A LINEAR APPROXIMATE-SIZE RANDOM SAMPLER FOR LABELLED PLANAR GRAPHS

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Abstract. This article introduces a new algorithm for the random generation of labelled planar graphs. Its principles rely on Boltzmann samplers, as recently developed by Duchon, Flajolet, Louchard, and Schaeffer. It combines the Boltzmann framework, a suitable use of rejection, a new combinatorial bijection found by Fusy, Poulalhon and Schaeffer, as well as a precise analytic description of the generating functions counting planar graphs, which was recently obtained by Giménez and Noy. This gives rise to an extremely efficient algorithm for the random generation of planar graphs. There is a preprocessing step of some fixed small cost. Then, the expected time complexity of generation is quadratic for exact-size uniform sampling and linear for approximate-size sampling. This greatly improves on the best previously known time complexity for exact-size uniform sampling of planar graphs with n vertices, which was a little over $\mathcal{O}(n^7)$.

1. Introduction

A graph is said to be planar if it can be embedded in the plane so that no two edges cross each other. In this article, we consider *labelled* planar graphs, i.e., a graph of size n has its vertices bearing distinct labels in $[1..n]$. Statistic properties of planar graphs have been intensively studied [5, 16, 17]. Very recently, Giménez and Noy [17] have solved *exactly* the difficult problem of the asymptotic enumeration of labelled planar graphs. They also provide exact analytic expressions for the asymptotic probability distribution of parameters such as the number of edges and the number of connected components. Since many other statistics on random planar graphs remain analytically and combinatorially untractable, it is an important issue to find efficient procedures to generate planar graphs at random. For example, random generation makes it possible to validate algorithms and programs on planar graphs, such as planarity testing, embedding algorithms, efficient procedures for finding geometric cuts, and so on.

Denise, Vasconcellos, and Welsh have proposed a first algorithm for the random generation of planar graphs [6], by defining a Markov chain on the set $\mathcal{G}_n$ of labelled planar graphs with n vertices. At each step, two different vertices v and v' are chosen at random. If they are adjacent, the edge (v, v') is deleted. If they are not adjacent and if the operation of adding (v, v') does not break planarity, then the edge (v, v') is added. By symmetry of the transition matrix of the Markov chain, the probability distribution converges to the uniform distribution on $\mathcal{G}_n$. This algorithm is very easy to describe but more difficult to implement, as there exists no simple linear-time planarity testing algorithm. More importantly, the rate of convergence to the uniform distribution is unknown.

A second approach for fixed-size uniform random generation has been developed by Bodirsky, Gröpl and Kang [4]. It relies on the *recursive method* introduced by Nijenhuis

Key words and phrases. Planar graphs, Random generation, Boltzmann sampling.

	Aux. mem.	Preproc. time	Time per generation	
Markov chains	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	*unknown*	{exact size}
Recursive method	$\mathcal{O}(n^5 \log n)$	$\mathcal{O}^*(n^7)$	$\mathcal{O}(n^3)$	{exact size}
Boltzmann sampler	$\mathcal{O}((\log n)^k)$	$\mathcal{O}((\log n)^k)$	$\mathcal{O}(n^2)$ $\mathcal{O}(n)$	{exact size} {approx. size}

FIGURE 1. Complexities of the random samplers of planar graphs ($\mathcal{O}^*$ stands for a big $\mathcal{O}$ taken up to logarithmic factors).

and Wilf [21] and formalised by Flajolet, Van Cutsem and Zimmermann [12]. The recursive method is a general framework for the random generation of combinatorial classes admitting a recursive decomposition. For such classes, producing an object of the class uniformly at random boils down to producing the *decomposition tree* corresponding to its recursive decomposition. Then, the branching probabilities that produce the decomposition tree with suitable (uniform) probability are computed using the *coefficients* counting the objects involved in the decomposition. As a consequence, this method requires a preprocessing step where large tables of large coefficients are calculated using the recursive relations they satisfy.

Bodirsky *et al* [4] apply the recursive method for planar graphs, which admit a well known combinatorial decomposition according to successive levels of connectivity. The coefficients enumerating planar graphs satisfy complicated recursive relations, so that the complexity of the preprocessing step is large. Precisely, the random generation of planar graphs with n vertices (and possibly also a fixed number m of edges), requires a preprocessing time of order $\mathcal{O}\left(n^7(\log n)^2(\log\log n)\right)$ and an auxiliary memory of size $\mathcal{O}(n^5 \log n)$. Once the tables have been computed, the complexity of each generation is $\mathcal{O}(n^3)$. A more recent optimisation of the recursive method by Denise and Zimmermann [7] —based on controlled real arithmetics— should be applicable; it would improve the time complexity somewhat, but the storage complexity would still be large.

In this article, we introduce a new algorithm for the random generation of labelled planar graphs, which relies on the same decomposition of planar graphs as the algorithm of Bodirsky *et al*. The main difference is that we translate this decomposition into a random generator using the framework of Boltzmann samplers, instead of the recursive method. (A preliminary description of our algorithm has been presented at the conference Analysis of Algorithms AofA'05 [13].) Boltzmann samplers are a powerful framework for random generation of decomposable combinatorial structures recently developed by Duchon, Flajolet, Louchard, and Schaeffer in [8]. The idea of Boltzmann sampling is to gain efficiency by relaxing the constraint of exact size sampling. As we will see, the gain is particularly significant in the case of planar graphs, where the decomposition is more involved than for classical classes such as trees. Given a combinatorial class, a *Boltzmann sampler* draws an object of size n with probability proportional to x^n (or proportional to $x^n/n!$ for labelled objects), where x is a certain *real* parameter that can be appropriately tuned. Accordingly, the probability distribution is spread over all objects of the class, with the property that objects of the same size have the same probability of occurring. In particular, the probability distribution is uniform when restricted to a fixed size. Like the recursive method, Boltzmann samplers can be produced for any combinatorial class admitting a recursive decomposition, as there are explicit sampling rules associated with each classical construction (Sum, Product, Set, Substitution). The

branching probabilities used to produce the decomposition tree of a random object are not based on the *coefficients* (recursive method) but on the *values* at x of the generating functions of the classes intervening in the decomposition.

In this article, we translate the decomposition of planar graphs into Boltzmann samplers and obtain very efficient random generators that produce planar graphs with a fixed number of vertices or with fixed numbers of vertices and edges uniformly at random. Furthermore, our samplers have an approximate-size version where a small tolerance, say a few percents, is allowed for the size of the output. For practical purpose, approximate-size random sampling often suffices. The approximate-size samplers we propose are very efficient as they have *linear time complexity*.

Theorem 1 (Samplers with respect to number of vertices)**.** *Let $n \in \mathbf{N}$ be a target size. There exists an* exact-size *sampler $\mathfrak{A}_n$ producing labelled planar graphs with n vertices uniformly at random. For any tolerance ratio $\epsilon > 0$, there exists an* approximate-size *sampler $\mathfrak{A}_{n,\epsilon}$ producing random planar graphs with number of vertices in $[n(1-\epsilon), n(1+\epsilon)]$ such that the distribution is uniform on each size $k \in [n(1-\epsilon), n(1+\epsilon)]$.*

Under a real-arithmetics complexity model, Algorithm $\mathfrak{A}_n$ is of expected complexity $\mathcal{O}(n^2)$. Algorithm $\mathfrak{A}_{n,\epsilon}$ is of expected complexity $\mathcal{O}(n)$, where the linearity constant depends on ϵ, being of order $1/\epsilon$ as $\epsilon \to 0$.

Theorem 2 (Samplers with respect to the numbers of vertices and edges)**.** *Let $n \in \mathbf{N}$ be a target size and $\mu \in (1,3)$ be a parameter of ratio edges-vertices. There exists an* exact-size *sampler $\mathfrak{A}_{n,\mu}$ producing planar graphs with n vertices and $\lfloor \mu n \rfloor$ edges uniformly at random. For any tolerance-ratio $\epsilon > 0$, there exists an* approximate-size *sampler $\mathfrak{A}_{n,\mu,\epsilon}$ producing random planar graphs with number of vertices in $[n(1-\epsilon), n(1+\epsilon)]$ and ratio edges/vertices in $[\mu(1-\epsilon), \mu(1+\epsilon)]$, such that the distribution is uniform for each fixed pair (number of vertices, number of edges).*

Under a real-arithmetics complexity model, for a fixed $\mu \in (1,3)$, Algorithm $\mathfrak{A}_{n,\mu}$ is of expected complexity $\mathcal{O}(n^{5/2})$, where the constant depends on μ. For fixed $\mu \in (1,3)$ and $\epsilon > 0$, Algorithm $\mathfrak{A}_{n,\mu,\epsilon}$ is of expected complexity $\mathcal{O}(n)$. The constant of linearity depends both on μ and ϵ, being of order $1/\epsilon$ as $\epsilon \to 0$ for any fixed $\mu \in (1,3)$.

The samplers are completely described in Section 5.3 and Section 5.4.

The real-arithmetic complexity model is that of the number of arithmetic operations (additions, comparisons) over real numbers assumed to be known *exactly*. The complexity of our algorithm is compared to the complexities of the two preceding random samplers in Figure 1.

Let us comment on the practical preprocessing complexity. The implementation of $\mathfrak{A}_{n,\epsilon}$ and $\mathfrak{A}_n$, as well as $\mathfrak{A}_{n,\mu,\epsilon}$ and $\mathfrak{A}_{n,\mu}$, requires the storage of a fixed number of real constants, which are special values of generating functions. The generating functions to be evaluated are those of several families of planar graphs (connected, 2-connected, 3-connected). A crucial result, recently established by O. Giménez and M. Noy [17], is that there exist exact analytic equations satisfied by these generating functions. Hence, their numerical evaluation can be performed efficiently, the complexity being of low polynomial degree k in the number of digits that need to be computed.

Fixed-size truncation of real numbers leads to algorithms with a probability of failure (caused by lack of precision) that can be made arbitrarily close to 0. No failure arises with a precision of 20 digits in practice, when we draw objects of size up to the million. In general, to draw objects of size n, the precision needed to make the probability of failure small is of order $\log(n)$ digits. Thus the preprocessing step to evaluate the generating

functions with a precision of $\log(n)$ digits has a complexity of order $\log(n)^k$. Notice that it is possible to achieve perfect uniformity by calling adaptive precision routines in case of failure, see Denise and Zimmermann [7] for a detailed discussion on similar problems. The following informal statement summarizes the discussion:

Fact. *With high probability, the auxiliary memory necessary to generate planar graphs of size n is of order $\mathcal{O}(\log(n))$ and the preprocessing time complexity is of order $\mathcal{O}(\log(n)^k)$ for some low integer k.*

We briefly discuss on the practical aspects of the implementation in the conclusion section.

2. Overview

The algorithm we propose relies on several tools. First, we extend the collection of constructions for Boltzmann samplers, as detailed in [8], and develop the more complicated case of substitution constructions, see Section 3. We describe in Section 4 the recursive decomposition of planar graphs according to successive levels of connectivity (also used in [4]) and adapt it to the Boltzmann framework. We start with the development of a Boltzmann sampler for (edge-rooted) 3-connected planar graphs. To do this, we use a recent result of bijective combinatorics found by the author, Poulalhon and Schaeffer [15], establishing that there exists a surprisingly simple correspondence between binary trees and edge-rooted 3-connected planar graphs. The realisation of a Boltzmann sampler for binary trees is straightforward and it yields, via the correspondence of [15] combined with rejection techniques, a Boltzmann sampler for edge-rooted 3-connected planar graphs. The next step is the realisation of a Boltzmann sampler for 2-connected planar graphs. A decomposition, due to Trakhtenbrot, ensures that edge-rooted 2-connected planar are assembled in a unique way from edge-rooted 3-connected planar graphs. Translating the decomposition yields a Boltzmann sampler for edge-rooted 2-connected planar graphs. Then we develop a Boltzmann sampler for connected planar graphs, using another decomposition ensuring that vertex-rooted connected planar graphs are assembled in a unique way from vertex-rooted 2-connected planar graphs. Finally, we obtain a Boltzmann sampler for (unconstrained) planar graphs, resulting from the decomposition of planar graphs into connected components. The corresponding Boltzmann sampler is denoted by $\Gamma G(x, y)$, where the variable x marks the number of vertices and the variable y marks the number of edges.

The Boltzmann sampler $\Gamma G(x, y)$ can unfortunately not be used directly to generate large planar graphs with a good time complexity. Indeed, the size distribution of $\Gamma G(x, y)$ is too concentrated on objects of small size. To improve the size distribution, we *point* the objects, in a way inspired by [8], which corresponds to a *derivation* of the associated generating function. The precise singularity analysis of the generating functions of planar graphs, recently performed in [17], indicates that we have to perform derivation of planar graphs three times in order to get a usable size distribution. In Section 5, we explain how to inject the derivative operator into the decomposition of planar graphs. This gives a Boltzmann sampler $\Gamma G'''(x, y)$ for "triply derived" planar graphs. Our random generators of planar graphs are finally obtained as *targetted samplers*, starting from $\Gamma G'''(x, y)$ and choosing well tuned values $x = x_n$ and $y = y(\mu)$ for each target size n and ratio edges/vertices $\mu \in (1, 3)$. The time complexity of the targetted samplers is analyzed in Section 6. This eventually yields the complexity results stated in Theorems 1 and 2. The general scheme of the planar graph generator is shown in Figure 2.

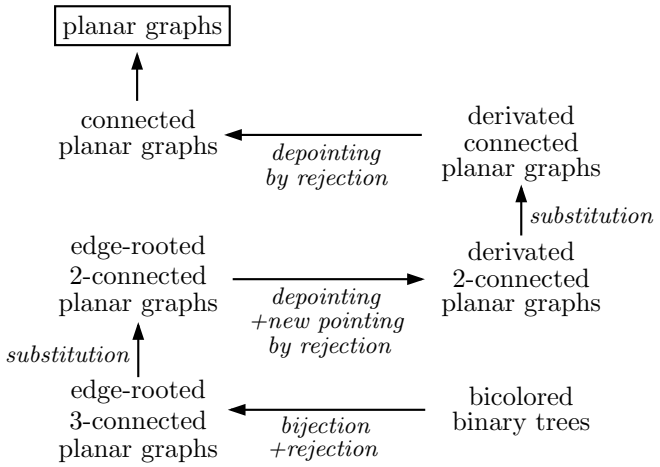

FIGURE 2. The chain of constructions from binary trees to planar graphs.

3. Boltzmann samplers

In this section, we define Boltzmann samplers and describe the main properties which we will need to develop a Boltzmann sampler for planar graphs in Section 4. In particular, we have to extend the framework to the case of *mixed classes*, meaning that the objects have two types of atoms. Indeed the decomposition of planar graphs involves both (labeled) vertices and (unlabeled) edges. The constructions needed to formulate the decomposition of planar graphs are classical ones in combinatorics: Sum, Product, Set, Substitutions [10]. For each of the constructions, we describe a *sampling rule*, so that Boltzmann samplers can be assembled for any class that admits a decomposition in terms of these constructions. Morevover, the decomposition of planar graphs involves rooting/unrooting operations. Taking these operations into account in the samplers makes it necessary to develop specific rejection techniques, as well as derivative operators, in the framework of Boltzmann samplers.

3.1. **Definition.** Boltzmann samplers, introduced and developed by Duchon *et al* in [8], constitute a general and efficient framework to produce a random generator on a combinatorial class $\mathcal{C}$ that admits an explicit decomposition. Instead of fixing a particular size for the random generation, objects are drawn under a probability distribution spread over the whole class. This distribution assigns to each object of a combinatorial class $\mathcal{C}$ a weight essentially proportional to the exponential of its size n. Precisely, if $\mathcal{C}$ is an unlabelled class, the *ordinary* generating function of $\mathcal{C}$ is

$$C(y) := \sum_{\gamma \in \mathcal{C}} y^{|\gamma|},$$

where $|\gamma|$ stands for the size (e.g., the number of nodes in a tree) of γ, and y is a variable marking the size. It is clear that the sum defining $C(y)$ converges if y is smaller than the radius of convergence ρ_C of $C(y)$, in which case y is said to be *coherent*. Then, the probability distribution assigining to each object γ of $\mathcal{C}$ a probability

$$\mathbf{P}_y(\gamma) = \frac{y^{|\gamma|}}{C(y)}$$

is a well defined distribution, called *ordinary Boltzmann distribution* of parameter y. An *ordinary Boltzmann sampler* of parameter y is a procedure $\Gamma C(y)$ that draws objects of $\mathcal{C}$ at random under the Boltzmann distribution $\mathbf{P}_y$. The authors of [8] provide a collection of rules to assemble Boltzmann samplers for combinatorial classes specified using basic combinatorial constructions, like Sum, Product, Sequence. The framework has been recently extended to constructions that are subject to symmetries, like Multiset, Powerset, Cycle [11]. An interesting application to random sampling of plane partitions is developed in [3].

Boltzmann samplers can similarly be assembled in the framework of labelled objects (e.g., graphs with labelled vertices). The *exponential* generating function of the class $\mathcal{C}$ is defined as

$$C(x) := \sum_{\gamma\in\mathcal{C}} \frac{x^{|\gamma|}}{|\gamma|!},$$

where $|\gamma|$ is the size of an object $\gamma \in \mathcal{C}$ (e.g., the number of vertices of a graph). The exponential Boltzmann distribution assigns to each object of $\mathcal{C}$ a weight

$$\mathbf{P}_x(\gamma) = \frac{x^{|\gamma|}}{|\gamma|!C(x)}\,.$$

Given a coherent value x, i.e., a value smaller than the radius of convergence of $C(x)$, a *Boltzmann sampler* for the labelled class $\mathcal{C}$ is a procedure $\Gamma C(x)$ that draws objects of $\mathcal{C}$ at random under the "labelled" Boltzmann distribution $\mathbf{P}_x$. As in the unlabelled framework, the authors of [8] give sampling rules associated to classical combinatorial constructions (Sum, Product, Set).

To assemble a Boltzmann sampler for planar graphs from their combinatorial decomposition, we need to extend the framework of Boltzmann samplers to the case of a *mixed* combinatorial class. In a mixed class $\mathcal{C} = \cup_{n,m}\mathcal{C}_{n,m}$, an object has n labelled "atoms" and m unlabelled "atoms", e.g., a graph with n labelled vertices and m unlabelled edges. For $\gamma \in \mathcal{C}$, we write $|\gamma|$ for the number of labelled atoms of γ and $||\gamma||$ for the number of unlabelled atoms of γ. The associated generating function $C(x, y)$ is defined as

$$C(x, y) := \sum_{\gamma\in\mathcal{C}} \frac{x^{|\gamma|}}{|\gamma|!} y^{||\gamma||}.$$

For a fixed real value $y > 0$, we denote by $\rho_C(y)$ the radius of convergence of the function $x \to C(x, y)$. A pair (x, y) is said to be *coherent* if $x \in (0, \rho_C(y))$, which means that $\sum_{\gamma\in\mathcal{C}} \frac{x^{|\gamma|}}{|\gamma|!} y^{||\gamma||}$ converges and that $C(x, y)$ is well defined. Given a coherent pair (x, y), the *mixed Boltzmann distribution* is the probability distribution $\mathbf{P}_{x,y}$ assigning to each object $\gamma \in \mathcal{C}$ probability

$$\mathbf{P}_{x,y}(\gamma) = \frac{1}{C(x, y)} \frac{x^{|\gamma|}}{|\gamma|!} y^{||\gamma||}.$$

An important property of this distribution is that two objects with the same parameters $(|\gamma|, ||\gamma||)$ have the same probability of occurring. A *mixed Boltzmann sampler* at (x, y) — called in short Boltzmann sampler hereafter— is a procedure $\Gamma C(x, y)$ that draws objects of $\mathcal{C}$ at random under the Boltzmann distribution $\mathbf{P}_{x,y}$. Observe that the development of the Boltzmann framework for mixed classes is an extension of the labelled case studied in [8]. Indeed, $\Gamma C(x, 1)$ is an exponential Boltzmann sampler for $\mathcal{C}$.

