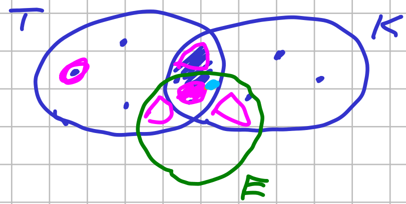
Spolky: tenisový $|T|=15$ $|T\cap K|=3$ $|T\cap K\cap E|=1$

kriketový $|K|=5$ $|K\cap E|=5$

egyptologický $|E|=11$ $|T\cap E|=2$



$|T\cup K| = \underset{15}{|T|} + \underset{5}{|K|} - \underset{3}{|T\cap K|} = 17$

$|T\cup K\cup E| = \underbrace{\underset{15}{|T|} + \underset{5}{|K|} + \underset{11}{|E|}}_{31} - \underbrace{\underset{3}{|T\cap K|} - \underset{2}{|T\cap E|}}_{10} - \underset{5}{|K\cap E|} + \underset{1}{|T\cap K\cap E|} = 22$

$\mathcal{S}$

$\bigcup_{A\in\mathcal{S}} A$

Věta (Princip inkluze a exkluze) Pro konečné množiny $A_1 - A_n$:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I\in\binom{\{1-n\}}{k}} \left|\bigcap_{i\in I} A_i\right|$$

Alternativně: $$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset\neq I\subseteq\{1-n\}} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right|$$

Dk: Pro každý prvek $x\in\bigcup_i A_i$ spočítáme příspěvky k levé a pravé straně
vždy 1 $\qquad p$

Nechť x patří do právě j množin z $A_1 - A_n$.

Průniky k-tic: ① $k>j$ … přispěje 0

② $k\leq j$ …. $(-1)^{k+1}\binom{j}{k}$

$p = \binom{j}{1} - \binom{j}{2} + \binom{j}{3} - \binom{j}{4} + \ldots - \ldots + (-1)^{j+1}\binom{j}{j}$

$p=1$ Q.E.D.

$0 = (1-1)^j = \binom{j}{0} - \binom{j}{1} + \binom{j}{2} - \ldots$

$0 = \binom{j}{0} - p$

$0 = 1 - p$

Druhý důkaz: $$\prod_{i=1}^{n}(1+x_i) = \sum_{I\subseteq\{1-n\}} \prod_{i\in I} x_i$$

$x_i := -c_{A_i}$

$$\underbrace{\prod_i (1-c_{A_i})}_{1-c_{\bigcup_i A_i}} = \left(\sum_{\emptyset\neq I\subseteq\{1-n\}} (-1)^{|I|} \underbrace{\prod_{i\in I} c_{A_i}}_{c_{\bigcap_{i\in I} A_i}}\right) + 1$$

$$\underbrace{\sum_a \cancel{1} + c_{\bigcup_i A_i}(a)}_{\left|\bigcup_i A_i\right|} = \underbrace{\sum_a \left(\sum_{\emptyset\neq I\subseteq\{1-n\}} (-1)^{|I|+1} \cdot c_{\bigcap_{i\in I} A_i}(a)\right) \cancel{+1}}_{\sum_I (-1)^{|I|+1} \underbrace{\sum_a c_{\bigcap_i A_i}(a)}_{\left|\bigcap_{i\in I} A_i\right|}}$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset\neq I\subseteq\{1-n\}} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right|$$

Nechť $A = \bigcup_i A_i$

Pro $B\subseteq A$: $c_B : A\to\{0,1\}$

Pro $a\in A$: $c_B(a) = \begin{cases} 0 & \\ 1 & \text{pokud } a\in B\end{cases}$

$c_{X\cap Y} = c_X\cdot c_Y$

$c_{\overline{X}} = 1 - c_X$ ($\overline{X} = A\setminus X$)

$\overline{X\cup Y} = \overline{X}\cap\overline{Y}$

$1 - c_{X\cup Y} = (1-c_X)(1-c_Y)$

• $\sum_{a\in A} c_X(a) = |X|$

Šatnářka

$S_n := \{\pi \mid \pi \text{ permutace na } \{1..n\}\}$

i dostal svůj $\iff \underbrace{\pi(i) = i}_{\text{i je pevný bod}}$

$\check{S}_n := |\{\pi \in S_n \mid \nexists i : \pi(i) = i\}|$

permutací bez pevného bodu na {1..n}

$\Pr[\text{náhodná } \pi \in S_n \text{ nemá pevný bod}] = \frac{\check{S}_n}{n!}$

