obtained by identifying r_1 and r_2 in the disjoint union of G_1 and G_2 . I believe that in the case when G_1 and G_2 are trees, the generalization of theorem 18 holds: $m(G_1 \text{ OR } t \cdot G_2) \rightarrow \max(m(G_1), m(G_2))$ as $t \rightarrow \infty$.

The theorem 1 implies that the measurable chromatic number $\chi_{\mathbb{R}^d}^m(D)$ grows exponentially in $|D|$ provided the elements of D grow fast enough. It is very likely that the usual chromatic number does not share this behavior. I conjecture that for any dimension d if the elements of D are algebraically independent over \mathbb{Q} , then the chromatic number $\chi_{\mathbb{R}^d}(D)$ is bounded independently of what D actually is. The conjecture is easily seen to be true when $d = 1$ because the finite subgraphs of $G_{\mathbb{R}^1}(D)$ are subgraphs of the $|D|$ -dimensional rectangular grid \boxplus . The *clique number* of a graph G , denoted $\omega(G)$, is the number of vertices in the largest complete subgraph of G . For $d \geq 2$ the only result that I can prove is

Theorem 21. *There is a function $f(d)$ such that if the elements of D are algebraically independent over \mathbb{Q} , then $\omega(G_{\mathbb{R}^d}(D)) < f(d)$.*

Proof. Denote by K_n the complete graph on n vertices. Suppose K_n is a subgraph of $G_{\mathbb{R}^d}(D)$. Then let $X = \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^d$ be the vertices of this complete subgraph. Let $A = (a_{i,j})_{i,j=1}^n$ be an $n \times n$ matrix whose entries are $a_{i,j} = \text{dist}(x_i, x_j)^2 = \langle x_i - x_j, x_i - x_j \rangle = \langle x_i, x_i \rangle + \langle x_j, x_j \rangle - 2\langle x_i, x_j \rangle$. The matrix $B = (b_{i,j})_{i,j=1}^n$ with $b_{i,j} = \langle x_i, x_i \rangle$ has rank 1. The matrix $C = (c_{i,j})_{i,j=1}^n$ with $c_{i,j} = \langle x_i, x_j \rangle$ has rank at most d . Thus the rank of $A = B + B^t - 2C$ is at most $d + 2$.

Consider any subset $X' \subset X$ of $d + 3$ elements. Let A' be the corresponding $(d + 3) \times (d + 3)$ submatrix of A . Let r_1, \dots, r_k be the non-zero elements that occur in A' . Since r_1, \dots, r_k are squares of algebraically independent numbers, they themselves are algebraically independent. Since A' is not of the full rank, $\det A' = 0$. Since the determinant is a polynomial function with rational coefficient in entries of A' , it follows that $\det A' = 0$ whenever $\{r_1, \dots, r_k\}$ is replaced by any set of k algebraically independent numbers. Therefore, $\det A'$ is zero as a polynomial in r_1, \dots, r_k . Since the matrix A' is a symmetric matrix, each r_i occurs at least twice. If each r_i occurred exactly twice, then $\det A'(r_1, \dots, r_k)$, being the determinant of the general symmetric matrix with the zeros on the diagonal, would not be the zero polynomial. Thus, in every set of $d + 3$ points at least one distance occurs twice.

Color the edge $x_i x_j$ of the complete graph on X by the distance between x_i and x_j . The above asserts that there is no K_{d+3} subgraph whose edges all colored differently. On the other hand, since the simplex on $d + 2$ vertices does not embed isometrically in \mathbb{R}^d , there is no monochromatically colored K_{d+2} subgraph. By the canonical Ramsey theorem [ER50] if n is large enough, then there is a $Y = \{y_1, \dots, y_{d+3}\} \subset X$ such that the color of an edge $y_i y_j$ for $i < j$ depends only on i . Let $t_i = \text{dist}(y_i, y_{i+1})^2$. The $(d + 3) \times (d + 3)$ matrix corresponding to Y is $M = (m_{i,j})_{i,j=1}^{d+3}$ where

$$m_{i,j} = \begin{cases} t_i, & \text{if } i < j, \\ t_j, & \text{if } i > j, \\ 0, & \text{if } i = j. \end{cases}$$

The matrix M is of the full rank. Indeed, if M was not of the full rank, then there would exist $\alpha_1, \dots, \alpha_{d+3}$ such that $\sum_i \alpha_i m_{i,j} = 0$ for all j . Then we would have $0 = \sum_i \alpha_i (m_{i,j} - m_{i,j+1}) = (\alpha_j - \alpha_{j+1})t_j$. Since $t_j \neq 0$, that would imply that $\alpha_1 = \dots = \alpha_{d+3}$. However $\sum m_{i,1}$ does not vanish since it is a sum of positive terms. The contradiction shows that M is of the full rank. However, from above we know that M has rank at most $d + 2$. The contradiction completes the proof. \square

Acknowledgements. I would like to thank Uri Andrews and Pablo Candela-Pokorna for discussions that inspired this work.

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