3.2. **Constructions.** The five constructions that follow serve to express the decomposition of planar graphs, see [10] for details. In particular, we need two specific substitution constructions, one at labelled atoms called x-substitution, the other at unlabelled atoms called y-substitution.

Sum. The sum $\mathcal{C} = \mathcal{A} + \mathcal{B}$ of two classes is meant as a *disjoint union*, i.e., it is the union of two distinct copies of $\mathcal{A}$ and $\mathcal{B}$. The generating function of $\mathcal{C}$ satisfies

$$C(x, y) = A(x, y) + B(x, y).$$

Product. The product $\mathcal{C} = \mathcal{A} \star \mathcal{B}$ is a classical cartesian product, combined with a relabelling step ensuring that the atoms of an object $\gamma \in \mathcal{A} \star \mathcal{B}$ bear distinct labels in $[1, \ldots, |\gamma|]$. The generating function of $\mathcal{C}$ satisfies

$$C(x, y) = A(x, y) \cdot B(x, y).$$

Set$_{\geq \mathbf{d}}$. For $d \geq 0$ and a class $\mathcal{A}$ having no object of size 0, $\mathcal{C} = \textsc{Set}_{\geq d}(\mathcal{A})$ is the class such that each object $\gamma \in \textsc{Set}_{\geq d}(\mathcal{A})$ is a finite set of at least d objects of $\mathcal{A}$, relabelled so that the atoms of γ bear distinct labels in $[1\,..\,|\gamma|]$. For $d = 0$, this corresponds to the classical construction Set. The generating function of $\mathcal{C}$ satisfies

$$C(x, y) = e_d(A(x, y)), \quad \text{where } e_d(Z) := \sum_{k \geq d} \frac{Z^k}{k!}.$$

x-substitution. Given $\mathcal{A}$ and $\mathcal{B}$ two classes such that $\mathcal{B}$ has no object of size 0, the class $\mathcal{C} = \mathcal{A} \circ_x \mathcal{B}$ is the class of objects that are obtained by taking an object $\rho \in \mathcal{A}$, called the *core-object*, substituting each labelled atom v of ρ by an object $\gamma_v \in \mathcal{B}$, and finally relabelling the atoms of $\cup_v \gamma_v$ with distinct labels from 1 to $\sum_v |\gamma_v|$. The generating function of $\mathcal{C}$ satisfies

$$C(x, y) = A(B(x, y), y).$$

y-substitution. Given $\mathcal{A}$ and $\mathcal{B}$ two classes such that $\mathcal{B}$ has no object of size 0, the class $\mathcal{C} = \mathcal{A} \circ_y \mathcal{B}$ is the class of objects that are obtained by taking an object $\rho \in \mathcal{A}$, called the *core-object*, substituting each unlabelled atom e of ρ by an object $\gamma_e \in \mathcal{B}$, and finally relabelling the atoms of $\rho \cup (\cup_e \gamma_e)$ with distinct labels from 1 to $|\rho| + \sum_e |\gamma_e|$. We assume here that the unlabelled atoms of an object of $\mathcal{A}$ are *distinguishable*. This property is satisfied in the case where $\mathcal{A}$ is a family of labelled graphs with no multiple edges, as two different edges are distinguished by the labels of their two incident vertices. The generating function of $\mathcal{C}$ satisfies

$$C(x, y) = A(x, B(x, y)).$$

3.3. **Sampling rules.** A nice feature of Boltzmann samplers is that the basic combinatorial constructions (Sum, Product, Set) give rise to simple rules for assembling the associated Boltzmann samplers. To describe these rules, we assume that the exact values of the generating functions at a given coherent pair (x, y) are known. We will also need two well-known probability distributions.

- A *Bernoulli law* of parameter $p \in (0, 1)$ is a random variable equal to 1 (or true) with probability p and equal to 0 (or false) with probability $1 - p$.

Construction		Boltzmann sampler
empty atom	$\mathbf{1}$	return $\mathbf{1}$
unit atom	$\mathcal{Z}$	return $\mathcal{Z}$
Sum	$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$\Gamma C(x,y)$: if Bern $\left(\frac{A(x,y)}{C(x,y)}\right)$, return $\Gamma A(x,y)$ else return $\Gamma B(x,y)$
Product	$\mathcal{C} = \mathcal{A} \star \mathcal{B}$	$\Gamma C(x,y)$: $\gamma \leftarrow (\Gamma A(x,y), \Gamma B(x,y))$ DistributeLabels(γ); return γ
$\text{Set}_{\geq d}$	$\mathcal{C} = \text{Set}_{\geq d}(\mathcal{A})$	$\Gamma C(x,y)$: $k \leftarrow \text{Pois}_{\geq d}(A(x,y))$ $\gamma \leftarrow (\Gamma A(x,y), \dots, \Gamma A(x,y))$ {*k ind. calls*} DistributeLabels(γ); return γ
x-subs	$\mathcal{C} = \mathcal{A} \circ_x \mathcal{B}$	$\Gamma C(x,y)$: $\gamma \leftarrow \Gamma A(B(x,y), y)$ for each labeled atom $v \in \gamma$ do replace v by $\gamma_v \leftarrow \Gamma B(x,y)$ od {*ind. calls*} DistributeLabels(γ); return γ
y-subs	$\mathcal{C} = \mathcal{A} \circ_y \mathcal{B}$	$\Gamma C(x,y)$: $\gamma \leftarrow \Gamma A(x, B(x,y))$ for each unlabeled atom $e \in \gamma$ do replace e by $\gamma_e \leftarrow \Gamma B(x,y)$ od {*ind. calls*} DistributeLabels(γ); return γ

FIGURE 3. The sampling rules associated with each of the five constructions.

- Given $\lambda > 0$ a real value and d a nonnegative integer, the *conditioned Poisson law* $\text{Pois}_{\geq d}(\lambda)$ is the probability distribution on $\mathbf{Z}_{\geq d}$ defined as follows:

$$\mathbb{P}(k) = \frac{1}{e_d(\lambda)} \frac{\lambda^k}{k!}, \text{ where } e_d(Z) := \sum_{k \geq d} \frac{Z^k}{k!}.$$

  For $d = 0$, this corresponds to the classical Poisson law, abbreviated as Pois.

For complexity analysis, a Bernoulli choice is assumed to have unit cost, and drawing from a conditioned Poisson law has cost equal to the value of the output. (Indeed, a conditioned Poisson law can be classically drawn using a loop executed k times if the result is k, see [8].)

Starting from combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ endowed with Boltzmann samplers $\Gamma A(x,y)$ and $\Gamma B(x,y)$, Figure 3 describes how to assemble a sampler for a class $\mathcal{C}$ obtained from $\mathcal{A}$ and $\mathcal{B}$ (or from $\mathcal{A}$ alone for the construction $\text{Set}_{\geq d}$) using the five constructions. The relabelling step, as mentioned in the definition of the constructions, is performed by an auxiliary procedure DistributeLabels. Given an object γ with its labelled atoms ranked from 1 to $|\gamma|$, DistributeLabels(γ) draws a permutation σ of $[1, .., |\gamma|]$ uniformly at random and gives label $\sigma(i)$ to the atom of rank i.

Proposition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two mixed combinatorial classes endowed with Boltzmann samplers $\Gamma A(x,y)$ and $\Gamma B(x,y)$. For the five constructions $\{+, \star, \text{Set}_{\geq d}, x-\text{subs}, y-\text{subs}\}$, the sampler $\Gamma C(x,y)$, as specified in Figure 3, is a valid Boltzmann sampler for the combinatorial class $\mathcal{C}$.*

Proof. 1) *Sum:* $\mathcal{C} = \mathcal{A} + \mathcal{B}$. An object of $\mathcal{A}$ has probability $\frac{1}{A(x,y)} \frac{x^{|\gamma|}}{|\gamma|!} y^{||\gamma||}$ (by definition of $\Gamma A(x,y)$) multiplied by $\frac{A(x,y)}{C(x,y)}$ (because of the Bernoulli choice) of being drawn by

$\Gamma C(x,y)$. Hence, it has probability $\frac{1}{C(x,y)}\frac{x^{|\gamma|}}{|\gamma|!}y^{||\gamma||}$ of being drawn. Similarly, an object of $\mathcal{B}$ has probability $\frac{1}{C(x,y)}\frac{x^{|\gamma|}}{|\gamma|!}y^{||\gamma||}$ of being drawn. Hence $\Gamma C(x,y)$ is a valid Boltzmann sampler for $\mathcal{C}$.

2) *Product:* $\mathcal{C} = \mathcal{A} \star \mathcal{B}$. Define a *generation scenario* as a pair $(\gamma_1 \in \mathcal{A}, \gamma_2 \in \mathcal{B})$, together with a relabelling permutation $\sigma \in \mathfrak{S}_{|\gamma_1|+|\gamma_2|}$ of the labelled atoms of (γ_1, γ_2). By definition, $\Gamma(\mathcal{A}\star\mathcal{B})(x,y)$ draws a generation scenario and returns the object $\gamma \in \mathcal{A}\star\mathcal{B}$ obtained by keeping the secondary (distributed) labels. Each generation scenario has probability

$$\left(\frac{1}{A(x,y)}\frac{x^{|\gamma_1|}}{|\gamma_1|!}y^{||\gamma_1||}\right)\left(\frac{1}{B(x,y)}\frac{x^{|\gamma_2|}}{|\gamma_2|!}y^{||\gamma_2||}\right)\frac{1}{(|\gamma_1|+|\gamma_2|)!}$$

of being drawn, the three factors corresponding respectively to $\Gamma A(x,y)$, $\Gamma B(x,y)$, and DistributeLabels(γ). Observe that this probability has the more compact form

$$\frac{1}{|\gamma_1|!|\gamma_2|!}\frac{1}{C(x,y)}\frac{x^{|\gamma|}}{|\gamma|!}y^{||\gamma||}.$$

Given $\gamma \in \mathcal{A}\star\mathcal{B}$, let γ_1 be its first component (in $\mathcal{A}$) and γ_2 be its second component (in $\mathcal{B}$). Any labelling of the labelled atoms of γ_1 from 1 to $|\gamma_1|$ and of the labelled atoms of γ_2 from 1 to $|\gamma_2|$ induces a unique generation scenario producing γ. Indeed, the two labellings determine unambiguously the relabelling permutation σ of the generation scenario. Hence, γ is produced from $|\gamma_1|!|\gamma_2|!$ different scenarios, each having probability $\frac{1}{|\gamma_1|!|\gamma_2|!C(x,y)}\frac{x^{|\gamma|}}{|\gamma|!}y^{||\gamma||}$. As a consequence, γ is drawn under the Boltzmann distribution.

3) *Set$_{\geq d}$*: $\mathcal{C} = \textsc{Set}_{\geq d}(\mathcal{A})$. In the case of the construction $\textsc{Set}_{\geq d}$, a *generation scenario* is defined as a sequence $(\gamma_1 \in \mathcal{A}, \ldots, \gamma_k \in \mathcal{A})$ with $k \geq d$, together with a relabelling permutation $\sigma \in \mathfrak{S}_{|\gamma_1|+\ldots+|\gamma_k|}$. Such a generation scenario produces an object $\gamma \in \textsc{Set}_{\geq d}(\mathcal{A})$. By definition of $\Gamma(\textsc{Set}_{\geq d}(\mathcal{A}))(x,y)$, the scenario has probability

$$\left(\frac{1}{e_d(A(x,y))}\frac{A(x,y)^k}{k!}\right)\left(\prod_{i=1}^{k}\frac{x^{|\gamma_i|}y^{||\gamma_i||}}{A(x,y)|\gamma_i|!}\right)\frac{1}{(|\gamma_1|+\ldots+|\gamma_k|)!},$$

the three factors corresponding respectively to drawing $\text{Pois}_{\geq d}(A(x,y))$, drawing the sequence, and the relabelling step. This probability has the simpler form

$$\frac{1}{k!C(x,y)}\frac{x^{|\gamma|}}{|\gamma|!}y^{||\gamma||}\prod_{i=1}^{k}\frac{1}{|\gamma_i|!}.$$

For $k \geq d$, an object $\gamma \in \textsc{Set}_{\geq d}(\mathcal{A})$ can be written as a sequence $\gamma_1, \ldots, \gamma_k$ in $k!$ different ways. In addition, by a similar argument as for the Product construction, a sequence $\gamma_1, \ldots, \gamma_k$ is produced from $\prod_{i=1}^{k}|\gamma_i|!$ different scenarios. As a consequence, γ is drawn under the Boltzmann distribution.

4) *x-substitution*: $\mathcal{C} = \mathcal{A} \circ_x \mathcal{B}$. For this construction, a *generation scenario* is defined as a core-object $\rho \in \mathcal{A}$, a sequence $\gamma_1, \ldots, \gamma_{|\rho|}$ of objects of $\mathcal{B}$, and a relabelling permutation $\sigma \in \mathfrak{S}_{|\gamma_1|+\ldots+|\gamma_{|\rho|}|}$ (γ_i stands for the object of $\mathcal{B}$ substituted at the atom i of ρ). This corresponds to the scenario of generation of an object $\gamma \in \mathcal{A} \circ_x \mathcal{B}$ by the algorithm $\Gamma(\mathcal{A} \circ_x \mathcal{B})$, and the scenario has probability

$$\left(\frac{1}{A(B(x,y),y)}\frac{B(x,y)^{|\rho|}}{|\rho|!}y^{||\rho||}\right)\left(\prod_{i=1}^{|\rho|}\frac{x^{|\gamma_i|}y^{||\gamma_i||}}{B(x,y)|\gamma_i|!}\right)\frac{1}{(|\gamma_1|+\ldots+|\gamma_{|\rho|}|)!}.$$

This expression has the simpler form

$$\frac{x^{|\gamma|}y^{||\gamma||}}{C(x,y)|\gamma|!}\frac{1}{|\rho|!}\prod_{i=1}^{|\rho|}\frac{1}{|\gamma_i|!}.$$

Given $\gamma \in \mathcal{A} \circ_x \mathcal{B}$, each labelling of the atoms of the core-object $\rho \in \mathcal{A}$ followed, for $1 \leq i \leq |\rho|$, by a relabelling of the atoms of γ_i from 1 to $|\gamma_i|$, induces a unique generation scenario producing γ. As a consequence, γ is produced from $|\rho|!\prod_{i=1}^{|\rho|}|\gamma_i|!$ scenarios, each having probability $\frac{x^{|\gamma|}y^{||\gamma||}}{C(x,y)|\gamma|!}\frac{1}{|\rho|!}\prod_{i=1}^{|\rho|}\frac{1}{|\gamma_i|!}$. Hence, γ is drawn under the Boltzmann distribution.

5) *y-substitution*: $\mathcal{C} = \mathcal{A} \circ_y \mathcal{B}$. A *generation scenario* is defined as a core-object $\rho \in \mathcal{A}$, a sequence $\gamma_1, \ldots, \gamma_{||\rho||}$ of objects of $\mathcal{B}$, and a relabelling permutation $\sigma \in \mathfrak{S}_{|\rho|+|\gamma_1|+\ldots+|\gamma_{||\rho||}|}$ (after giving a rank to each unlabelled atom of ρ, γ_i stands for the object of $\mathcal{B}$ substituted at the unlabelled atom of rank i). This corresponds to the scenario of generation of an object $\gamma \in \mathcal{A} \circ_y \mathcal{B}$ by the algorithm $\Gamma(\mathcal{A} \circ_y \mathcal{B})$, and the scenario has probability

$$\left(\frac{1}{A(x,B(x,y))}\frac{x^{|\rho|}}{|\rho|!}B(x,y)^{||\rho||}\right)\left(\prod_{i=1}^{||\rho||}\frac{x^{|\gamma_i|}y^{||\gamma_i||}}{B(x,y)|\gamma_i|!}\right)\left(\frac{1}{(|\rho|+|\gamma_1|+\ldots+|\gamma_{||\rho||}|)!}\right).$$

This expression has the simpler form

$$\frac{x^{|\gamma|}y^{||\gamma||}}{C(x,y)|\gamma|!}\frac{1}{|\rho|!}\prod_{i=1}^{||\rho||}\frac{1}{|\gamma_i|!}.$$

Given $\gamma \in \mathcal{A} \circ_y \mathcal{B}$, each labelling of the atoms of the core-object $\rho \in \mathcal{A}$ followed, for $1 \leq i \leq ||\rho||$, by a relabelling of the atoms of γ_i from 1 to $|\gamma_i|$, induces a unique generation scenario producing γ. As a consequence, γ is produced from $|\rho|!\prod_{i=1}^{||\rho||}|\gamma_i|!$ scenarios, each having probability $\frac{x^{|\gamma|}y^{||\gamma||}}{C(x,y)|\gamma|!}\frac{1}{|\rho|!}\prod_{i=1}^{||\rho||}\frac{1}{|\gamma_i|!}$. Hence, γ is drawn under the Boltzmann distribution.

□

Example. Consider the class $\mathcal{C}$ of labelled binary trees where the atoms are the inner nodes. The class $\mathcal{C}$ has the following decomposition grammar,

$$\mathcal{C} = (\mathcal{C} + \mathbf{1}) \star \mathcal{Z} \star (\mathcal{C} + \mathbf{1}).$$

Accordingly, the series $C(x)$ counting binary trees satisfies $C(x) = x\,(1 + C(x))^2$. Thus, $C(x)$ can be easily evaluated for a fixed real parameter $x < \rho_C = \frac{1}{4}$.

Using the sampling rules for Sum and Product, we obtain the following Boltzmann sampler for binary trees,

$\Gamma C(x)$: return $(\Gamma(1+C)(x), \mathcal{Z}, \Gamma(1+C)(x))$ {independent calls}
$\Gamma(1+C)(x)$: if Bern $\left(\frac{1}{1+C(x)}\right)$ return $\varnothing$ {leaf}
else return $\Gamma C(x)$

Remark 1. The procedure DistributeLabels(γ) throws distinct labels uniformly at random on the atoms of γ that support labels. The fact that the relabelling permutation is always chosen uniformly ensures that the call to DistributeLabels can be postponed till the end of the algorithm, i.e., we can apply the labelling to the finally output object (this is also mentioned by Flajolet *et al* [12, Sec 3]). Hence, the labels

do not really matter and induce no additional complexity to the Boltzmann samplers: for a class $\mathcal{C}$ whose combinatorial decomposition involves the five constructions, we just have to generate the unlabelled *shape* of an object γ produced by $\Gamma C(x,y)$; then we call DistributeLabels(γ).

