A short proof of the existence of the Jordan normal form of a matrix

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Theorem 1 Let $V$ be an $n$-dimensional vector space and $\Phi : V \to V$ be a linear mapping of $V$ into itself. Then there is a basis of $V$ such that the matrix representing $\Phi$ with respect to the basis is

$$
\begin{pmatrix}
J_1 & & \\
& J_2 & \\
& & \ddots & \\
& & & J_k
\end{pmatrix}
$$

where empty space is filled by 0’s and $J_1, \ldots, J_k$ are square matrices, called Jordan blocks, of the form

$$
J_i = 
\begin{pmatrix}
\lambda_i & 1 \\
& \lambda_i & 1 \\
& & \ddots & \ddots \\
& & & \lambda_i & 1 \\
& & & & \lambda_i
\end{pmatrix}
$$

for $i = 1, \ldots, k$, where $\lambda_1, \ldots, \lambda_k$ are complex numbers and empty space is filled by 0’s.

Conclusion 1 (Jordan’s normal form of a matrix) Let $A$ be a square matrix; there is a regular matrix $P$ such that the matrix $P^{-1}AP$ has the form described in the preceding theorem.

The matrix form shown in the theorem is called Jordan canonical form or Jordan normal form.

Remark: The numbers $\lambda_1, \ldots, \lambda_k$ of the theorem need not be distinct. E.g., the unit matrix is a matrix in Jordan canonical form, where Jordan blocks are matrices of size $1 \times 1$ equal to (3), i.e. with $\lambda_1 = \cdots = \lambda_k = 1$.

We need one definition
Definition 1 We say that a vector space $V$ is a direct sum of its subspaces $V_1, \ldots, V_m$, if for each vector $v \in V$ there is the unique sequence of vectors $v_1, \ldots, v_m$ such that $v_i \in V_i$ for $i = 1, \ldots, m$ and $v = v_1 + \cdots + v_m$. In such a case we write $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$.

Uniqueness in the definition means that it must be $V_i \cap V_j = \{0\}$ for any two different $i$ and $j$ in the range $1 \leq i, j \leq m$, because if a non-zero vector $v$ was a member of both $V_i$ and $V_j$ then the uniqueness of the sequence $v_1, \ldots, v_m$ is corrupted: it would be possible to choose $v_i = v$ and the other vector equal to $0$, of $v_j = v$ and others vectors equal to the null vector.

Thus, $\dim(V) = \dim(V_1) + \cdots + \dim(V_m)$.

The proof of the theorem is based of the following two lemmae:

Lemma 1 Let $V$ be an $n$-dimensional vector space and $\Phi : V \to V$ be a linear mapping of $V$ into itself. Let $\lambda_1, \ldots, \lambda_r$ be different eigenvalues of $\Phi$. Then there are integer $s_1, \ldots, s_r$ such that $V = \text{Ker}(\Phi - \lambda_1 I)^{s_1} \oplus \cdots \oplus \text{Ker}(\Phi - \lambda_r I)^{s_r}$.

Proof Choose first one of the eigenvalues of $\Phi$ and denote it by $\lambda$.

Part 1 Define $W_i = \text{Ker}(\Phi - \lambda I)^i$ for each natural number $i$. It is clear that $W_1 \subset W_2 \subset W_3 \subset \cdots W_i \subset \cdots$

Since we suppose that $V$ has finite dimension, the sequence could not be strictly increasing forever, but there must be a number $t$ such that $W_t = W_{t+1}$. Assume that $t$ is the smallest among such numbers. It is almost obvious that this would imply $W_{t+1} = W_{t+2} = W_{t+3} = \cdots$.

Part 2 We will prove that $\text{Ker}(\Phi - \lambda I)^t \cap \text{Im}(\Phi - \lambda I)^t = \{0\}$.

Assume that a non-zero vector $v$ belongs to $\text{Ker}(\Phi - \lambda I)^t \cap \text{Im}(\Phi - \lambda I)^t$.

This implies that there exists $w \in V$ such that $v = (\Phi - \lambda I)^t(w)$ (because $v \in \text{Im}(\Phi - \lambda I)^t$) and also $(\Phi - \lambda I)^t(v) = 0$ (because $v \in \text{Ker}(\Phi - \lambda I)^t$).