$A := \{\pi \in S_n \mid \pi \underbrace{\text{ má pevný bod}}_{\exists i : \pi(i) = i}\}$

$A_i := \{\pi \in S_n \mid \pi(i) = i\}$, $i = 1..n$

$A = \bigcup_i A_i$

$|A_i| = (n-1)!$

$|A_i \cap A_j| = (n-2)!$ pro $i \neq j$

$\vdots$

P/E

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$\underbrace{\left|\bigcup_{i=1}^{n} A_i\right|}_{|A|} = \underbrace{\sum_{k=1}^{n} (-1)^{k+1}}\ \underbrace{\sum_{I \in \binom{\{1..n\}}{k}} \underbrace{\left|\bigcap_{i \in I} A_i\right|}_{(n-k)!}}_{\underbrace{\binom{n}{k} \cdot (n-k)!}_{\frac{n!}{k!}}}$$

$$|A| = \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!}$$

$$|A| = n! \cdot \sum_{k=1}^{n} (-1)^{k+1}/k!$$

$$|A| = n!\left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^{n+1}}{n!}\right)$$

$$\check{S}_n = n! - |A|$$

$$\check{S}_n = n! \cdot \left(\underbrace{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}}_{\Pr[\dots]}\right)$$

$\Pr[\dots] \to 1/e$

$$\check{S}_n = n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Odhady

(0) $2^{n-1} \le n! \le n^n$

(1) $n^{n/2} \le n! \le \left(\frac{n+1}{2}\right)^n$

log: $\frac{n}{2} \log n \le \log n! \le n \cdot \log \frac{n+1}{2}$

$\log n! \in \Theta(n \log n)$

$n! \in 2^{\Theta(n \log n)}$

(2*) $e\left(\frac{n}{e}\right)^n \le n! \le en \cdot \left(\frac{n}{e}\right)^n$

(2**) $n! \sim \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$

Stirlingova formule

$f \sim g \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$

(3) $\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le n^k$

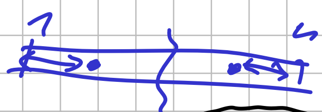
(4*) $\binom{n}{k} \le \left(\frac{en}{k}\right)^k$

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

$(n!)^2 = 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot \ldots \cdot n \cdot n$

$(1 \cdot n)(2 \cdot (n-1))(3(n-2)) \ldots (n \cdot 1)$

$n! = \underbrace{\sqrt{1 \cdot n}}_{\ge \sqrt{n}}\ \underbrace{\sqrt{2(n-1)}}_{\ge \sqrt{n}}\ \sqrt{3(n-2)} \ldots \sqrt{n \cdot 1}$



$i \cdot (n-i+1) \ge n$

$i = 1, i = n$: $= n$

jinak: min ≥ 2, max $\ge n/2$

$n! \ge (\sqrt{n})^n = (n^{1/2})^n = n^{n/2}$

AG nerovnost: pro $x, y > 0$: $\sqrt{x \cdot y} \le \frac{x+y}{2}$

$\sqrt{\underset{x}{i}\,\underset{y}{(n-i+1)}} \le \frac{i + n - i + 1}{2} = \frac{n+1}{2}$

$n! \le \left(\frac{n+1}{2}\right)^n$

Dk: $0 \le (a-b)^2 = a^2 - 2ab + b^2$

$2ab \le a^2 + b^2$

$ab \le \frac{a^2 + b^2}{2}$

$a \leftarrow \sqrt{x}$

$b \leftarrow \sqrt{y}$

$\sqrt{x} \cdot \sqrt{y} \le \frac{x+y}{2}$

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot 1}$$

$\frac{n-i}{k-i} \ge \frac{n}{k}$

$\cancel{nk} - ik \ge \cancel{nk} - in$

$ik \le in$

$k \le n$

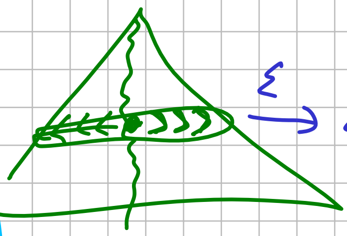
(5) $\frac{2^{2n}}{2n+1} \le \binom{2n}{n} \le 2^{2n} = 4^n$

min ≤ průměr ≤ max

max ≤ Σ

(6*) $\frac{4^n}{2\sqrt{n}} \le \binom{2n}{n} \le \frac{4^n}{\sqrt{2n}}$

$\sqrt{2}$-krát



$\xrightarrow{\Sigma} 2^{2n} = 4^n$

$\binom{2n}{n} \in \Theta\left(\frac{4^n}{\sqrt{n}}\right)$