3.4. **Additional techniques for Boltzmann sampling.**

3.4.1. *Derivation, y-derivation, and edge-rooting.* In the following sections, we will make much use of the *derivative* operator. Given a mixed (or labelled) combinatorial class $\mathcal{C} = \cup_{n,m}\mathcal{C}_{n,m}$, an object of the *derived class* $\mathcal{C}'$ is obtained by removing the label n of an object of $\mathcal{C}$ of size n, so that the obtained object has size $n-1$ (the atom n can be considered as a *pointed* atom that does not count in the size). As a consequence, $\mathcal{C}'_{n-1,m} \simeq \mathcal{C}_{n,m}$, so that the generating function $C'(x,y)$ of $\mathcal{C}'$ satisfies

$$C'(x,y) = \sum_{n,m} |\mathcal{C}_{n,m}| \frac{x^{n-1}}{(n-1)!} y^m = \frac{\partial C}{\partial x}(x,y). \tag{1}$$

The y-derivative of $\mathcal{C}$ is the class $\overline{\mathcal{C}}$ of objects of $\mathcal{C}$ having a marked unlabelled atom that does not count in the size. Thus, the generating function $\overline{C}(x,y)$ of $\overline{\mathcal{C}}$ satisfies

$$\overline{C}(x,y) = \frac{\partial C}{\partial y}(x,y). \tag{2}$$

For the particular case of planar graphs, we will also consider *edge-rooted* objects, i.e., planar graphs where an edge is "marked" (distinguished) and oriented. In addition, the root edge is not counted as unlabelled atom, and the two extremities of the root do not count as labelled atoms (i.e., are not labelled). The edge-rooted class of $\mathcal{C}$ is denoted by $\overrightarrow{\mathcal{C}}$. The generating function $\overrightarrow{C}(x,y)$ of $\overrightarrow{\mathcal{C}}$ satisfies

$$\overrightarrow{C}(x,y) = \frac{2}{x^2}\frac{\partial C}{\partial y}(x,y). \tag{3}$$

3.4.2. *Rejection.* Great flexibility results from combining Boltzmann sampling with *rejection*, making it possible to adjust the obtained distributions of the samplers.

Lemma 1 (Rejection)**.** *Given a combinatorial class $\mathcal{C}$, let $W : \mathcal{C} \to \mathbf{R}^+$ and $p : \mathcal{C} \to [0,1]$ be two functions, called* weight-function *and* rejection-function*, respectively. Assume that W is summable, i.e., $\sum_{\gamma\in\mathcal{C}} W(\gamma)$ is finite. Let $\mathfrak{A}$ be a random generator on $\mathcal{C}$ that draws each object $\gamma \in \mathcal{C}$ with probability proportional to $W(\gamma)$. Then, the procedure*

$$\mathfrak{A}_{\text{rej}} : \text{repeat } \mathfrak{A} \to \gamma \text{ until Bern}(p(\gamma)); \text{ return } \gamma$$

is a random generator on $\mathcal{C}$, which draws each object $\gamma \in \mathcal{C}$ with probability proportional to $W(\gamma)p(\gamma)$.

Proof. Define $W := \sum_{\gamma\in\mathcal{C}} W(\gamma)$. By definition, $\mathfrak{A}$ draws an object $\gamma \in \mathcal{C}$ with probability $P(\gamma) := W(\gamma)/W$. Let p_{rej} be the probability of rejection of $\mathfrak{A}_{\text{rej}}$ at each try. The probability $P_{\text{rej}}(\gamma)$ that γ is drawn by $\mathfrak{A}_{\text{rej}}$ satisfies $P_{\text{rej}}(\gamma) = P(\gamma)p(\gamma) + p_{\text{rej}}P_{\text{rej}}(\gamma)$, where the first (second) term is the probability that γ is drawn at the first try (at least at the second try, respectively). Hence, $P_{\text{rej}}(\gamma) = P(\gamma)p(\gamma)/(1-p_{\text{rej}}) = W(\gamma)p(\gamma)/(W\cdot(1-p_{\text{rej}}))$, i.e., $P_{\text{rej}}(\gamma)$ is proportional to $W(\gamma)p(\gamma)$. □

4. Decomposition of planar graphs and Boltzmann samplers

The classical method to count planar graphs consists in decomposing a planar graph into planar components that have higher degree of connectivity. (Recall that a graph is k-connected if it has at least k vertices and if the removal of any set of $k-1$ vertices and their incident edges does not disconnect the graph.) The decomposition is stopped at connectivity degree 3, where the graphs have a unique planar embedding (up to continuous deformation and reflection), according to a theorem of Whitney [29]. The generation method we describe *follows* the decomposition, i.e., planar graphs are assembled according to their decomposition.

First, by uniqueness of the embedding, generating 3-connected planar graphs is equivalent to generating 3-connected maps, where a map is a planar graph endowed with an explicit topological planar embedding. Following the general bijective approach introduced by Schaeffer [24], Fusy *et al* develop a bijective method to enumerate 3-connected maps [15], recovering the counting formulas originally obtained by Mullin and Schellenberg [20]. As described in Section 4.1, the bijection yields an explicit Boltzmann sampler for (rooted) 3-connected maps. The next step is to generate 2-connected planar graphs from 3-connected ones. We take advantage of a decomposition of 2-connected planar graphs into 3-connected planar components, which has been formalised by Trakhtenbrot [26] and later used by Walsh [28] to count 2-connected planar graphs and by Bender, Gao, Wormald to obtain asymptotic enumeration [1]. Finally, connected planar graphs are generated from 2-connected ones by using a well-known decomposition at separating vertices, and planar graphs are generated from connected ones by choosing the number of connected components and then generating each component. Notice that these steps translate to explicit equations relating the generating functions of 2-connected, connected, and unconstrained planar graphs. Starting from these equations, Giménez and Noy have solved the asymptotic enumeration of planar graphs, using analytic methods and clever integral manipulations [17].

Notations. In the sequel, the number of vertices and the number of edges of a planar graphs γ are respectively denoted by $V(\gamma)$ and $E(\gamma)$. Notice that $V(\gamma)$ may not always be equal to $|\gamma|$ and $E(\gamma)$ might not be equal to $||\gamma||$, e.g., an edge-rooted planar graph γ satisfies $V(\gamma) = |\gamma| + 2$ and $E(\gamma) = ||\gamma|| + 1$.

4.1. Boltzmann sampler for 3-connected planar graphs.

The development of a Boltzmann sampler for (edge-rooted) 3-connected planar graphs goes in two steps. First, we take advantage of a result of Whitney ensuring that Boltzmann sampling of edge-rooted 3-connected planar graphs is equivalent to Boltzmann sampling of so-called *rooted 3-connected maps*, where the terminology of map refers to an explicit embedding. Then we use an explicit bijection relating the families of 3-connected maps and the (very simple) family of binary trees. Via the bijection, a Boltzmann sampler for rooted binary trees is translated to a Boltzmann sampler for rooted 3-connected maps.

4.1.1. *Equivalence with rooted 3-connected maps.*

A well known result due to Whitney [29] ensures that a 3-connected planar graph has a unique embedding on the sphere up to continuous deformation and reflection (in general a planar graph can have many embeddings). A *3-connected map* is an unlabelled 3-connected planar graph embedded on the sphere up to continuous deformation. A 3-connected map is *rooted* by marking and orienting an edge of the map. Equivalently —use stereographic projection from the face on the right of the root— a rooted 3-connected map is a 3-connected planar graph embedded in the plane

up to continuous deformation, the root having the infinite face on its right. The class of rooted 3-connected maps is denoted by $\mathcal{M} = \cup_{i,j}\mathcal{M}_{i,j}$ where i is the number of vertices different from the two end points of the rooted edge and j is the number of edges without counting the rooted one. The associated generating function is $M(z,w) = \sum_{i,j} |\mathcal{M}_{i,j}| z^i w^j$ (as both vertices and edges are unlabelled, the series is ordinary in the two variables). Whitney's theorem ensures that a labelled edge-rooted 3-connected planar graphs has two different labelled embeddings on the sphere (up to continuous deformation), the two embeddings differing by a reflection. In other words, if we define a *labelled rooted map* as a map where the i vertices different from the two root extremities carry distinct labels in $[1..i]$, then we have the identity

$$\mathcal{M}_{i,j} \times i! \simeq 2\overrightarrow{\mathcal{G}_{3(i,j)}}, \tag{4}$$

which can be written compactly as

$$\mathcal{M} \simeq 2\overrightarrow{\mathcal{G}_3}, \qquad M(z,w) = \frac{4}{z^2}\frac{\partial G_3}{\partial w}(z,w) \tag{5}$$

Definition 1. Let $\mathcal{M} = \cup_{i,j}\mathcal{M}_{i,j}$ be a class with two types of unlabelled atoms, called an *ordinary mixed class*. A *Boltzmann sampler* for $\mathcal{M}$ is a random generator $\Gamma M(x,y)$ drawing each object $\gamma \in \mathcal{M}_{i,j}$ with probability

$$\mathbb{P}(\gamma) = \frac{x^i y^j}{M(x,y)},$$

where $M(x) = \sum_{i,j} |\mathcal{M}_{i,j}| x^i y^j$ is the generating function of $\mathcal{M}$, which is ordinary in the two variables. The derived class $\mathcal{M}'$ and y-derived class $\overline{\mathcal{M}}$ are defined in the same way as for mixed classes; $\mathcal{M}'$ ($\overline{\mathcal{M}}$) is the class of objects of $\mathcal{M}$ having a marked atom of the first type (second type, respectively) that does not count in the size.

Equation (5) ensures that rooted 3-connected maps correspond to the unlabelled shape of edge-rooted 3-connected labelled planar graphs. In addition, according to Remark 1, it is sufficient to draw only the unlabelled shape of the objects, so that we have the following result.

Lemma 2. *Finding a Boltzmann sampler $\Gamma\overrightarrow{G_3}(z,w)$ for edge-rooted 3-connected planar graphs is equivalent to finding a Boltzmann sampler $\Gamma M(z,w)$ for rooted 3-connected maps.*

Proof. If $\mathcal{M}$ is endowed with a Boltzmann sampler $\Gamma M(z,w)$, then the algorithm

$\boxed{\Gamma\overrightarrow{G_3}(z,w)\text{: } \gamma \leftarrow \Gamma M(z,w);\ \textsc{DistributeLabels}(\gamma);\ \text{return } \gamma;}$

draws each labelled 3-connected planar graph with probability proportional to $z^i w^j$ (because of $\Gamma M(z,w)$) multiplied by $1/i!$ (because of $\textsc{DistributeLabels}(\gamma)$). Hence the procedure $\Gamma\overrightarrow{G_3}(z,w)$ is a mixed Boltzmann sampler for $\overrightarrow{\mathcal{G}_3}$. □

4.1.2. *Bijection between binary trees and some dissections.* A fruitful bijective method to enumerate planar maps has been developed by Schaeffer in his thesis [24]. Several families of rooted maps are counted in this way [22, 23, 14]; in each case the family of maps is proved to be in bijection with an explicit family of trees. The advantage compared to symbolic methods, as developed by Tutte [27], is that the bijections yield efficient (linear-time) generators for maps, as random sampling of maps is reduced to the much easier task of random sampling of trees, see the survey [25]. The method has been recently applied to enumerate the family of 3-connected maps, which is of interest here. Precisely, there is

a bijection between binary trees and some specific dissections of the hexagon [15], these dissections being closely related to 3-connected maps. We make use of a formulation of the bijection as an unbiased correspondence between rooted bicolored binary trees and rooted dissections.

Definition 2. A *bicolored* binary tree is defined as a rooted binary tree —each node has a left son and a right son that are possibly empty— whose nodes are partitioned into black and white nodes, with the property that any pair of adjacent nodes have different colors. For i, j two integers, the set of bicolored binary trees with i black nodes and j white nodes is denoted by $\mathcal{T}_{i,j}$.

Definition 3. An *irreducible dissection* is a planar map with hexagonal outer face, quadrangular inner faces, and no filled 4-cycle, i.e., the interior of each 4-cycle is a face. As all face degrees are even, there exists a bicoloration of vertices, say in black and white, such that adjacent vertices have different colors; and the bicoloration is unique up to color choice of the first vertex. A rooted dissection is endowed with the unique vertex bicoloration such that the root-vertex is black. For i, j two integers, we denote by $\mathcal{D}_{i,j}$ the set of rooted irreducible dissections having i inner black vertices and j inner white vertices.

Proposition 2 (Fusy *et al* [15])**.** *For i and j two integers, there exists a mapping, called* closure-mapping*, that establishes a bijection between the sets*

$$\mathcal{T}_{i,j} \times \{1, 2, 3\} \equiv \mathcal{D}_{i,j} \times [1..j + 2].$$

The construction of a dissection from a binary tree takes linear time.

Starting from a binary tree, the closure mapping consists in completing edges incident to leaves (considered as made of a unique half-edge) into compete edges (made of two half-edges) so as to close quadrangular faces. At the end a hexagon is added outside of the obtained figure, and the leaves attached to remaining non-completed edges are merged with vertices of the hexagon so as to form only quadrangular faces, see [15] for a detailed description.

4.1.3. *Boltzmann sampler for binary trees.* Notice that bicolored binary trees admit a recursive decomposition, so that a Boltzmann sampler is easily derived. Precisely, the class $\mathcal{T}$ of bicolored binary trees is partioned into the class $\mathcal{T}_\bullet$ of black-rooted binary trees and the class $\mathcal{T}_\circ$ of white-rooted binary trees. The associated ordinary generating functions with respect to the number of black nodes (variable z) and the number of nodes (variable w) are denoted by $T(z, w)$, $T_\bullet(z, w)$, and $T_\circ(z, w)$. The decomposition at the root of a bicolored binary tree yields the following decomposition grammar, where $\mathcal{Z}_\bullet$ and $\mathcal{Z}_\circ$ stand for a black and a white node, respectively.

(6)
$$\begin{cases} \mathcal{T} &=& \mathcal{T}_\bullet + \mathcal{T}_\circ \\ \mathcal{T}_\bullet &=& (\mathbf{1} + \mathcal{T}_\circ) \star \mathcal{Z}_\bullet \star (\mathbf{1} + \mathcal{T}_\circ) \\ \mathcal{T}_\circ &=& (\mathbf{1} + \mathcal{T}_\bullet) \star \mathcal{Z}_\circ \star (\mathbf{1} + \mathcal{T}_\bullet) \end{cases} \qquad \begin{cases} T(z,w) &=& T_\bullet(z,w) + T_\circ(z,w) \\ T_\bullet(z,w) &=& zw\,(1 + T_\circ(z,w))^2 \\ T_\circ(z,w) &=& w\,(1 + T_\bullet(z,w))^2 \end{cases}.$$

The decomposition grammar of bicolored binary trees is directly translated to the following Boltzmann sampler $\Gamma T(z, w)$ for bicolored binary trees, based on the remark given just after:

$\Gamma T(z,w)$: if Bern $\left(\frac{T_\bullet(z,w)}{T(z,w)}\right)$ return $\Gamma T_\bullet(z,w)$
else return $\Gamma T_\circ(z,w)$

$\Gamma T_\bullet(z,w)$: return $(\Gamma(1+T_\circ)(z,w), \mathcal{Z}_\bullet, \Gamma(1+T_\circ)(z,w))$

$\Gamma T_\circ(z,w)$: return $(\Gamma(1+T_\bullet)(z,w), \mathcal{Z}_\circ, \Gamma(1+T_\bullet)(z,w))$

$\Gamma(1+T_\circ)(z,w)$: if Bern $\left(\frac{1}{1+T_\circ(z,w)}\right)$ return $\varnothing$ (leaf)
else return $\Gamma T_\circ(z,w)$

$\Gamma(1+T_\bullet)(z,w)$: if Bern $\left(\frac{1}{1+T_\bullet(z,w)}\right)$ return $\varnothing$ (leaf)
else return $\Gamma T_\bullet(z,w)$

Remark 2. We consider here ordinary mixed classes, i.e., classes with two types of unlabelled atoms, a case not covered by the rules given in Figure 3, where the classes considered have both labeled atoms and unlabeled atoms. However, an easy adaptation of the proof of Proposition 1 ensures that the sampling rules for *Sum* and *Product* are also valid in the case of a class with two types of unlabelled atoms, i.e., a Boltzmann sampler for $\mathcal{C} = \mathcal{A} + \mathcal{B}$ is obtained by calling $\Gamma A(x,y)$ with probability $A(x,y)/C(x,y)$ and calling $\Gamma B(x,y)$ otherwise; and a Boltzmann sampler for $\mathcal{C} = \mathcal{A} \star \mathcal{B}$ consists of two independent calls to $\Gamma A(x,y)$ and $\Gamma B(x,y)$.

4.1.4. *Boltzmann sampler for rooted irreducible dissections.* The bijection stated in Proposition 2 yields the following sampler for rooted irreducible dissections,

$\Gamma I(z,w)$: repeat $u \leftarrow \mathrm{rnd}(0,1)$; max_size $\leftarrow \lfloor 1/u \rfloor$;
$\tau \leftarrow \Gamma T(z,w)$;
abort and restart as soon as $\#\mathrm{nodes}(\tau) + 2 > \mathrm{max\_size}$
until (generation finishes)
return closure$(\tau, \mathrm{rnd}(1,2,3))$

Lemma 3. *The procedure $\Gamma I(z,w)$ is a Boltzmann sampler for rooted irreducible dissections.*

Proof. The sampler $\Gamma T(z,w)$ draws each binary tree $\tau \in \mathcal{T}_{i,j}$ with probability proportional to $z^i w^j$. Hence, Proposition 2 ensures that the algorithm
repeat $\tau \leftarrow \Gamma T(z,w)$
until (Bern $(1/(\#\mathrm{nodes}(\tau)+2))$)
return closure$(\tau, \mathrm{rnd}(1,2,3)$
is a Boltzmann sampler for rooted irreducible dissections with respect to the number of black vertices (variable z) and the number of vertices (variable w), according to Lemma 1. However, this sampler can be made more efficient by "simulating" the Bernoulli choice all along the generation instead of waiting that the entire object is drawn. This yields the sampler $\Gamma I(z,w)$, equivalent in distribution but more efficient. As we will see in the analysis, we need this improved version using an *early abort technique* to obtain the complexity results stated in Theorems 1 and 2. □

4.1.5. *From dissections to 3-connected maps.* Irreducible dissections are closely related to 3-connected maps, via a classical correspondence between maps and quadrangulations. Given a bicolored rooted quadrangulation Q, the *primal map* of Q is the rooted map M

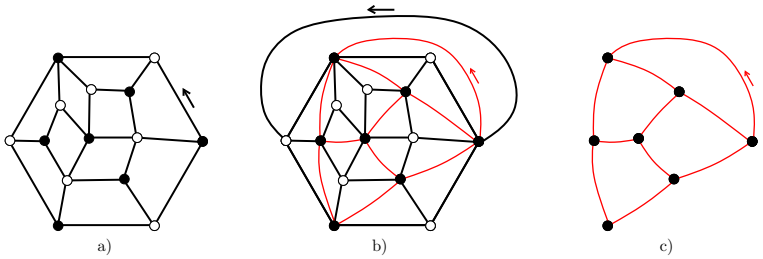

FIGURE 4. An admissible rooted irreducible dissection (Fig.a), and the associated rooted 3-connected map (Fig.c).

whose vertex set is the set of black vertices of Q, each face f of Q giving rise to an edge of M connecting the two (opposite) black vertices of f. The map M is naturally rooted with the same root-vertex as Q. A quadrangulation is called *irreducible* if the interior of every 4-cycle, expect the outer one, is a face. It is well known that the primal-map construction restricts to a bijection between rooted irreducible quadrangulations with i black vertices and j white vertices, and rooted 3-connected maps with i vertices and j faces. Moreover, irreducible dissections are close to irreducible quadrangulations. Indeed, deleting the root edge of a rooted irreducible quadrangulation Q yields a rooted irreducible dissection δ, the root of δ being naturally chosen so that Q and δ have the same root-vertex. The inverse operation consists in adding an outer edge, the new root, connecting the root-vertex of a rooted dissection to the opposite outer vertex.