Thus, $(\Phi - \lambda I)^{2t}(w) = (\Phi - \lambda I)^t(v) = 0$, and hence $w \in W_{2t}$. But since $W_t = W_{2t}$, it is also $w \in W_t = \text{Ker}(\Phi - \lambda I)^t$, and hence $v = (\Phi - \lambda I)^t(w) = 0$.

Part 3 We already know that $\dim(V) = \dim(\text{Ker}(\Phi - \lambda I)^t) + \dim(\text{Im}(\Phi - \lambda I)^t)$. Moreover, we know that if $V_1$ and $V_2$ are subspaces of $V$, then the subspace that spans both $V_1$ and $V_2$ has the dimension $\dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$. Applying this to $V_1 = \text{Ker}(\Phi - \lambda I)^t$ and $V_2 = \text{Ker}(\Phi - \lambda I)^t$ (i.e., $\dim(V_1 \cap V_2) = 0$), we
obtain that the dimension of the subspace of $V$ that spans both $\ker(\Phi - \lambda I)^t$ and $\text{Im}(\Phi - \lambda I)^t$ is equal to $\dim(V)$, and hence

$$V = \ker(\Phi - \lambda I)^t \oplus \text{Im}(\Phi - \lambda I)^t.$$  

**Part 4**

Both $\ker(\Phi - \lambda I)^t$ and $\text{Im}(\Phi - \lambda I)^t$ are invariant subspaces of $\Phi$ (a subspace $U$ of $V$ is an invariant subspace of $\Phi$, if $v \in U$ implies $\Phi(v) \in U$).

Note that

$$\Phi(\Phi - \lambda I) = \Phi\Phi - \lambda(\Phi I) = (\Phi - \lambda I)\Phi.$$  

This implies that

if $v \in \ker(\Phi - \lambda I)^t$, then $(\Phi - \lambda I)^t(v) = 0$, and

$$0 = \Phi(0) = \Phi(\Phi - \lambda I)^t(v) = (\Phi - \lambda I)^t\Phi(v),$$  

and hence $\Phi(v) \in \ker(\Phi - \lambda I)^t$, and

if $v \in \text{Im}(\Phi - \lambda I)^t$, then $v = (\Phi - \lambda I)^t(w)$ for some $w \in V$, and

$$\Phi(v) = \Phi(\Phi - \lambda I)^t(w) = (\Phi - \lambda I)^t\Phi(w),$$  

i.e., $\Phi(v) \in \text{Im}(\Phi - \lambda I)^t$.

**Part 5**

Now, the lemma can be proved by the induction on the number of different eigenvalues of $\Phi$: if $\lambda_1, \ldots, \lambda_r$ are different eigenvalues of $\Phi$ and we put $\lambda$ of Parts 1-4 to be $\lambda_1$, then the eigenvalues of the restriction of $\Phi$ to $\text{Im}(\Phi - \lambda I)^t$ are $\lambda_2, \ldots, \lambda_r$, and, by the induction hypothesis,

$$\text{Im}(\Phi - \lambda I)^t = \ker(\Phi - \lambda_2 I)^{s_2} \oplus \cdots \oplus \ker(\Phi - \lambda_r I)^{s_r},$$  

for some $s_2, \ldots, s_r$. ♣

The second lemma that we will use in order to prove the Jordan form theorems is

**Lemma 2** (Mark Wildon[1]) *Let $V$ be an $n$-dimensional vector space and $T : V \to V$ be a linear mapping of $V$ into itself such that $T^s = 0$ for some natural number $s$. Then there are vectors $u_1, \ldots, u_k$ and natural numbers $a_1, \ldots, a_k$ such that

$$T^{a_i}(u_i) = 0 \quad \text{for } i = 1, \ldots, k,$$

and the vectors

$$u_1, T(u_1), \ldots, T^{a_1-1}(u_1), \ldots, u_k, T(u_k), \ldots, T^{a_k-1}(u_k)$$

are non-zero vectors that form a basis of $V$.***
**Proof** If $T$ itself maps all vectors to 0, then it is sufficient to put $u_1, \ldots, u_k$ to be a basis of $V$ and $a_1 = \cdots = a_k = 1$.