Definition 4. A rooted irreducible dissection δ is called *admissible* if the operation of adding an edge connecting the root vertex to the opposite outer vertex yields a (rooted) irreducible quadrangulation Q. If so, the primal map of Q is called (by extension) the *primal map* of the dissection and is denoted by Primal(δ), see Figure 4. By extension also, a pair $(\tau, a) \in \mathcal{T} \times \{1, 2, 3\}$ is called *admissible* if closure(τ, a) is an admissible rooted dissection. The set of admissible pairs (τ, a) such that τ has i black nodes and j white nodes is denoted by $\Pi_{i,j}$; and the whole class of admissible pairs is denoted by Π.

It is easily shown that a rooted irreducible dissection δ is admissible iff there exists no path of length 3 that connects the root-vertex of δ to the opposite outer vertex, and passes by an inner vertex of δ. Hence, testing admissibility has linear time complexity. To sum up, 1) rooted irreducible dissections are in bijection with a superset of rooted 3-connected maps, via edge-addition in the outer face and primal-map construction; 2) the dissections associated to 3-connected maps are called admissible; 3) testing admissibility has linear time complexity.

4.1.6. *Boltzmann sampler for rooted 3-connected maps.* From the Boltzmann sampler for rooted irreducible dissections, the correspondence stated in Proposition 2 yields the following sampler for rooted 3-connected maps:

$\Gamma M(z, w)$: repeat $\delta \leftarrow \Gamma I(z, w)$ until δ is admissible
return Primal(δ)

The Boltzmann sampler $\Gamma M(z, w)$ for rooted 3-connected maps is also a mixed Boltzmann sampler $\Gamma\overrightarrow{G_3}(z, w)$ for edge-rooted 3-connected planar graphs, according to the equivalence stated in Lemma 2.

4.2. **Boltzmann sampler for 2-connected planar graphs.** The next step of our sampler is to realise a Boltzmann sampler for 2-connected planar graphs from the Boltzmann sampler for edge-rooted 3-connected planar graphs obtained in Section 4.1. Precisely, we first decribe a Boltzmann sampler for edge-rooted 2-connected planar graphs, and subsequently obtain a Boltzmann sampler for the derived class of 2-connected planar graphs, by using rejection techniques.

To generate edge-rooted 2-connected planar graphs, we use a well-known decomposition, due to Trakhtenbrot [26] and called *network-decomposition*, which ensures that an edge-rooted 2-connected planar graph can be assembled from edge-rooted 3-connected planar components. Precisely, Trakhtenbrot's decomposition deals with so-called *networks*, where a network is defined as a connected graph N with two distinguished vertices 0 and ∞ called *poles*, such that the graph N^* obtained by adding an edge between 0 and ∞ is a 2-connected planar graph. For the enumeration, the two poles are not counted in the size.

We rely on [28] for the description of Trakhtenbrot's decomposition. A *series-network* or s-network is a network made of at least 2 networks connected *in chain* at their poles, the ∞-pole of a network coinciding with the 0-pole of the following network in the chain. A *parallel network* or p-network is a network made of at least 2 networks connected *in parallel*, so that their respective ∞-poles and 0-poles coincide. A network N such that N^* is 3-connected and the poles are not adjacent is called a *pseudo-brick*. A *polyhedral network* or h-network is a network obtained by taking a pseudo-brick and substituting each edge e of the pseudo-brick by a network N_e (these networks put the bridge between 2-connected and 3-connected planar graphs).

Proposition 3 (Trakhtenbrot)**.** *Networks with at least 2 edges are partitioned into s-networks, p-networks and h-networks.*

Let us explain how to obtain a recursive decomposition involving the different families of networks. Let $\mathcal{D}$, $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{H}$ be respectively the classes of networks, s-networks, p-networks, and h-networks. Let $D(z,y)$, $S(z,y)$, $P(z,y)$, $H(z,y)$ be the associated mixed generating functions with respect to the number of non-pole vertices (variable z) and the number of edges (variable y). We recall that $\mathcal{L}$ is the family consisting only of the link-network, i.e., the graph with one edge connecting the two poles. Proposition 3 ensures that

$$\mathcal{D} = \mathcal{L} + \mathcal{S} + \mathcal{P} + \mathcal{H}.$$

An s-network can be uniquely decomposed into a non-s-network (the head of the chain) followed by a network (the trail of the chain), which yields

$$\mathcal{S} = (\mathcal{L} + \mathcal{P} + \mathcal{H}) \star \mathcal{Z} \star \mathcal{D}.$$

A p-network has a unique *maximal* parallel decomposition into a set of components that are not p-networks. Observe that we consider here graphs without multiple edges, so that at most one of these components is an edge. Whether there is one or no such edge-component yields

$$\mathcal{P} = \mathcal{L} \star \textsc{Set}_{\geq 1}(\mathcal{S} + \mathcal{H}) + \textsc{Set}_{\geq 2}(\mathcal{S} + \mathcal{H}).$$

By definition, the class of h-networks corresponds to an y-substitution of networks in pseudo-bricks. We write $\mathcal{G}_3$ for the family of labelled 3-connected planar graphs and denote by $G_3(z,w)$ the associated generating function with respect to vertices and edges. By definition, a pseudo-brick is an edge-rooted 3-connected plane graph. As a consequence,

$$\mathcal{H} = \overrightarrow{\mathcal{G}_3} \circ_y \mathcal{D}.$$

Finally, Trakhtenbrot's decomposition yields the following decomposition grammar relating networks and edge-rooted 3-connected planar graphs:

$$\text{(N)} \qquad \begin{cases} \mathcal{D} = \mathcal{L} + \mathcal{S} + \mathcal{P} + \mathcal{H} \\ \mathcal{S} = (\mathcal{L} + \mathcal{P} + \mathcal{H}) \star \mathcal{Z} \star \mathcal{D} \\ \mathcal{P} = \mathcal{L} \star \textsc{Set}_{\geq 1}(\mathcal{S} + \mathcal{H}) + \textsc{Set}_{\geq 2}(\mathcal{S} + \mathcal{H}) \\ \mathcal{H} = \overrightarrow{\mathcal{G}_3} \circ_y \mathcal{D} \end{cases}$$

The decomposition grammar (N) is directly translated to a Boltzmann sampler $\Gamma D(z, y)$ for networks, using the sampling rules given in Figure 3. The only terminal nodes of the decomposition grammar are the classes $\mathcal{Z}$, $\mathcal{L}$ (which are explicit), and the class $\overrightarrow{\mathcal{G}_3}$. Thus, the sampler $\Gamma D(z, y)$ and the auxiliary samplers $\Gamma S(z, y)$, $\Gamma P(z, y)$, and $\Gamma H(z, y)$ are recursively specified in terms of $\Gamma\overrightarrow{G_3}(z, w)$, where $w = D(z, y)$.

Observe that each edge-rooted 2-connected planar graph different from the link graph gives rise to two networks, obtained respectively by counting or not counting the root-edge. This yields the identity

$$(1 + \mathcal{L}) \star \overrightarrow{\mathcal{B}} \simeq (1 + \mathcal{D}), \tag{7}$$

where $\mathcal{D}$ is the class of networks and $\mathcal{L}$ is the one-element class made of the link-graph. From that point, a Boltzmann sampler is easily obtained for the family of edge-rooted 2-connected planar graphs. Define a procedure AddRootEdge that adds an edge connecting the two poles 0 and ∞ of a network if they are not already adjacent, and roots the obtained graph at the edge $(0, \infty)$ oriented from 0 to ∞. Equation (7) translates to the following Boltzmann sampler for $\overrightarrow{\mathcal{B}}$,

$\Gamma\overrightarrow{B}(z, y)$: $\gamma \leftarrow \Gamma(1 + D)(z, y)$; AddRootEdge($\gamma$); return γ
$\Gamma(1 + D)(z, y)$: if Bern $\left(\frac{1}{1+D(z,y)}\right)$ return link-graph else return $\Gamma D(z, y)$;

The last step is to obtain a Boltzmann sampler for derived 2-connected planar graphs from the Boltzmann sampler for edge-rooted 2-connected planar graphs (indeed, derived 2-connected planar graphs are the building blocks needed to construct connected planar graphs). This requires a simple rejection loop:

$\Gamma B'(z, y)$: repeat $\gamma \leftarrow \Gamma\overrightarrow{B}(z, y)$ until Bern $\left(\frac{V(\gamma)}{2E(\gamma)}\right)$;
DistributeLabels(γ); return γ

Lemma 4. *The procedure $\Gamma B'(z, y)$ is a Boltzmann sampler for derived 2-connected planar graphs.*

Proof. By definition, $\Gamma\overrightarrow{B}(z, y)$ draws each graph $\gamma \in \overrightarrow{B}$ with probability proportional to $\frac{z^{|\gamma|}}{|\gamma|!} y^{||\gamma||}$, i.e., proportional to $\frac{z^{V(\gamma)}}{(V(\gamma)-2)!} y^{E(\gamma)}$ (because $V(\gamma) = |\gamma|!+2$ and $E(\gamma) = ||\gamma||+1$). It is easily checked that the procedure: [1) $\gamma \leftarrow \Gamma\overrightarrow{B}(z, y)$; 2) DistributeLabels($\gamma$); 3) return γ;] draws each graph $\gamma \in \mathcal{B}$ with probability proportional to $2E(\gamma)\frac{z^{V(\gamma)}}{V(\gamma)!} y^{E(\gamma)}$, the proof relying on bi-labelled objects (as in the proof of Proposition 1) and on the fact that there are $2E(\gamma)$ ways to root a graph $\gamma \in \mathcal{B}$. Lemma 1 ensures that $\Gamma B'(z, y)$

draws a 2-connected planar graph $\gamma \in \mathcal{B}$ with probability proportional to $\frac{z^{V(\gamma)}}{(V(\gamma)-1)!}y^{E(\gamma)}$, i.e., is a Boltzmann sampler for derived 2-connected planar graphs (using the identity $\mathcal{C}'_{n-1,m} \simeq \mathcal{C}_{n,m}$). Moreover, it easily follows from Euler's relation that a 2-connected planar graph γ satisfies $V(\gamma) \leq 2E(\gamma)$, so that the Bernoulli choice is valid. In fact, the factor 2 is necessary only for the case of the link-graph, i.e., the graph having one edge, otherwise $V(\gamma) \leq E(\gamma)$. Notice that the call to DistributeLabels(γ) can be removed according to Remark 1, its presence is only useful to carry out the proof. □

4.3. **Boltzmann sampler for connected planar graphs.** To obtain a Boltzmann sampler for connected planar graphs, we translate a decomposition linking derived connected and derived 2-connected planar graphs to a Boltzmann sampler for derived connected planar graphs. Then, a further rejection step yields a Boltzmann sampler for connected planar graphs. The *block-decomposition* (see [18, p.10] for a detailed description) is specified as follows. Each derived connected planar graph can be uniquely constructed by composition in the following way: take a set of derived 2-connected planar graphs and attach them, by merging their marked vertices into a unique marked vertex. Then, for each unmarked vertex v of each 2-connected component, take a derived connected planar graph γ_v and merge the marked vertex of γ_v with v (this operation corresponds to an x-substitution). Writing $\mathcal{B}$ for the class of 2-connected planar graphs and $B(z, y)$ for its mixed generating function with respect to vertices and edges, the block-decomposition implies

$$\mathcal{C}' = \textsc{Set}\left(\mathcal{B}' \circ_x (\mathcal{Z} \star \mathcal{C}')\right), \qquad C'(x, y) = \exp(B'(xC'(x, y), y)). \tag{8}$$

The block-decomposition translates to the following sampler for derived connected planar graphs

```
ΓC'(x,y): k ← Pois(B'(z,y))  [with z = xC'(x,y)]
          γ ← (ΓB'(z,y), ..., ΓB'(z,y)) {k independent calls}
          merge the k components of γ at their marked vertices
          for each unmarked vertex v of γ
              γ_v ← ΓC'(x,y)
              merge the marked vertex of γ_v with v
          return γ.
```

Lemma 5. *The sampler $\Gamma C'(x, y)$ is a Boltzmann sampler for derived connected planar graphs.*

Proof. Using the sampling rules for Set and x-substitution in Figure 3, the block decomposition (8) is directly translated to a Boltzmann sampler for derived connected planar graphs, which is recursively specified in terms of $\Gamma B'(z, y)$, where $z = xC'(x, y)$. □

A Boltzmann sampler for connected planar graphs is simply obtained from $\Gamma C'(x, y)$ by using a rejection step so as to adjust the probability distribution:

```
ΓC(x,y): repeat γ ← ΓC'(x,y) until Bern(1/V(γ)); return γ.
```

Lemma 6. *The sampler $\Gamma C(x, y)$ is a Boltzmann sampler for connected planar graphs.*

Proof. Lemma 1 ensures that the probability of a graph γ to be drawn by $\Gamma C(x, y)$ is proportional to $\frac{x^{V(\gamma)-1}}{(V(\gamma)-1)!}y^{||\gamma||}$ (because of $\Gamma C'(x)$ and of the identity $\mathcal{C}'_{n-1,m} \equiv \mathcal{C}_{n,m}$)

multiplied by $\frac{1}{V(\gamma)}$ (because of the rejection step). Hence the probability is proportional to $\frac{x^{V(\gamma)}}{V(\gamma)!}y^{||\gamma||}$, i.e., $\Gamma C(x,y)$ is a valid Boltzmann sampler for connected planar graphs. □

4.4. **Boltzmann sampler for planar graphs.** Let $\mathcal{G}$ be the class of planar graphs and $\mathcal{C}$ the class of connected planar graphs. Let $G(x,y)$ and $C(x,y)$ be the mixed generating functions of $\mathcal{G}$ and $\mathcal{C}$ with respect to the number of vertices and edges. A planar graph is decomposed into the set of its connected components, yielding

$$\mathcal{G} = \textsc{Set}(\mathcal{C}), \qquad G(x,y) = \exp(C(x,y)), \tag{9}$$

which translates to the sampler

$\Gamma G(x,y)$: $k \leftarrow \text{Pois}(C(x,y))$
return $(\Gamma C(x,y), \dots, \Gamma C(x,y))$ {k independent calls}

Proposition 4. *The procedure $\Gamma G(x,y)$ is a Boltzmann sampler for planar graphs.*

Proof. As $\Gamma C(x,y)$ is a Boltzmann sampler for connected planar graphs, the sampling rule for the construction Set, given in Figure 3, ensures that $\Gamma G(x,y)$ is a Boltzmann sampler for planar graphs □

5. Deriving an efficient sampler

The preceding section has provided the complete description of a Boltzmann sampler for planar graphs. However more is needed to achieve the complexity stated in Theorems 1 and 2, as shown here.

5.1. **Size distribution.** In the last section, we have described a method to produce a mixed Boltzmann sampler $\Gamma G(x,y)$ for labelled planar graphs. In particular, $\Gamma G(x,1)$ is a Boltzmann sampler for labelled planar graphs, drawing two planar graphs with the same number of vertices with equal probability. For practical purpose, a *target size* n is chosen by the user (e.g., $n = 100,000$), and the sampler is required to return a random planar graph whose size is around n up to a few percents, or even exactly n. As a consequence, the size distribution of planar graphs output by $\Gamma G(x,y)$ has to be studied. Typically, we need to *tune* the real parameter x in order to ensure that the size distribution is concentrated around the target value n. The validity of this tuning operation depends on the *singularity type* of $G(x)$.

Definition 2. Given $\alpha \in \mathbf{R}\backslash\mathbf{Z}_{\geq 0}$, a generating function $f(x)$ is said to be α-singular if the following expansion holds in a Δ-neighbourhood (i.e., an indented disk) of its dominant singularity ρ (see [8] for technical conditions of such neighbourhoods),

$$f(x) \underset{x\to\rho}{=} P(x) + c_\alpha \left(1 - \frac{x}{\rho}\right)^\alpha + o\left(1 - \frac{x}{\rho}\right)^\alpha,$$

where $P(x)$ is a polynomial and c_α is a non-zero real value.

Remark 3. Clearly, an α-singular function converges at ρ if $\alpha > 0$; and diverges if $\alpha < 0$: $f(x) \sim (1 - x/\rho)^\alpha$. Moreover, if a function is α singular, then its derivative is $(\alpha - 1)$-singular [10, ch.6]. These properties will be useful in the analysis of the time complexity (Section 6).

The following lemma, Theorem 6.3 of [8], ensures that the tuning operation mentioned above applies well when $f(x)$ is α-singular with $\alpha < 0$. We state it in a slightly more general version, extended to mixed Boltzmann samplers.

Lemma 7. (Duchon *et al* [8]) *Let $\mathcal{F}$ be a mixed combinatorial class endowed with a Boltzmann sampler $\Gamma F(x,y)$. Let $F(x,y)$ be the mixed generating function of $\mathcal{F}$. Given $y>0$, assume that the function $x \to F(x,y)$ is α-singular with $\alpha<0$. For each integer n, define $x_n = \rho_G(y)\left(1+\frac{\alpha}{n}\right)$, $\rho_G(y)$ being the radius of convergence of $x \to F(x,y)$. Let X_n be the random variable defined as the labelled size of an object output by $\Gamma F(x_n,y)$.*

Then, for each fixed tolerance-ratio $\epsilon>0$,

$$\mathbf{P}\left(X_n \in [n(1-\epsilon), n(1+\epsilon)]\right) \to p_\epsilon \quad \text{as } n\to\infty,$$

where p_ϵ is a positive constant depending on ϵ, being of order ϵ as $\epsilon \to 0$: $p_\epsilon \sim_{\epsilon\to 0} \sigma\cdot\epsilon$ for some constant σ.

Moreover,

$$\mathbf{P}\left(X_n = n\right) \sim \frac{\sigma}{n} \quad \text{as } n\to\infty, \text{ for the same constant } \sigma>0.$$

The following lemma indicates that we have to "derive 3 times" the Boltzmann sampler $\Gamma G(x,y)$ for planar graphs, so that the size distribution of the output gets the good behaviour stated in Lemma 7.

Lemma 8. *Let $G(x,y)$ be the mixed generating function of labelled planar graphs. Then, for each $y>0$, the function $x \to G'''(x,y)$ is $(-1/2)$-singular.*

Proof. It has been shown by Giménez and Noy [17] that, for each $y>0$, the function $x\to G(x,y)$ is 5/2-singular. Moreover, the derivative of an α-singular function is $(\alpha-1)$-singular (see Remark 3). As a consequence, the function $x\to G'''(x,y)$ is $(-1/2)$-singular. □

5.2. **The derivative operator and the decomposition of planar graphs.** The derivative operator is easily injected in the 5 constructions used to describe the decomposition of planar graphs,

$$\left\{\begin{array}{rcl} (\mathcal{A}+\mathcal{B})' &=& \mathcal{A}'+\mathcal{B}' \\ (\mathcal{A}\star\mathcal{B})' &=& \mathcal{A}'\star\mathcal{B}+\mathcal{A}\star\mathcal{B}' \\ \textsc{Set}_{\geq d}(\mathcal{A})' &=& \mathcal{A}'\star\textsc{Set}_{\geq d-1}(\mathcal{A}) \\ (\mathcal{A}\circ_x\mathcal{B})' &=& \mathcal{B}'\star\mathcal{A}'\circ_x\mathcal{B} \\ (\mathcal{A}\circ_y\mathcal{B})' &=& \mathcal{A}'\circ_y\mathcal{B}+\mathcal{B}'\star\overline{\mathcal{A}}\circ_y\mathcal{B}, \end{array}\right. \tag{10}$$

where we recall that $\overline{\mathcal{A}}$ stands for the y-derived class of $\mathcal{A}$. As a consequence, the derivative operator can be injected in the chain of decompositions, in order to assemble a Boltzmann sampler for triply derived planar graphs, as explained next.

5.2.1. *Boltzmann samplers for derived 3-connected planar graphs.* Given a bicolored binary tree τ, we denote by $|\tau|_\bullet$ the number of black nodes of τ and by $|\tau|$ the number of nodes of τ. Let $\Gamma T(z,w)$ be a Boltzmann sampler for bicolored binary trees and $\Gamma T'(z,w)$ be a Boltzmann sampler for the class $\mathcal{T}'$ of bicolored binary trees with a pointed black nodes that does not count in the size. In other words, $\Gamma T'(z,w)$ draws a bicolored binary tree τ with probability proportional to $|\tau|_\bullet z^{|\tau|_\bullet} w^{|\tau|}$. The class $\mathcal{T}'$ has a complete recursive decomposition, obtained by deriving the decomposition grammar of $\mathcal{T}$ with respect to z,

$$\left\{\begin{array}{rcl} \mathcal{T}' &=& \mathcal{T}'_\bullet+\mathcal{T}'_\circ \\ \mathcal{T}'_\bullet &=& \mathcal{T}'_\circ\star\mathcal{Z}_\bullet\star(1+\mathcal{T}_\circ)+(1+\mathcal{T}_\circ)\star(1+\mathcal{T}_\circ)+(1+\mathcal{T}_\circ)\star\mathcal{Z}_\bullet\star\mathcal{T}'_\circ \\ \mathcal{T}'_\circ &=& \mathcal{T}'_\bullet\star\mathcal{Z}_\circ\star(1+\mathcal{T}_\bullet)+(1+\mathcal{T}_\bullet)\star\mathcal{Z}_\circ\star\mathcal{T}'_\bullet, \end{array}\right. \tag{11}$$

which is translated to a Boltzmann sampler $\Gamma T'(z,w)$ using the sampling rules for Sum and Product.

The correspondence *binary-trees↔rooted-3-connected-maps* stated in Proposition 2 and Lemma 1 (rejection lemma) ensure that the following algorithms are Boltzmann samplers for the derived classes of rooted 3-connected maps up to order 2.

$\Gamma M'(z,w)$: repeat $\tau \leftarrow \Gamma T(z,w)$
until ($M \leftarrow \text{closure}(\tau, \text{rnd}(1,2,3))$ is admissible and Bern $\left(\frac{|\tau|_\bullet+1}{|\tau|+2}\right)$);
return M

$\Gamma\overline{M}(z,w)$: repeat $\tau \leftarrow \Gamma T(z,w)$
until ($M \leftarrow \text{closure}(\tau, \text{rnd}(1,2,3))$ is admissible and Bern $\left(\frac{3}{4}\frac{|\tau|+3}{|\tau|+2}\right)$);
return M

$\Gamma M''(z,w)$: repeat $\tau \leftarrow \Gamma T'(z,w)$
until ($M \leftarrow \text{closure}(\tau, \text{rnd}(1,2,3))$ is admissible and Bern $\left(\frac{|\tau|_\bullet+1}{|\tau|+2}\right)$);
return M

$\Gamma\overline{M}'(z,w)$: repeat $\tau \leftarrow \Gamma T'(z,w)$
until ($M \leftarrow \text{closure}(\tau, \text{rnd}(1,2,3))$ is admissible and Bern $\left(\frac{3}{8}\frac{(|\tau|_\bullet+1)(|\tau|+3)}{|\tau|_\bullet(|\tau|+2)}\right)$);
return M

$\Gamma\overline{\overline{M}}(z,w)$: repeat $\tau \leftarrow \Gamma T'(z,w)$
until ($M \leftarrow \text{closure}(\tau, \text{rnd}(1,2,3))$ is admissible and Bern $\left(\frac{1}{7}\frac{|\tau|+3}{|\tau|_\bullet}\right)$);
return M

For instance, Proposition 2 and Lemma 1 imply that, in the sampler $\Gamma\overline{M}'(z,w)$, each admissible pair $(\tau,a) \in \Pi$ (with $a \in \{1,2,3\}$) is drawn with probability proportional to $(|\tau|_\bullet+1)(|\tau|+3)z^{|\tau|_\bullet}w^{|\tau|}$ (the 3/8 factor ensuring that the Bernoulli parameter is not larger than 1). Moreover, if $\gamma = \text{closure}(\tau, a)$, then $|\tau|_\bullet+1 = V(\gamma)-2$ and $|\tau|+3 = E(\gamma)-1$. As a consequence, $\Gamma\overline{M}'(z,w)$ draws a rooted 3-connected map γ with probability proportional to $(E(\gamma)-1)(V(\gamma)-2)z^{V(\gamma)}w^{E(\gamma)}$, i.e, is a Boltzmann sampler for $\overline{\mathcal{M}}'$. In the four samplers given above, the probabilities of the Bernoulli choices are always away from 0 by a fixed constant, a consequence of the inequalities $|\tau|_\bullet \leq |\tau|$ and $|\tau| \leq 3|\tau|_\bullet + 1$. This property is crucial to obtain the complexities stated in Theorems 1 and 2. It is possible to improve the constant factors in the Bernoulli choices (e.g., 1/7 for $\Gamma\overline{\overline{M}}(z,w)$) by treating graphs with few vertices separately.

5.2.2. *Boltzmann samplers for derived 2-connected planar graphs.* Starting from the 4-lines decomposition grammar (N) of networks and deriving two times, we obtain successively

$$\text{(N)}\quad \begin{cases} \mathcal{D} = \mathcal{L} + \mathcal{S} + \mathcal{P} + \mathcal{H} \\ \mathcal{S} = (\mathcal{L} + \mathcal{P} + \mathcal{H}) \star \mathcal{Z} \star \mathcal{D} \\ \mathcal{P} = \mathcal{L} \star \textsc{Set}_{\geq 1}(\mathcal{S} + \mathcal{H}) + \textsc{Set}_{\geq 2}(\mathcal{S} + \mathcal{H}) \\ \mathcal{H} = \overrightarrow{\mathcal{G}_3} \circ_y \mathcal{D} \end{cases}$$

$$\text{(N')}\quad \begin{cases} \mathcal{D}' = \mathcal{S}' + \mathcal{P}' + \mathcal{H}' \\ \mathcal{S}' = (\mathcal{P}' + \mathcal{H}') \star \mathcal{Z} \star \mathcal{D} + (\mathcal{L} + \mathcal{P} + \mathcal{H}) \star (\mathcal{D} + \mathcal{Z} \star \mathcal{D}') \\ \mathcal{P}' = \mathcal{L} \star (\mathcal{S}' + \mathcal{H}') \star \textsc{Set}(\mathcal{S} + \mathcal{H}) + (\mathcal{S}' + \mathcal{H}') \star \textsc{Set}_{\geq 1}(\mathcal{S} + \mathcal{H}) \\ \mathcal{H}' = \overrightarrow{\mathcal{G}_3}' \circ_y \mathcal{D} + \mathcal{D}' \star \overrightarrow{\overrightarrow{\mathcal{G}_3}} \circ_y \mathcal{D} \end{cases}$$

$$\text{(N'')}\quad \begin{cases} \mathcal{D}'' = \mathcal{S}'' + \mathcal{P}'' + \mathcal{H}'' \\ \mathcal{S}'' = (\mathcal{P}''+\mathcal{H}'') \star \mathcal{Z} \star \mathcal{D} + 2(\mathcal{P}'+\mathcal{H}') \star (\mathcal{D}+\mathcal{Z}\star\mathcal{D}') + (\mathcal{L}+\mathcal{P}+\mathcal{H}) \star (2\mathcal{D}'+\mathcal{Z}\star\mathcal{D}'') \\ \mathcal{P}'' = (\mathcal{L}\star(\mathcal{S}''+\mathcal{H}'') + (1+\mathcal{L}) \star (\mathcal{S}'+\mathcal{H}')^2) \star \textsc{Set}(\mathcal{S}+\mathcal{H}) + (\mathcal{S}''+\mathcal{H}'') \star \textsc{Set}_{\geq 1}(\mathcal{S}+\mathcal{H}) \\ \mathcal{H}'' = \overrightarrow{\mathcal{G}_3}'' \circ_y \mathcal{D} + 2\mathcal{D}' \star \overrightarrow{\overrightarrow{\mathcal{G}_3}}' \circ_y \mathcal{D} + \mathcal{D}'^2 \star \overrightarrow{\overrightarrow{\mathcal{G}_3}} \circ_y \mathcal{D} + \mathcal{D}'' \star \overrightarrow{\overrightarrow{\mathcal{G}_3}} \circ_y \mathcal{D} \end{cases}$$

In these three systems taken together, the only terminal nodes are the class $\overrightarrow{\mathcal{G}_3}$ and its derived classes up to order 2, which are isomorphic to the class $\mathcal{M}$ of rooted 3-connected maps and its derived classes up to order 2, via the identity $\mathcal{M} \simeq 2\overrightarrow{\mathcal{G}_3}$. In addition, we have obtained in Section 5.2.1 Boltzmann samplers for the derived classes of rooted 3-connected planar maps up to order 2. Hence, using the sampling rules of Figure 3 , the three systems for networks, derived networks, and doubly derived networks are translated respectively to Boltzmann samplers $\Gamma D(z,y)$, $\Gamma D'(z,y)$, and $\Gamma D''(z,y)$, which are recursively specified in terms of the Boltzmann samplers for $\mathcal{M}$ and its derived classes up to order 2, taken at $(z, w = D(z,y))$.

Then, Boltzmann samplers for derived edge-rooted 2-connected planar graphs are easily obtained. Indeed, Equation (7) yields successively

$$(1+\mathcal{L}) \star \overrightarrow{\mathcal{B}} \simeq (1+\mathcal{D}), \quad (1+\mathcal{L}) \star \overrightarrow{\mathcal{B}}' \simeq \mathcal{D}', \quad (1+\mathcal{L}) \star \overrightarrow{\mathcal{B}}'' \simeq \mathcal{D}'',$$

which translates to

$\Gamma\overrightarrow{B}(z,y)$: $\gamma \leftarrow \Gamma(1+D)(z,y)$; AddRootEdge($\gamma$); return γ
$\Gamma\overrightarrow{B}'(z,y)$: $\gamma \leftarrow \Gamma D'(z,y)$; AddRootEdge($\gamma$); return γ
$\Gamma\overrightarrow{B}''(z,y)$: $\gamma \leftarrow \Gamma D''(z,y)$; AddRootEdge($\gamma$); return γ

Finally, the rejection technique allows us to obtain Boltzmann samplers for derived 2-connected planar graphs from Boltzmann samplers for (derived) edge-rooted 2-connected planar graphs; the following samplers $\Gamma B'(z,y)$, $\Gamma B''(z,y)$, and $\Gamma B'''(z,y)$ are valid Boltzmann samplers for $\mathcal{B}'$, $\mathcal{B}''$, and $\mathcal{B}'''$ (after a call to DistributeLabels(γ)).

(12)
$\Gamma B'(z,y)$: repeat $\gamma \leftarrow \Gamma\overrightarrow{B}(z,y)$ until Bern $\left(\frac{V(\gamma)}{2E(\gamma)}\right)$; return γ
$\Gamma B''(z,y)$: repeat $\gamma \leftarrow \Gamma(\overrightarrow{B} + z\overrightarrow{B}')(z,y)$ until Bern $\left(\frac{V(\gamma)}{2E(\gamma)}\right)$; return γ
$\Gamma(\overrightarrow{B} + z\overrightarrow{B}')(z,y)$: if Bern $\left(\frac{\overrightarrow{B}(z,y)}{\overrightarrow{B}(z,y)+z\overrightarrow{B}'(z,y)}\right)$ return $\Gamma\overrightarrow{B}(z,y)$ else return $\Gamma\overrightarrow{B}'(z,y)$
$\Gamma B'''(z,y)$: repeat $\gamma \leftarrow \Gamma(2\overrightarrow{B}' + z\overrightarrow{B}'')(z,y)$ until Bern $\left(\frac{V(\gamma)}{E(\gamma)}\right)$; return γ
$\Gamma(2\overrightarrow{B}' + z\overrightarrow{B}'')(z,y)$: if Bern $\left(\frac{2\overrightarrow{B}'(z,y)}{2\overrightarrow{B}'(z,y)+z\overrightarrow{B}''(z,y)}\right)$ return $\Gamma\overrightarrow{B}'(z,y)$ else return $\Gamma\overrightarrow{B}''(z,y)$

The proof is similar to the proof of Lemma 4. For instance, the sampler $[\gamma \leftarrow \Gamma(\overrightarrow{B} + z\overrightarrow{B}')(z,y)$; DistributeLabels($\gamma$); return $\gamma]$ draws each 2-connected planar graph γ with

probability proportional to $E(\gamma)(V(\gamma)-1)\frac{z^{V(\gamma)}}{V(\gamma)!}y^{E(\gamma)}$. Hence, Lemma 1 ensures that the sampler $\Gamma B''(z,y)$ draws each 2-connected planar graph $\gamma \in \mathcal{B}$ with probability proportional to $V(\gamma)(V(\gamma)-1)\frac{z^{V(\gamma)}}{V(\gamma)}y^{E(\gamma)}$, i.e., is a Boltzmann sampler for $\mathcal{B}''$. Notice that the factor $1/2$ before $V(\gamma)/E(\gamma)$ in the Bernoulli choice is only useful for the link-graph (otherwise $V(\gamma) \leq E(\gamma)$). Using a starting switch deciding if the generated graph is the link-graph makes it possible to remove the factor $1/2$, improving the expected time complexity of the sampler.

5.2.3. *Boltzmann samplers for derived connected planar graphs.* Starting from Equation (8), the derivative rules (10) yield successively

(13)
$$\begin{cases} \mathcal{C}' = \textsc{Set}\left(\mathcal{B}' \circ_x (\mathcal{Z} \star \mathcal{C}')\right), \\ \mathcal{C}'' = (\mathcal{C}' + \mathcal{Z} \star \mathcal{C}'') \star \mathcal{B}'' \circ_x (\mathcal{Z} \star \mathcal{C}') \star \mathcal{C}', \\ \mathcal{C}''' = (2\mathcal{C}'' + \mathcal{Z} \star \mathcal{C}''') \star \mathcal{B}'' \circ_x (\mathcal{Z} \star \mathcal{C}') \star \mathcal{C}' + (\mathcal{C}' + \mathcal{Z} \star \mathcal{C}'')^2 \star \mathcal{B}''' \circ_x (\mathcal{Z} \star \mathcal{C}') \star \mathcal{C}' + (\mathcal{C}' + \mathcal{Z} \star \mathcal{C}'') \star \mathcal{B}'' \circ_x (\mathcal{Z} \star \mathcal{C}') \star \mathcal{C}''. \end{cases}$$

Using the sampling rules of Figure 3, these decompositions translate to Boltzmann samplers $\Gamma C'(x,y)$, $\Gamma C''(x,y)$, and $\Gamma C'''(x,y)$, which are recursively specified in terms of the Boltzmann samplers $\Gamma B'''(z,y)$, $\Gamma B''(z,y)$, and $\Gamma B'(z,y)$, where $z = xC'(x,y)$.

5.2.4. *Boltzmann samplers for derived planar graphs.* Starting from $\mathcal{G} = \textsc{Set}(\mathcal{C})$, the derivative rules (10) yield successively

(14)
$$\begin{cases} \mathcal{G} &= \textsc{Set}(\mathcal{C}), \\ \mathcal{G}' &= \mathcal{C}' \star \mathcal{G}, \\ \mathcal{G}'' &= \mathcal{C}'' \star \mathcal{G} + \mathcal{C}' \star \mathcal{G}', \\ \mathcal{G}''' &= \mathcal{C}''' \star \mathcal{G} + 2\mathcal{C}'' \star \mathcal{G}' + \mathcal{C}' \star \mathcal{G}''. \end{cases}$$

Again, using the sampling rules of Figure 3, these decompositions translate to Boltzmann samplers $\Gamma G'(x,y)$, $\Gamma G''(x,y)$, and $\Gamma G'''(x,y)$, which are specified in terms of the Boltzmann samplers $\Gamma C'''(x,y)$, $\Gamma C''(x,y)$, $\Gamma C'(x,y)$, and $\Gamma C(x,y)$. (The Boltzmann sampler $\Gamma C(x,y)$ has already been obtained from $\Gamma C'(x,y)$ using rejection, see Lemma 6).

The complete algorithmic scheme, from binary trees to triply derived planar graphs, is summarized in Figure 5 and Figure 6.

5.3. **Samplers according to the number of vertices.** The random sampler of planar graphs we use is the "triply derived" Boltzmann sampler $\Gamma G'''(x_n, 1)$ with the value $x_n = \rho_G\left(1 - \frac{1}{2n}\right)$ tuned as indicated in Lemma 7, ρ_G being the radius of convergence of $G(x,1)$. The exact-size sampler is

$\mathfrak{A}_n$: repeat $\gamma \leftarrow \Gamma G'''(x_n, 1)$ until $V(\gamma) = n$; return γ.

For any $\epsilon > 0$, the approximate-size sampler is

$\mathfrak{A}_{n,\epsilon}$: repeat $\gamma \leftarrow \Gamma G'''(x_n, 1)$ until $V(\gamma) \in [n(1-\epsilon), n(1+\epsilon)]$; return γ.