Now, the proof is by induction on the dimension of $V$. Suppose first that the dimension of $V$ is 1: in this case $T^s$ could be a constant mapping to 0 only if $T$ is, and we use the previous statement.

Let us suppose that the lemma holds for all cases when the dimension is smaller than $n$, and we will prove the lemma for $n$. Consider the vector space $\text{Im}(T)$. If $\dim(\text{Im}(T)) = 0$, then $T$ is a zero mapping and the lemma follows. The assumption $\dim(\text{Im}(T)) = n$ would imply that $T$ is a one-to-one mapping, which would contradict to the assumption that $T^s = 0$ for some $s$. Thus, we can assume that $0 < \dim(\text{Im}(T)) < n$ and, by the induction hypothesis, there are vectors $v_1, \ldots, v_\ell$ and natural numbers $b_1, \ldots, b_\ell$ such that

$$T^{b_i}(v_i) = 0 \quad \text{for } i = 1, \ldots, \ell,$$

and

$$v_1, T(v_1), \ldots, T^{b_i-1}(v_1), \ldots, v_\ell, T(v_\ell), \ldots, T^{b_\ell-1}(v_\ell) \quad (1)$$

form a basis of $\text{Im}(T)$.

For each $i = 1, \ldots, \ell$, $v_i \in \text{Im}(T)$, and hence we can choose $w_i \in V$ such that $T(w_i) = v_i$. Vectors $T^{b_i-1}(v_1), \ldots, T^{b_\ell-1}(v_\ell)$ are linearly independent vectors in $\text{Ker}(T)$. Steinitz theorem says that we can extend these vectors to a basis

$$T^{b_1-1}(v_1), \ldots, T^{b_\ell-1}(v_\ell), z_1, \ldots, z_m \quad (2)$$

of $\text{Ker}(T)$.

Note that in our notation, $T^i(w_i) = T^{j-1}(v_i)$ for all relevant $i$ and $j$.

Now it is sufficient to prove that the vectors

$$w_1, T(w_1), \ldots, T^{b_1}(w_1), \ldots, w_\ell, T(w_\ell), \ldots, T^{b_\ell}(w_\ell), z_1, \ldots, z_m \quad (3)$$

form a basis of $V$.

We will first prove their linear independence. Assume that

$$\alpha_1 w_1 + \alpha_1 T(w_1) + \cdots + \alpha_{b_1} T^{b_1}(w_1) + \cdots + \alpha_\ell,0 w_\ell + \cdots + \alpha_\ell,0 T^{b_\ell}(w_\ell) +$$

$$+ \beta_1 z_1 + \cdots + \beta_m z_m = 0.$$

Apply the linear mapping $T$ to the equation to get

$$\alpha_1 T(w_1) + \alpha_1 T^2(w_1) + \cdots + \alpha_{b_1-1} T^{b_1}(w_1) + \cdots + \alpha_\ell,0 T(w_\ell) + \cdots + \alpha_\ell,0 T^{b_\ell}(w_\ell) = 0$$

i.e.,

$$\alpha_1 v_1 + \alpha_1 T(v_1) + \cdots + \alpha_{b_1-1} T^{b_1-1}(v_1) + \cdots + \alpha_\ell,0 v_\ell + \cdots + \alpha_\ell,0 T^{b_\ell-1}(v_\ell) = 0$$

and since the left side of the last equation is a linear combination of elements of a basis (1) of $\text{Im}(T)$, the corresponding $\alpha$’s must be 0.
Putting $\alpha_{1,0} = \alpha_{1,1} = \cdots = \alpha_{1,b_1-1} = \cdots = \alpha_{\ell,0} = \cdots = \alpha_{\ell,b_\ell-1} = 0$ into the original equation, we get

$$\alpha_{1,b_1} T^{b_1}(w_1) + \cdots + \alpha_{\ell,b_\ell} T^{b_\ell}(w_\ell) + \beta_1 z_1 + \cdots + \beta_m z_m = 0,$$

but the left side of this equation is a linear combination of elements of a basis (2) of Ker$(T)$, and hence even $\alpha$’s in the last equation are equal to 0, which proves the linear independence of the original system of vectors listed in (3).