5.4. **Samplers according to the numbers of vertices and edges.** For any $y > 0$, we denote by $\rho_G(y)$ the radius of convergence of $x \to G(x,y)$. Let $\mu(y)$ be the function defined as

$$\mu(y) := -y\frac{\mathrm{d}\rho_G}{\mathrm{d}y}(y)/\rho_G(y).$$

It has been shown by Giménez and Noy [17] that the function $\mu(y)$ is strictly increasing on $(0, +\infty)$, with $\lim \mu(y) = 1$ as $y \to 0$ and $\lim \mu(y) = 3$ as $y \to +\infty$. As a consequence,

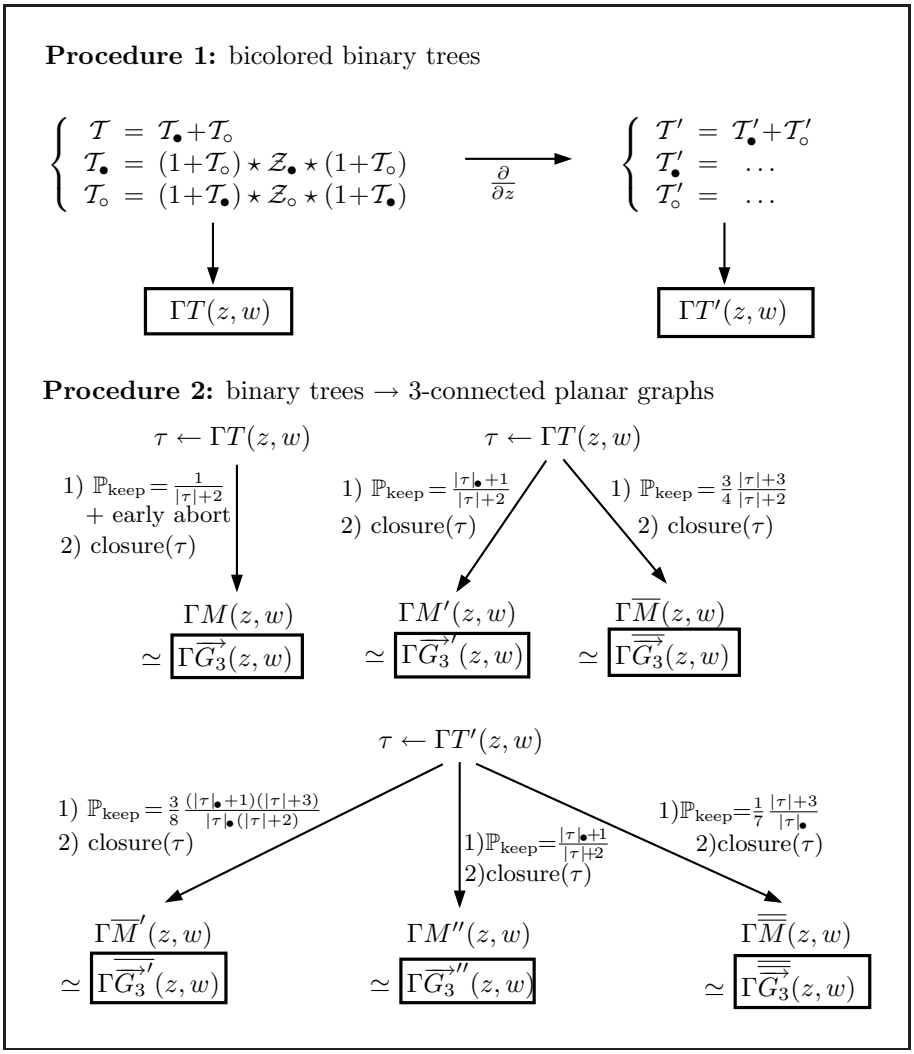

FIGURE 5. The algorithmic scheme producing Boltzmann samplers for 3-connected planar graphs from Boltzmann samplers for bicolored binary trees.

$\mu(y)$ has an inverse function $y(\mu)$ defined on $(1,3)$. We define $x_n(\mu) := \rho_G(y(\mu))(1-\frac{1}{2n})$. The exact size sampler we propose is

$\mathfrak{A}_{n,\mu}$: repeat $\gamma \leftarrow \Gamma G'''(x_n(\mu), y(\mu))$ until $(V(\gamma)=n$ and $E(\gamma)=\lfloor \mu n \rfloor)$; return γ.

For any $\epsilon > 0$, the approximate-size sampler is

$\mathfrak{A}_{n,\mu,\epsilon}$: repeat $\gamma \leftarrow \Gamma G'''(x_n(\mu), y(\mu))$
until $(V(\gamma) \in [n(1-\epsilon), n(1+\epsilon)]$ and $\frac{E(\gamma)}{V(\gamma)} \in [\mu(1-\epsilon), \mu(1+\epsilon)])$;
return γ.

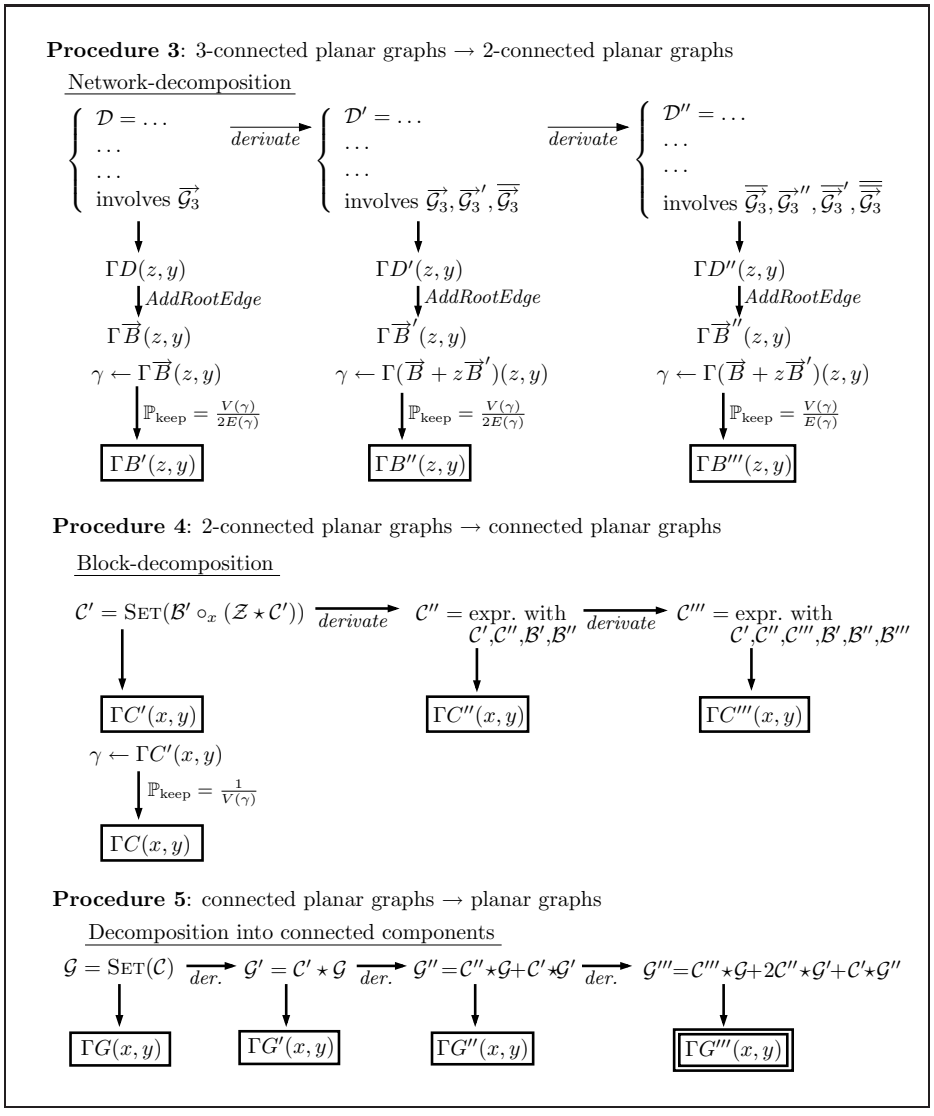

FIGURE 6. The algorithmic scheme producing a Boltzmann sampler for triply derived planar graphs from Boltzmann samplers for 3-connected planar graphs.

6. Analysis of the time complexity

6.1. **Preliminaries.** This section is devoted to the proof of the time complexities of the planar graph generators, as stated in Theorems 1 and 2. For this purpose, we need explict rules to compute the expected complexity of a Boltzmann sampler obtained from the constructions given in Figure 3. Given a mixed combinatorial class $\mathcal{C}$ endowed with a Boltzmann sampler $\Gamma C(x,y)$, we denote by $\Lambda C(x,y)$ the expected complexity of a call to $\Gamma C(x,y)$. We also define $|C|_{(x,y)}$ ($||C||_{(x,y)}$) as the expected number of labelled (unlabelled, respectively) atoms of an object of $\mathcal{C}$ drawn under the Boltzmann distribution at (x,y).

Notice that

$$|C|_{(x,y)} = x\frac{\partial C}{\partial x}(x,y)/C(x,y), \qquad ||C||_{(x,y)} = y\frac{\partial C}{\partial y}(x,y)/C(x,y). \tag{15}$$

Then we have the following computation rules for each of the sampling rules given in Figure 3:

$$\begin{cases} \mathcal{C} = \mathcal{A} + \mathcal{B} & \Lambda C(x,y) = \frac{A(x,y)}{C(x,y)}\Lambda A(x,y) + \frac{B(x,y)}{C(x,y)}\Lambda B(x,y) \\ \mathcal{C} = \mathcal{A} \star \mathcal{B} & \Lambda C(x,y) = \Lambda A(x,y) + \Lambda B(x,y) \\ \mathcal{C} = \textsc{Set}_{\geq d}(\mathcal{A}) & \Lambda C(x,y) = \frac{e_{d-1}(A(x,y))}{e_d(A(x,y))} A(x,y)\cdot(1+\Lambda A(x,y)) \\ \mathcal{C} = \mathcal{A} \circ_x \mathcal{B} & \Lambda C(x,y) = \Lambda A(B(x,y),y) + |A|_{(B(x,y),y)}\Lambda B(x,y) \\ \mathcal{C} = \mathcal{A} \circ_y \mathcal{B} & \Lambda C(x,y) = \Lambda A(x,B(x,y)) + ||A||_{(x,B(x,y))}\Lambda B(x,y) \end{cases} \tag{16}$$

Let us comment on the computation rule for $\textsc{Set}_{\geq d}(\mathcal{A})$. The sampler $\Gamma(\textsc{Set}_{\geq d}(\mathcal{A}))$ draws a Poisson law $k \leftarrow \text{Pois}_{\geq d}(A(x,y))$ and then performs k independent calls to $\Gamma A(x,y)$. As already mentioned in Section 3.3, the cost of drawing $\text{Pois}_{\geq d}(A(x,y))$ is equal to its output k. The expected complexity of k calls to $\Gamma A(x,y)$ is $k\ \Lambda A(x,y)$. Moreover, for $\lambda \geq 0$, the expectation of $\text{Pois}_{\geq d}(\lambda)$ is $e_{d-1}(\lambda)/e_d(\lambda)\cdot\lambda$. The result follows.

We also need a computation rule associated with a rejection sampler: the following lemma is the counterpart of Lemma 1 for complexity analysis.

Lemma 9 (rejection complexity)**.** *Let $\mathfrak{A}$ be a random sampler on a combinatorial class $\mathcal{C}$ according to a probability distribution $\mathbb{P}$, and let $p : \mathcal{C} \to [0,1]$ be a function on $\mathcal{C}$, called rejection function. Consider the rejection algorithm*

$\mathfrak{A}_{\text{rej}}$: repeat $\gamma \leftarrow \mathfrak{A}$ until $\text{Bern}(p(\gamma))$ return γ.

Then the expected complexity $\mathbb{E}(\mathfrak{A}_{\text{rej}})$ of $\mathfrak{A}_{\text{rej}}$ and the expected complexity $\mathbb{E}(\mathfrak{A})$ of $\mathfrak{A}$ are related by

$$\mathbb{E}(\mathfrak{A}_{\text{rej}}) = \frac{1}{p_{\text{acc}}}\mathbb{E}(\mathfrak{A}), \tag{17}$$

where $p_{\text{acc}} := \sum_{\gamma\in\mathcal{C}} \mathbb{P}(\gamma)p(\gamma)$ is the probability of success of $\mathfrak{A}$ at each trial.

Proof. The quantity $\mathbb{E}(\mathfrak{A}_{\text{rej}})$ satisfies the recursive equation

$$\mathbb{E}(\mathfrak{A}_{\text{rej}}) = \mathbb{E}(\mathfrak{A}) + (1-p_{\text{acc}})\mathbb{E}(\mathfrak{A}_{\text{rej}}).$$

Indeed, a first trial, with expected complexity $\mathbb{E}(\mathfrak{A})$, is always needed; and in case of rejection, occuring with probability $(1-p_{\text{acc}})$, the sampler restarts in the same way as when it is launched. □

As detailed from Section 6.2 to Section 6.6, the computation rules given by (16) and Lemma 9 allow us to obtain the asymptotic order of $\Lambda G'''(x,y)$ when (x,y) is close to a singularity. We claim that this is sufficient to establish the time complexities of the planar graph generators. We recall that, given a generating function $C(x,y)$ and a fixed $y > 0$, $\rho_C(y)$ stands for the radius of convergence of $x \to C(x,y)$. In the sequel, all convergence statements such as $x \to \rho_C(y)$ are meant "from below", i.e., with $x < \rho_C(y)$.

Claim 1. *Proving the time complexities of the planar graph generators, as stated in Theorems 1 and 2, reduces to proving the following asymptotic result:*

$$\text{for each fixed y} > 0, \ \ \Lambda G'''(x,y) = \underset{x\to\rho_G(y)}{\mathcal{O}}\left(\frac{1}{1-x/\rho_G(y)}\right). \tag{18}$$

Proof. Let $\pi_{n,\epsilon}$ be the probability that the output of $\Gamma G'''(x_n, 1)$ (with $x_n = \rho_G(1-1/2n)$) has size in $[n(1-\epsilon), n(1+\epsilon)]$ and π_n the probability that the output has size n. According to Lemma 9, the expected complexity of the exact size and approximate size samplers with respect to vertices —as described in Section 5.3— satisfy

$$\mathbb{E}(\mathfrak{A}_{\mathfrak{n}}) = \frac{\Lambda G'''(x_n, 1)}{\pi_n}, \qquad \mathbb{E}(\mathfrak{A}_{\mathfrak{n},\epsilon}) = \frac{\Lambda G'''(x_n, 1)}{\pi_{n,\epsilon}}.$$

Assume that (18) is true. Then $\Lambda G'''(x_n, 1) = \mathcal{O}(n)$ as $n \to \infty$. Moreover, Lemma 7 and Lemma 8 ensure that π_n is of order $1/n$ as $n \to \infty$ and $\pi_{n,\epsilon} \to p_\epsilon$ as $n \to \infty$, the constant p_ϵ being of order ϵ as $\epsilon \to 0$. Thus, proving (18) is sufficient to prove the expected complexities of the samplers with respect to vertices stated in Theorem 1.

The proof for the samplers with respect to vertices and edges is similar. Let $\pi_{n,\mu}$ be the probability that the output of $\Gamma G'''(x_n(\mu), y(\mu))$ (with $x_n(\mu)$ and $y(\mu)$ as given in Section 5.4) has n vertices and $\lfloor \mu n \rfloor$ edges, and let $\pi_{n,\mu,\epsilon}$ be the probability that the output $\gamma \leftarrow \Gamma G'''(x_n(\mu), y(\mu))$ satisfies $V(\gamma) \in [n(1-\epsilon), n(1+\epsilon)]$ and $E(\gamma)/V(\gamma) \in [\mu(1-\epsilon), \mu(1+\epsilon)]$. Lemma 9 ensures that

$$\mathbb{E}(\mathfrak{A}_{\mathfrak{n},\mu}) = \frac{\Lambda G'''(x_n(\mu), y(\mu))}{\pi_{n,\mu}}, \qquad \mathbb{E}(\mathfrak{A}_{\mathfrak{n},\mu,\epsilon}) = \frac{\Lambda G'''(x_n(\mu), y(\mu))}{\pi_{n,\mu,\epsilon}}.$$

Notice that $\pi_{n,\mu} = \mathbb{P}(||\gamma|| = \lfloor \mu n \rfloor \mid |\gamma| = n) \cdot \mathbb{P}(|\gamma| = n)$, where $\mathbb{P}(.)$ is the Boltzmann distribution on $\mathcal{G}'''$ at $(x_n(\mu), y(\mu))$. For a fixed $\mu \in (1, 3)$, it has been shown by Giménez and Noy [17] (based on the quasi-power theorem) that $\mathbb{P}(||\gamma|| = \lfloor \mu n \rfloor \mid |\gamma| = n)$ is of order $n^{-1/2}$ as $n \to \infty$. Moreover, Lemma 7 and Lemma 8 ensure that $\mathbb{P}(|\gamma| = n)$ is of order $1/n$ as $n \to \infty$. Hence, $\pi_{n,\mu} \sim cn^{-3/2}$ for some constant c that depends on μ. If (18) is true, then $\Lambda G'''(x_n(\mu), y(\mu)) = \mathcal{O}(n)$, so that $\mathbb{E}(\mathfrak{A}_{n,\mu}) = \mathcal{O}(n^{5/2})$. Similarly, $\pi_{n,\mu,\epsilon} = \mathbb{P}\left(\frac{E(\gamma)}{V(\gamma)} \in I_{\mu,\epsilon} \mid V(\gamma) \in I_{n,\epsilon}\right) \cdot \mathbb{P}(V(\gamma) \in I_{n,\epsilon})$, where $I_{\mu,\epsilon} = [\mu(1-\epsilon), \mu(1+\epsilon)]$ and $I_{n,\epsilon} = [n(1-\epsilon), n(1+\epsilon)]$. Based on the result of Giménez and Noy, it is easily proved that, for fixed $\mu \in (1, 3)$ and $\epsilon > 0$, the first factor converges to 1 as $n \to \infty$. In addition, Lemma 7 and Lemma 8 ensure that the second factor converges to a constant p_ϵ, the constant being of order ϵ as $\epsilon \to 0$. Assuming that (18) is true, this yields the expected linear time complexity of $\mathfrak{A}_{n,\mu,\epsilon}$, as stated in Theorem 2. □

The method to prove (18) is to inject the computation rules for complexities, as given by (16) and Lemma 9, into the decomposition of planar graphs. In this way we obtain successively the asymptotic expected complexities of the Boltzmann samplers for 3-connected, 2-connected, connected, and (finally) unconstrained planar graphs.

6.2. **Binary trees.** The decomposition grammars of binary trees and derived binary trees are translated to Boltzmann samplers using the sampling rules for Sum and Product. The obtained samplers have no rejection involved: the tree is built progressively based on the results of the Bernoulli choices. To simplify, we assume unit cost for each Bernoulli choice possibly followed by a node addition. In this model, the complexity of generating a tree is equal to its size (number of nodes) all along the generation.

6.3. **From binary trees to 3-connected planar graphs.** The Boltzmann samplers for rooted 3-connected maps —as given in Section 5.2.1— perform a call to the Boltzmann samplers for binary trees and apply the closure-mapping to the obtained tree. The procedure is combined with a rejection step to obtain the Boltzmann distribution. For each of the families $\mathcal{M}$, $\mathcal{M}'$, $\overline{\mathcal{M}}$, $\mathcal{M}''$, $\overline{\mathcal{M}}'$, and $\overline{\overline{\mathcal{M}}}$, we show in this section that the

expected complexity of the Boltzmann sampler has the same asymptotic order as the expected size of the output. Precisely, the asymptotic is a constant for $\Gamma M(z,w)$, is of order $(1-z/\rho_M(w))^{-1/2}$ for the derived samplers $\Gamma M'(z,w)$ and $\Gamma\overline{M}(z,w)$, and is of order $(1-z/\rho_M(w))^{-1}$ for the doubly derived samplers $\Gamma M''(z,w)$, $\Gamma\overline{M}'(z,w)$, and $\Gamma\overline{\overline{M}}(z,w)$. This means that the rejection loops do not make the complexity order increase.

Lemma 10. *Let I be a compact (i.e., closed and bounded) interval contained in $(0,\infty)$. Then there exists a constant $c>0$ such that, for $w\in I$ and $\rho_M(w)/2<z<\rho_M(w)$,*

$$\Lambda M(z,w)\le c,$$

$$\Lambda M'(z,w)\le \tfrac{c}{\sqrt{1-z/\rho_M(w)}},\qquad \Lambda\overline{M}(z,w)\le \tfrac{c}{\sqrt{1-z/\rho_M(w)}},\tag{19}$$

$$\Lambda M''(z,w)\le \tfrac{c}{1-z/\rho_M(w)},\quad \Lambda\overline{M}'(z,w)\le \tfrac{c}{1-z/\rho_M(w)},\quad \Lambda\overline{\overline{M}}(z,w)\le \tfrac{c}{1-z/\rho_M(w)}.$$

Proof. Proof of the bound on $\Lambda M(z,w)$. The Boltzmann sampler $\Gamma M(z,w)$, as given in the proof of Lemma 3, is a rejection sampler calling the Boltzmann sampler $\Gamma I(z,w)$ for rooted irreducible dissections until the dissection generated is admissible. Moreover, $\Gamma I(z,w)$ calls the following tree generator

$\mathfrak{A}$: $u\leftarrow \mathrm{rnd}(0,1)$; $\mathrm{max\_size}\leftarrow\lfloor 1/u\rfloor$;
$\tau\leftarrow \Gamma T(z,w)$;
abort as soon as $\#\mathrm{nodes}(\tau)+2>\mathrm{max\_size}$

until the generation finishes, and then returns $\mathrm{closure}(\tau,\mathrm{rnd}(1,2,3))$, where τ is the tree generated.

Recall that the closure-mapping has linear time complexity, assumed here (for the sake of simplicity) to be exactly λn for a tree of size n, with $\lambda>0$ a fixed constant. Let $\tau\in\mathcal{T}$ be a tree of size n. For $1\le k<n$, the probability that $\mathfrak{A}$ aborts at size k knowing that τ is generated is

$$\mathbb{P}_k(\tau)=\frac{1}{k}-\frac{1}{k+1}=\frac{1}{k(k+1)},$$

and the probability that τ is completely generated is $1/n$. As a consequence, the expected complexity of $\mathfrak{A}$ knowing that τ is generated satisfies

$$\mathbb{E}_{\mathfrak{A}}(\tau)=\sum_{k=1}^{n-1}k\mathbb{P}_k(\tau)+(n+\lambda n)\frac{1}{n}=\sum_{k=1}^{n-1}k\frac{1}{k(k+1)}+1+\lambda=H_n+\lambda,$$

where $H_n:=\sum_{k=1}^{n}1/k$ is the nth harmonic number. For each $w\in I$, define $a_k(w)$ as the kth coefficient of the one-variable series $z\to T(z,w)$. Singularity analysis of the "tree-type" series $T(z,w)$ [10] and transfer theorems [9, 10] (based on Cauchy's integral formula) ensure that there exists a constant $c_0>0$ such that

$$a_k(w)\le c_0\rho_T(w)^{-k}k^{-3/2}\text{ for }k\ge 1\text{ and }w\in I.$$

It is shown in [2] that, for each $w>0$, $\rho_T(w)=\rho_M(w)$, so that $\rho_T(w)$ can be replaced by $\rho_M(w)$ in the bound on $a_k(w)$. The probability that $\Gamma T(z,w)$ draws a tree with k black nodes is equal to $T(z,w)^{-1}a_k(w)z^k$ for $z<\rho_M(w)$. Moreover, a tree τ having k black nodes has at most $3k+1$ nodes, so that $\mathbb{E}_{\mathfrak{A}}(\tau)\le H_{3k+1}+\lambda$. As a consequence, the expected complexity of $\mathfrak{A}$ satisfies

$$\mathbb{E}(\mathfrak{A})\le\frac{1}{T(z,w)}c_0\sum_{k\ge1}k^{-3/2}(H_{3k+1}+\lambda)\left(\frac{z}{\rho_M(w)}\right)^k\quad\text{for }w\in I\text{ and }z<\rho_M(z).$$

Lemma 9 ensures that the expected complexity of the Boltzmann sampler for rooted irreducible dissections satisfies

$$\Lambda I(z,w) = \frac{\mathbb{E}(\mathfrak{A})}{p_{\text{acc}}},$$

where p_{acc} is the success probability of $\mathfrak{A}$. By definition of $\mathfrak{A}$, each tree $\tau \in \mathcal{T}$ is accepted with probability $1/(|\tau|+2)$ if τ is admissible and with probability 0 otherwise. As a consequence

$$p_{\text{acc}} = \frac{1}{T(z,w)} \sum_{\tau \in \mathcal{T}} \frac{1}{|\tau|+2} z^{|\tau|_\bullet} w^{|\tau|}.$$

Let $I(z,w)$ be the generating function of rooted irreducible dissections with respect to the number of inner black nodes (variable z) and the number of inner white nodes (variable w). According to Proposition 2,

$$p_{\text{acc}} = \frac{I(z,w)}{3T(z,w)}.$$

Moreover, the probability that a dissection drawn by $\Gamma I(z,w)$ is admissible is clearly equal to $M(z,w)/(zw^3 I(z,w))$, because of the bijection between admissible rooted dissections and rooted 3-connected maps. Hence, we obtain

$$\begin{aligned} \Lambda M(z,w) &= \frac{zw^3 I(z,w) \Lambda I(z,w)}{M(z,w)} \\ &\leq \frac{3zw^3}{M(z,w)} c_0 \sum_{k\geq 1} k^{-3/2}(H_{3k+1}+\lambda) \left(\frac{z}{\rho_M(w)}\right)^k, \quad \text{for } w \in I \text{ and } z < \rho_M(w). \end{aligned}$$

This clearly gives a uniform bounding constant on $\Lambda M(z,w)$ for $w \in I$ and $\frac{\rho_M(w)}{2} < z < \rho_M(w)$, because the sum $\sum_{k\geq 1} k^{-3/2}(H_{3k+1}+\lambda)$ is convergent and $M(z,w) \geq M(\frac{1}{2}\rho_M(\min_I), \min_I) > 0$.

Proof of the bound on $\Lambda M'(z,w)$ and $\Lambda \overline{M}(z,w)$. The proof of the bounds on the expected complexities of the derived samplers is easier. Indeed, these samplers do not use early abort rejection. For instance, the sampler $\Gamma M'(z,w)$, as given in Section 5.2.1, satisfies

$$\Lambda M'(z,w) = \frac{\lambda \cdot \Lambda T(z,w)}{p_{\text{acc}}}, \text{ with } p_{\text{acc}} = \frac{1}{3T(z,w)} \sum_{(\tau,a)\in\Pi} \frac{|\tau|_\bullet + 1}{|\tau|+2} z^{|\tau|_\bullet} w^{|\tau|}.$$

Recall that, if $(\gamma) = \text{Primal}(\text{closure}(\tau, a))$, with $(\tau, a) \in \Pi$, then $V(\gamma) = |\tau|_\bullet + 3$ and $E(\gamma) = |\tau| + 4$. Proposition 2 yields

$$p_{\text{acc}} = \frac{1}{3T(z,w)} \sum_{\gamma \in \mathcal{M}} (V(\gamma)-2) z^{V(\gamma)-3} w^{E(\gamma)-4} = \frac{1}{3w^3 T(z,w)} \frac{\partial M}{\partial z}(z,w).$$

Moreover, we have seen in Section 6.2 that the complexity of generating a tree τ using $\Gamma T(z,w)$ is equal to the size of τ. Hence,

$$\Lambda T(z,w) = \sum_{\tau \in \mathcal{T}} \frac{1}{T(z,w)} |\tau| z^{|\tau|_\bullet} w^{|\tau|} = w \frac{\partial T}{\partial w}(z,w)/T(z,w).$$

This yields

$$\Lambda M'(z,w) = 3\lambda w^4 \frac{\partial T}{\partial w}(z,w) / \frac{\partial M}{\partial z}(z,w).$$

The function $T(z,w)$ has a tree-like singularity type [10, ch.6]. As a consequence, its derivative satisfies:

$$\forall w \in I,\ \exists\, c(w) > 0 \text{ such that } \frac{\partial T}{\partial w}(z,w) \sim \frac{c(w)}{\sqrt{1 - z/\rho_M(w)}} \quad \text{as } z \to \rho_M(w).$$

In addition, the constant $c(w)$ varies continuously with w, so that $c(w)$ is bounded on I. Moreover, $\frac{\partial M}{\partial z}(z,w)$ has positive coefficients, so that it is clearly bounded from below for $w \in I$ and $\rho_M(w)/2 < z < \rho_M(w)$. This concludes the proof of the bound on $\Lambda M'(z,w)$. The proof of the bound on $\Lambda\overline{M}(z,w)$ is similar.

Proof of the bound on $\Lambda M''(z,w)$, $\Lambda\overline{M}'(z,w)$ *and* $\Lambda\overline{\overline{M}}(z,w)$. Let us finally sketch the proof of the upper bound on $\Lambda M''(z,w)$. Using a similar approach as for $\Lambda M'(z,w)$, we obtain the following expression of $\Lambda M''(z,w)$,

$$\Lambda M''(z,w) = 6\lambda w^4 \frac{\partial^2 T}{\partial z \partial w}(z,w) / \frac{\partial^2 M}{\partial z^2}(z,w).$$

The singularity analysis of the tree-like series $T(z,w)$ and $\frac{\partial M}{\partial z}(z,w)$ ensures that, for each $w > 0$, there exist a constant $c_0(w) > 0$ and a constant $c_1(w) > 0$ such that

$$\frac{\partial^2 T}{\partial z \partial w}(z,w) \underset{z \to \rho_M(w)}{\sim} \frac{c_0(w)}{(1 - z/\rho_M(w))^{3/2}}, \quad \frac{\partial^2 M}{\partial z^2}(z,w) \underset{z \to \rho_M(w)}{\sim} \frac{c_1(w)}{\sqrt{1 - z/\rho_M(w)}}.$$

In addition, the two constants $c_0(w)$ and $c_1(w)$ vary continuously with w, so that they are upper and lower bounded on I. This concludes the proof of the bound on $\Lambda M''(z,w)$. The bounds on $\Lambda\overline{M}'(z,w)$ and $\Lambda\overline{\overline{M}}(z,w)$ are proved similarly. □

Finally, using the correspondence $\mathcal{M} \simeq 2\overrightarrow{\mathcal{G}_3}$, the complexity results stated in Lemma 10 apply to the Boltzmann samplers of $\overrightarrow{\mathcal{G}_3}$ and of the derived families of $\overrightarrow{\mathcal{G}_3}$ up to order 2.

6.4. **From 3-connected to 2-connected planar graphs.** The first step is the analysis of the expected complexities of the Boltzmann samplers for networks.

Lemma 11. *For each fixed* $y > 0$,

$$\begin{aligned}
\Lambda D(z,y) &= \underset{z \to \rho_D(y)}{\mathcal{O}}(1), \\
\Lambda D'(z,y) &= \underset{z \to \rho_D(y)}{\mathcal{O}}\left(\frac{1}{\sqrt{1 - z/\rho_D(y)}}\right), \\
\Lambda D''(z,y) &= \underset{z \to \rho_D(y)}{\mathcal{O}}\left(\frac{1}{1 - z/\rho_D(y)}\right).
\end{aligned}$$

Proof. Proof of the bound on $\Lambda D(z,y)$. Recall that the Boltzmann sampler $\Gamma D(z,y)$ for networks results from the translation of the decomposition grammar (N) given in Section 5.2.2. Clearly, the labelled vertices created during the generation are either vertices separating the head component and the trail component of an s-network (call to $\Gamma S(z,y)$, with $\mathcal{S} = (\mathcal{L}+\mathcal{P}+\mathcal{H}) \times \mathcal{Z} \times \mathcal{D}$); or they belong to one of the 3-connected components of the network generated (call to $\Gamma M(z, D(z,y))$). Let $\Lambda M_{\text{rej}}(z,w)$ be the expected complexity of $\Gamma M(z,w)$ without counting the last (successful) trial. Given a fixed $\gamma \in \mathcal{M}$, the expected complexity of $\Gamma M(z,w)$ knowing that γ is finally output satisfies $\Lambda M_\gamma(z,w) = \Lambda M_{\text{rej}}(z,w) + \lambda|\gamma|$. Indeed, the first term corresponds to the failing trials and the second term corresponds to the cost of generating γ (the factor λ standing for the complexity of the closure mapping). It is shown in [1] that $z - \rho_M(D(z,y))$ is negative and converges

to 0 when $z \to \rho_D(y)$. In addition $D(z,y)$ converges to a positive value when $z \to \rho_D(y)$, hence $D(z,y)$ stays in a compact interval $I \subset (0,\infty)$ for z close to $\rho_D(y)$. Therefore Lemma 10 applies, giving a uniform upper bound $\Lambda M(z, D(z,y)) \leq c$ for z close to $\rho_D(y)$. Clearly, $\Lambda M_{\text{rej}}(z,w) \leq \Lambda M(z,w)$, so that $\Lambda M_\gamma(z, D(z,y)) \leq c + \lambda|\gamma|$. Thus there exists a constant $\lambda_0 \geq 1$ such that $\Lambda M_\gamma(z, D(z,y)) \leq \lambda_0|\gamma|$ for z close to $\rho_D(y)$.

The cost of a network generation consists of the node additions when calling $\Gamma S(z,y)$, of drawing the Poisson laws when calling $\Gamma P(z,y)$, and of generating the 3-connected components. As mentioned in Section 3.3, the cost of drawing a Poisson law is equal to its output. The sum of the results of the Poisson laws is clearly bounded by the number of components, itself bounded by the number of edges of the network generated. Given a network $\gamma \in \mathcal{D}$ of size n, let k be the number of vertices of γ that are separating in a s-network, and let $\beta_1, \ldots, \beta_l$ be the 3-connected components of γ. Observe that $n = k + |\beta_1| + \ldots + |\beta_l|$. When γ is generated, the cost of addition of separating vertices is k, and the total cost of the Poisson laws is bounded by $E(\gamma)$, hence is at most $3n$ according to Euler's relation. Hence, the expected complexity of $\Gamma D(z,y)$ knowing that γ is generated satisfies

$$\Lambda D_\gamma(z,y) \leq k + 3n + \sum_{i=1}^{l} \Lambda M_{\beta_i}(z, D(z,y)) \leq k + 3n + \lambda_0(|\beta_1| + \ldots + |\beta_l|) \leq (\lambda_0 + 3)n.$$

Let a_n be the nth coefficient of the function $z \to D(z,y)$. Observe that $\frac{1}{D(z,y)} a_n z^n$ is the probability that the output of $\Gamma D(z,y)$ has size n. It follows from the bound $\Lambda D_\gamma(z,y) \leq (\lambda_0 + 3)|\gamma|$ that

$$\Lambda D(z,y) \leq \frac{\lambda_0 + 3}{D(z,y)} \sum_{n \geq 1} n\ a_n z^n, \text{ for } z < \rho_D(y).$$

It is shown in [1] that the function $z \to D(z,y)$ is 3/2-singular. Hence, according to transfer theorems [9], $a_n \sim c_0 \rho_D(y)^{-n} n^{-5/2}$ for some constant $c_0 > 0$. In particular $a_n = \mathcal{O}(\rho_D(y)^{-n} n^{-5/2})$. Therefore, there exists a constant $c > 0$ such that

$$\Lambda D(z,y) \leq \frac{c}{D(z,y)} \sum_{n \geq 1} n^{-3/2} \left(\frac{z}{\rho_D(y)}\right)^n \text{ for } z < \rho_D(y).$$

This clearly gives a bounding constant on $\Lambda D(z,y)$ as $z \to \rho_D(y)$, because $\sum n^{-3/2}$ is finite and $D(z,y)$ converges to a positive value as $z \to \rho_D(y)$.

Proof of the bound on $\Lambda D'(z,y)$. Let $X := (\Lambda D'(z,y), \Lambda S'(z,y), \Lambda P'(z,y), \Lambda H'(z,y))$. Using the computation rules (16), the decomposition grammar (N') of derived networks, as given in Section 5.2.2, is translated to a linear system

$$X = AX + L,$$

where A is a 4×4-matrix and L is a 4-vector. Precisely, the components of A are rational expressions in terms of series of networks and their derivatives: all these quantities converge as $z \to \rho_D(y)$ because the series of networks are 5/2-singular. Hence A converges to a matrix A_0 as $z \to \rho_D(y)$. In addition, observe that A is a substochastic matrix, i.e., a matrix with nonnegative coefficients and with sum at most 1 in each row. Indeed, the entries in each of the 4 rows of A correspond to probabilities of a Bernoulli switch when calling $\Gamma D'(z,y)$, $\Gamma S'(z,y)$, $\Gamma P'(z,y)$, and $\Gamma H'(z,y)$, respectively. Hence, the limit matrix A_0 is also substochastic. It is easily checked that A_0 is indeed strictly substochastic, i.e., at least one row has sum < 1 (in our case, the first and third row add up to 1, whereas the

second and 4th row add up to < 1). In addition, A_0 is irreducible, i.e., the dependency graph induced by positive coefficients of A_0 is strongly connected. A well known result of Markov chain theory ensures that $(I - A_0)$ is invertible [19]. Hence, $(I - A)$ is invertible for z close to $\rho_D(y)$, and $(I - A)^{-1}$ converges to the finite matrix $(I - A_0)^{-1}$. Moreover, the components of L are of the form $L = (a, b, c, d \cdot \Lambda M'(z, D(z, y)) + e \cdot \Lambda\overline{M}(z, D(z, y))$, where $\{a, b, c, d, e\}$ are expressions involving the series of networks, their derivatives, and the quantities $\{\Lambda D, \Lambda S, \Lambda P, \Lambda H\}$, which have already been shown to be bounded as $z \to \rho_D(y)$. As a consequence, a, b, c, d, e are bounded as $z \to \rho_D(y)$. Lemma 10 ensures that there exists a constant c_0 such that, if $w = D(z, y)$, then

$$\Lambda M'(z, w) \leq \frac{c_0}{\sqrt{1 - z/\rho_M(w)}}, \qquad \Lambda\overline{M}(z, w) \leq \frac{c_0}{\sqrt{1 - z/\rho_M(w)}} \quad \text{for } z \text{ close to } \rho_D(y).$$

Let $w_0 = \lim D(z, y)$ and $\alpha = \lim D'(z, y)$ as $z \to \rho_D(y)$. Then

$$z - \rho_M(w) \sim z - \rho_M(w_0) - \rho'_M(w_0)(w - w_0) \sim z - \rho_M(w_0) - \rho'_M(w_0)\alpha(z - \rho_D(y)).$$

The property $\rho_M(w_0) = \rho_D(y)$, proved in [1], ensures that $z - \rho_M(w) \sim (1 - \rho'_M(w_0)\alpha)(z - \rho_D(y))$. Clearly, the function $\rho_M(y)$ is nonincreasing, so that $\rho'_M \leq 0$; and $D'(z, y)$ has positive coefficients, so that $\alpha > 0$. This gives $(1 - \rho'_M(w_0)\alpha) \geq 1$, ensuring that $(z - \rho_M(D(z, y))) \sim c(z - \rho_D(y))$ as $z \to \rho_D(y)$, with c a positive constant. As a consequence the components of L, which are $\mathcal{O}(1 - z/\rho_M(D(z, y)))^{-1/2}$, are also $\mathcal{O}(1 - z/\rho_D(y))^{-1/2}$ as $z \to \rho_D(y)$. Hence, the components of $X = (I - A)^{-1}L$, in particular $\Lambda D'(z, y)$, are $\mathcal{O}(1 - z/\rho_D(y))^{-1/2}$ as $z \to \rho_D(y)$.