In order to prove that the system (3) forms a basis of $V$ we just need to prove that the number of vectors in (3) is equal to the dimension of $V$. The system (1) is a basis of Im$(T)$, which means that dim(Im$(T)$) = $b_1 + \cdots + b_\ell$. Moreover, the system (2) is a basis of ker$(T)$, i.e., dim(Ker$(T)$) = $\ell + m$. Using the theorem on the dimension of the image and the kernel of a linear mapping, we get that

$$\dim(V) = \dim(\text{Im}(T)) + \dim(\text{Ker}(T)) = b_1 + \cdots + b_\ell + \ell + m = (1 + b_1) + \cdots + (1 + b_\ell) + m,$$

which is exactly the number of vectors of the system (3).

An example for the Wildon’s lemma: Let $V$ be a vector space of the dimension 3 and $T(x_1, x_2, x_3) = (x_2 + x_3, 0, 0)$. Then Im$(T)$ is one-dimensional vector space generated by the vector $(1, 0, 0)$. We can easily choose \( b_1 = 1, \; v_1 = (1, 0, 0), \) and $a_1 = 1$.

Now, there are two important vectors that $T$ maps to $v_1$, namely $(0, 1, 0)$ and $(0, 0, 1)$. Moreover, any vector $(x_1, x_2, 1 - x_2)$ maps into $v_1$ as well. We choose one of them as $w_1$, e.g., $(0, 0, 1)$. Now, what about the vector $(0, 1, 0)$ and other vectors that map into $v_1$? If $T(w) = v_1$ for some vector $w$ other than $w_1$ (e.g., if $w = (0, 1, 0)$), then $T(w - w_1) = v_1 - v_1 = 0$, and hence $w - w_1$ is a member of Ker$(T)$ that was not included in Im$(T)$, and we can choose that vector as $z_1$, an additional member of a basis of Ker$(T)$. Thus, we obtain the basis $w_1 = (0, 0, 1), \; v_1 = (1, 0, 0), \) and $z_1 = (0, 1, -1)$, and we know that $T(w_1) = v_1, T(v_1) = 0$, and we also have $T(z_1) = 0$.

Proof of the Theorem:
Using the first lemma, there are integer $s_1, \ldots, s_r$ such that

$$V = \text{Ker}(\Phi - \lambda_1 I)^{s_1} \oplus \cdots \oplus \text{Ker}(\Phi - \lambda_r I)^{s_r},$$

where $\lambda_1, \ldots, \lambda_r$ are different eigenvalues of $\Phi$.

Assume a basis of $V$ obtained so that we concatenate bases of Ker$(\Phi - \lambda_1 I)^{s_1}, \ldots, \text{Ker}(\Phi - \lambda_r I)^{s_r}$. With respect to such a basis, the matrix represen-
tation of $\Phi$ is a matrix of the form

$$
\begin{pmatrix}
J_1 & J_2 & \cdots & J_k \\
\vdots & \ddots & \ddots & \vdots \\
& \cdots & \ddots & J_k \\
\end{pmatrix},
$$

where $J_1, \ldots, J_k$ are general square matrices; $J_i$ is the matrix of the restriction of $\Phi$ to $\text{Ker}(\Phi - \lambda_i I)^{s_i}$ with respect to the chosen basis.

However, if the basis of $\text{Ker}(\Phi - \lambda_i I)^{s_i}$ was constructed using Wildon’s lemma, then each $J_i$ turns to be

$$
\begin{pmatrix}
J_{i,1} & J_{i,2} & \cdots & J_{i,s_i} \\
& \ddots & \ddots & \vdots \\
& & \ddots & J_{i,s_i} \\
\end{pmatrix},
$$

where each $J_{i,j}$ is a Jordan block with $\lambda_i$ on the diagonal; each Jordan block corresponds to one chain of vectors $v_j, T(v_j), \ldots, T^{a_j-1}$, where $T = (\Phi - \lambda_i I)^{s_i}$.

References