Proof of the bound on $\Lambda D''(z, y)$. Again we define $X := (\Lambda D'', \Lambda S'', \Lambda P'', \Lambda H'')$. Using the computation rules (16), the decomposition grammar (N'') of doubly derived networks, as given in Section 5.2.2, is translated to a linear system

$$X = AX + L,$$

where A is a 4×4-matrix and L is a 4-vector. The coefficients of A involve series of networks, derived networks and doubly derived networks. All series of doubly derived networks are $(-1/2)$-singular, so that they diverge, of order $(1 - z/\rho_D(y))^{-1/2}$ as $z \to \rho_D(y)$. However, it can be checked that the coefficients of A all converge to explicit constants as $z \to \rho_D(y)$ because series of doubly derived networks come in pair (numerator, denominator), for instance $A_{12} = \frac{S''}{D''}$. Hence, A converges to a matrix A_0 as $z \to \rho_D(y)$. Again, it is readily seen that A is substochastic for each $z < \rho_D(y)$, so that the limit A_0 is also substochastic. In addition, it is easily checked that A_0 is strictly substochastic (each of the three first rows adds up to 1, whereas the 4th row adds up to < 1), and is irreducible. Hence $(I - A_0)$ is invertible, so that $(I - A)$ is invertible for z close to $\rho_D(y)$ and $(I - A)^{-1}$ converges to the finite matrix $(I - A_0)^{-1}$ as $z \to \rho_D(y)$. Moreover, it can be checked that the coefficient of L are $\mathcal{O}(1 - z/\rho_D(y))^{-1}$ using the following properties: 1) the complexity results of Lemma 10, 2) the fact that $1 - z/\rho_M(D(z, y)) \sim c(1 - z/\rho_D(y))$ for a non-zero constant c (property also used for the proof of the bound on $\Lambda D'$), 3) the series $D(z, y)$, $S(z, y)$, $P(z, y)$, $H(z, y)$ of networks are 5/2-singular given a fixed $y > 0$, and the series $M(z, w)$ is 5/2-singular given a fixed $w > 0$. Hence these series and their derivatives converge when $z \to \rho_D(y))$ (when $z \to \rho_M(w)$), whereas their bi-derived series are of order $(1 - z/\rho_D(y))^{-1/2}$ as $z \to \rho_D(y)$ (of order $(1 - z/\rho_M(w))^{-1/2}$ as $z \to \rho_M(w)$, respectively).

Finally, it follows from $X = (I - A)^{-1}L$ that the coefficients of X, in particular $\Lambda D''(z, y)$, are $\mathcal{O}(1 - z/\rho_D(y))^{-1}$ as $z \to \rho_D(y)$. □

Then, we can easily deduce from the complexities of network generation the complexities of the Boltzmann samplers for 2-connected planar graphs.

Lemma 12. *For each fixed $y > 0$,*

$$\begin{aligned}
\Lambda B'(z,y) &= \underset{z\to\rho_B(y)}{\mathcal{O}}(1),\\
\Lambda B''(z,y) &= \underset{z\to\rho_B(y)}{\mathcal{O}}\left(\frac{1}{\sqrt{1-z/\rho_B(y)}}\right),\\
\Lambda B'''(z,y) &= \underset{z\to\rho_B(y)}{\mathcal{O}}\left(\frac{1}{1-z/\rho_B(y)}\right).
\end{aligned}$$

Proof. First, the samplers for edge-rooted 2-connected planar graphs are directly obtained from the Boltzmann samplers of networks (see Section 5.2.2). By construction, we have

$$\begin{aligned}
\Lambda\overrightarrow{B}(z,y) &= \Lambda(1+D)(z,y) = \underset{z\to\rho_D(y)}{\mathcal{O}}(1),\\
\Lambda\overrightarrow{B}'(z,y) &= \Lambda D'(z,y) = \underset{z\to\rho_D(y)}{\mathcal{O}}\left((1-z/\rho_D(y))^{-1/2}\right),\\
\Lambda\overrightarrow{B}''(z,y) &= \Lambda D''(z,y) = \underset{z\to\rho_D(y)}{\mathcal{O}}\left((1-z/\rho_D(y))^{-1}\right).
\end{aligned}$$

It is shown in [1] that the singularities of $x \to D(x,y)$ and of $x \to B(x,y)$ are equal, so that $\rho_D(y)$ can be replaced by $\rho_B(y)$ in these asymptotic bounds. The Boltzmann samplers for $\mathcal{B}'$, $\mathcal{B}''$ and $\mathcal{B}'''$ combine the samplers for edge-rooted 2-connected graphs with a rejection choice: the graph γ generated is kept with probability $V(\gamma)/2E(\gamma)$ for $\Gamma B'(z,y)$ and $\Gamma B''(z,y)$, and kept with probability $V(\gamma)/E(\gamma)$ for $\Gamma B'''(z,y)$. The crucial point is that the graphs we consider are planar, so that Euler's relation gives the bound $V(\gamma)/E(\gamma) \geq 1/3$. As a consequence, the success probability in $\Gamma B'(z,y)$ and $\Gamma B''(z,y)$ at each trial is at least $1/6$ and the success probability in $\Gamma B'''(z,y)$ at each trial is at least $1/3$. Lemma 9 yields

$$\begin{aligned}
\Lambda B'(z,y) &\leq 6\,\Lambda\overrightarrow{B}(z,y) = \underset{z\to\rho_B(y)}{\mathcal{O}}(1),\\
\Lambda B''(z,y) &\leq 6\,\Lambda(\overrightarrow{B} + z\overrightarrow{B}')(z,y) = \underset{z\to\rho_B(y)}{\mathcal{O}}\left((1-z/\rho_D(y))^{-1/2}\right),\\
\Lambda B'''(z,y) &\leq 3\,\Lambda(z\overrightarrow{B}' + \overrightarrow{B}'')(z,y) = \underset{z\to\rho_B(y)}{\mathcal{O}}\left((1-z/\rho_D(y))^{-1}\right).
\end{aligned}$$

□

6.5. **From 2-connected to connected planar graphs.** The next step of our bottom-to-top approach is to go from the complexities of the Boltzmann samplers for 2-connected planar graphs to the complexities of the Boltzmann samplers for connected planar graphs.

Lemma 13. *For each fixed $y > 0$,*

$$\begin{aligned}
\Lambda C(x,y) &= \underset{x\to\rho_C(y)}{\mathcal{O}}(1),\\
\Lambda C'(x,y) &= \underset{x\to\rho_C(y)}{\mathcal{O}}(1),\\
\Lambda C''(x,y) &= \underset{x\to\rho_C(y)}{\mathcal{O}}\left(\frac{1}{\sqrt{1-x/\rho_C(y)}}\right),\\
\Lambda C'''(x,y) &= \underset{x\to\rho_C(y)}{\mathcal{O}}\left(\frac{1}{1-x/\rho_C(y)}\right).
\end{aligned}$$

Proof. Proof of the bound on $\Lambda C'(x,y)$. The equation $\mathcal{C}' = \textsc{Set}(\mathcal{B}' \circ_x (\mathcal{Z}\times\mathcal{C}'))$ translates to

$$\Lambda C'(x,y) = B'(z,y)(1+\Lambda B'(z,y) + |B'|_{(z,y)}\Lambda C'(x,y)), \quad \text{where } z = xC'(x,y),$$

As a consequence,

$$\Lambda C'(x,y) = \frac{B'(z,y)(1+\Lambda B'(z,y))}{1-B'(z,y)|B'|_{(z,y)}}.$$

It is shown in [1] that $z = xC'(x,y)$ converges (from below) to $\rho_B(y)$ when $x \to \rho_C(y)$. Moreover, the function $z \to B(z,y)$ is 5/2-singular, so that $B(z,y)$, $B'(z,y)$ and $B''(z,y)$ converge when $z \to \rho_B(y)$. It remains to prove that $B'(z,y)|B'|_{(z,y)}$ converges to a value different from 1 in order to ensure that the quantity $B'(z,y)/(1-B'(z,y)|B'|_{(z,y)})$ converges. Observe that the expected size $|C'|_{(x,y)}$ of $\Gamma C'(x,y)$ satisfies

$$|C'|_{(x,y)} = B'(z,y)|B'|_{(z,y)}(|C'|_{(x,y)}+1).$$

(Indeed, the expected size $|C|_{(x,y)}$ satisfies computation rules similar to (16)). Notice that all quantities in this equality converge to positive values as $x \to \rho_C(y)$ because C', C'', B' and B'' converge at singularities. As a consequence, $B'(z,y)|B'|_{(z,y)}$ converges to a constant smaller than 1 when $z = xC'(x,y)$ and $x \to \rho_C(y)$. Hence, $B'(z)/(1-B'(z,y)|B'|_{(z,y)})$ converges to a finite constant. Moreover Lemma 12 ensures that $\Lambda B'(z,y)$ is bounded when $z \to \rho_B(y)$. Thus $\Lambda C'(x,y)$ is bounded when $x \to \rho_C(y)$.

Proof of the bound on $\Lambda C(x,y)$. The sampler $\Gamma C(x,y)$ calls $\Gamma C'(x,y)$ and keeps the generated graph γ with probability $1/V(\gamma)$. Lemma 9 yields

$$\Lambda C(x,y) = \frac{\Lambda C'(x,y)}{p_{\text{acc}}},$$

where $p_{\text{acc}} = \frac{1}{C'(x,y)}\sum_{\gamma\in\mathcal{C}'}\frac{1}{V(\gamma)}\frac{x^{|\gamma|}}{|\gamma|!}y^{||\gamma||}$. Notice that a graph $\gamma \in \mathcal{C}'$ satisfies $V(\gamma) = |\gamma|+1$ and $E(\gamma) = ||\gamma||$. Using the correspondence, $\mathcal{C}_{n,m} \simeq \mathcal{C}'_{n-1,m}$, we obtain $p_{\text{acc}} = C(x,y)/(xC'(x,y))$, giving

$$\Lambda C(x,y) = \frac{xC'(x,y)}{C(x,y)}\Lambda C'(x,y).$$

The function $x \to C(x,y)$ is 5/2-rational, so that $x \to C'(x,y)$ is 3/2-singular. Hence, $C(x,y)$ and $C'(x,y)$ converge when $x \to \rho_C(y)$ (see Remark 3). In addition, we have proved that $\Lambda C'(x,y)$ is bounded when $x \to \rho_C(y)$. Hence, $\Lambda C(x,y)$ is also bounded as $x \to \rho_C(y)$.

Proof of the bound on $\Lambda C''(x,y)$. The second line of the system (13) translates to (we use the generic notation F to abbreviate $F(x,y)$)

$$\begin{aligned}\Lambda C'' &= \frac{C'}{C'+xC''}\Lambda C' + \frac{xC''}{C'+xC''}\Lambda C'' \\ &\quad +\Lambda B''(z,y) + |B''|_{(z,y)}\Lambda C' + \Lambda C', \text{ where } z = xC'(x,y).\end{aligned}$$

As a consequence,

$$\Lambda C'' = \frac{C'+xC''}{C'}\Lambda B''(z,y) + \left(\frac{2C'+xC''}{C'} + |B''|_{(z,y)}\frac{C'+xC''}{C'}\right)\Lambda C', \text{ where } z = xC'(x,y).$$

The function C is 5/2-singular. According to Remark 3, C' is 3/2-singular and C'' is 1/2-singular, so that C' and C'' converge to positive constants as $x \to \rho_C(y)$. Similarly, $z \to B(z,y)$ is 5/2-singular, so that $z \to B''(z,y)$ and $z \to B'''(z,y)$ are respectively 1/2-singular and $(-1/2)$-singular. In particular, the expected size $|B''|_{(z,y)} = zB'''(z,y)/B''(z,y)$ is of order $(1 - z/\rho_B(y))^{-1/2}$ as $z \to \rho_B(y)$. We have proved in Lemma 12 that $\Lambda B''(z,y) = \mathcal{O}(1 - z/\rho_B(y))^{-1/2}$ as $z \to \rho_B(y)$. Thus, the expression of $\Lambda C''$ given above has the following asymptotic bound

$$\Lambda C''(x,y) = \underset{x\to\rho_C(y)}{\mathcal{O}}((1 - z/\rho_B(y))^{-1/2}), \text{ where } z = xC'(x,y).$$

Define $F(x,y) := xC'(x,y)$, $z_0 := \lim F(x,y)$, and $\alpha := \lim \frac{\partial}{\partial x}F(x,y)$ as $x \to \rho_C(y)$. It is shown in [17] that $z_0 = \rho_B(y)$. As a consequence, we have $z - \rho_B(y) = z - z_0 \sim \alpha(x - \rho_C(y))$ when $z = F(x,y)$ and $z \to \rho_C(y)$. In addition, $\alpha > 0$ because the series $\frac{\partial F}{\partial x}$ has positive coefficients. Hence, $1 - z/\rho_B(y)$ can be replaced by $1 - x/\rho_C(y)$ in the asymptotic bound on $\Lambda C''(x,y)$, i.e.,

$$\Lambda C''(x,y) = \underset{x\to\rho_C(y)}{\mathcal{O}}((1 - z/\rho_C(y))^{-1/2}).$$

Proof of the bound on $\Lambda C'''(x,y)$.

The third line of the decomposition grammar (13) translates to an equation of the form

$$\Lambda C''' = a\Lambda B''(z,y) + b\Lambda B'''(z,y) + c\Lambda C' + d\Lambda C'' + e\Lambda C''', \text{ where } z = xC'(x,y),$$

and the coefficients a, b, c, d and e involve $B''(z,y)$, $B'''(z,y)$, $C'(x,y)$, $C''(x,y)$, and $C'''(x,y)$. Again we use the property of α-singularity of the functions to derive the asymptotic order of the coefficients a, b, c and d as $x \to \rho_C(y)$. Indeed the functions $x \to C'(x,y)$ and $x \to B'(z,y)$ are 5/2-singular. Hence $C'(x,y)$ and $C''(x,y)$ converge when $x \to \rho_C(y)$, and $C'''(x,y) \sim c_0(1 - x/\rho_C(y))^{-1/2}$; similarly $B'(z,y)$ and $B''(z,y)$ converge when $z \to \rho_B(y)$, and $B'''(z,y) \sim b_0(1 - z/\rho_B(y))^{-1/2}$. Observe that b and e are probabilities of the Bernoulli switch when calling $\Gamma C'''(x,y)$. In particular $b + e \leq 1$. In addition, b converge to a positive value because it is the quotient of two $(-1/2)$-singular functions (the denominator being $C'''(x,y)$). Hence, e converges to a value < 1, which ensures that

$$\Lambda C''' = \underset{x\to\rho_C(y)}{\mathcal{O}}(a\Lambda B''(z,y) + b\Lambda B'''(z,y) + c\Lambda C' + d\Lambda C'').$$

The asymptotic order of the coefficients $\{a,b,c,d\}$, of $\{\Lambda B''(z,y), \Lambda B'''(z,y)\}$ (obtained in Lemma 12), and of $\{\Lambda C', \Lambda C''\}$ (obtained in the foregoing proof) can be injected in the expression of $\Lambda C'''$ given above. This gives an asymptotic bound of the form

$$\Lambda C'''(x,y) = \underset{x\to\rho_C(y)}{\mathcal{O}}\left(\frac{1}{\sqrt{1 - x/\rho_C(y)}} + \frac{1}{1 - z/\rho_B(y)}\right), \text{ where } z = xC'(x,y).$$

Again we us the property that $(1 - x/\rho_C(y)) \sim c(1 - z/\rho_B(y))$ (with $c \neq 0$) when $z = xC'(x, y)$. As a consequence,

$$\Lambda C'''(x, y) = \underset{x \to \rho_C(y)}{\mathcal{O}} \left(\frac{1}{1 - x/\rho_C(y)} \right).$$

□

6.6. **From connected planar graphs to planar graphs.** The last step is to derive from Lemma 13 the asymptotic expected complexity of the samplers $\Gamma G(x, y)$, $\Gamma G'(x, y)$, $\Gamma G''(x, y)$ and $\Gamma G'''(x, y)$ when $x \to \rho_G(y)$.

Lemma 14. *For each fixed $y > 0$,*

$$\begin{aligned}
\Lambda G(x, y) &= \underset{x \to \rho_G(y)}{\mathcal{O}}(1), \\
\Lambda G'(x, y) &= \underset{x \to \rho_G(y)}{\mathcal{O}}(1), \\
\Lambda G''(x, y) &= \underset{x \to \rho_G(y)}{\mathcal{O}} \left(\frac{1}{\sqrt{1 - x/\rho_C(y)}} \right), \\
\Lambda G'''(x, y) &= \underset{x \to \rho_G(y)}{\mathcal{O}} \left(\frac{1}{1 - x/\rho_C(y)} \right).
\end{aligned}$$

Proof. Using the computation rules (16), the first line of the system (14) translates to

$$\Lambda G(x, y) = C(x, y)(1 + \Lambda C(x, y)).$$

We have seen that $C(x, y)$ converges and $\Lambda C(x, y)$ is bounded when $x \to \rho_C(y)$. In addition, it is proved in [17] that $\rho_C(y) = \rho_G(y)$. Hence $\Lambda G(x, y)$ is $\mathcal{O}(1)$ as $x \to \rho_G(y)$.

The second line of (14) translates to $\Lambda G'(x, y) = \Lambda C'(x, y) + \Lambda G(x, y)$. These two terms are bounded as $x \to \rho_G(y)$, so that $\Lambda G'(x, y)$ is also bounded.

The third line of (14) translates to $\Lambda G'' = \frac{C''G}{G''}(\Lambda C'' + \Lambda G) + \frac{C'G'}{G''}(\Lambda C' + \Lambda G')$. Observe that the quantities $\frac{C''G}{G''}$ and $\frac{C'G'}{G''}$ are bounded by 1 (these are the probabilities of the Bernoulli switch). Hence, the asymptotic bounds on $\Lambda C'$, $\Lambda C''$ (obtained in Lemma 13) and on ΛG and $\Lambda G'$ (obtained in the foregoing proof) ensure that $\Lambda G'' = \mathcal{O}((1 - z/\rho_G(y))^{-1/2})$.

Similarly, translating the 4th line of (14), it is easily checked that $\Lambda G'''(x, y) = \mathcal{O}((1 - x/\rho_G(y))^{-1})$. □

The asymptotic bound on $\Lambda G'''(x, y)$ exactly corresponds to Assertion (18). According to Claim 1, this concludes the proof of the time complexities of the planar graph generators stated in Theorems 1 and 2.

7. Conclusion

Using a well known decomposition of planar graphs by increasing degree of connectivity, we have developed in this article very efficient samplers for planar graphs. The translation of the decomposition into a planar graph generator relies on the recently introduced framework of Boltzmann sampling. Our sampler is built progressively. The first step is the realisation of a Boltzmann sampler for 3-connected planar graphs, which is derived from an explicit bijection with the well understood family of binary trees. Then, from the sampler of 3-connected planar graphs, we have obtained successively Boltzmann samplers

for 2-connected planar graphs, connected planar graphs, and finally unconstrained planar graphs, by taking advantage of explicit decompositions relating these families.

Notice that the samplers developed on the way to planar graphs are interesting on their own. For each of the families {3-connected, 2-connected, connected, unconstrained} planar graphs, there results from our study the existence of approximate-size uniform samplers with expected linear time complexity and of fixed-size uniform samplers with expected quadratic time complexity.

Regarding the practical aspects of the implementation, the evaluation of the generating functions of planar graphs has been carried out with the mathematical software Maple, based on the analytic expressions given by Giménez and Noy [17]. Then, the random generator has been implemented in Java, with a precision of 64 bits for the values of generating functions ("double" type). Using the approximate-size sampler, planar graphs with size of order 100,000 are generated in a few seconds with a machine clocked at 1GHz. In contrast, the recursive method of Bodirsky *et al* is currently limited to sizes of about 100.

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