Linear Algebra (not only) for computer scientists

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Chapter 9 Determinants

Determinants were developped for solution of a sqqare system of linear equations and give an explicit formula for their solution (see Theorem 9.13). Gtoofried Wilhelm Liebnitz is supposed to be the author od determinants and they were also discovered independently by Japonese mathematician Seki Kowa in the same year 1683. Their approach (a bit different from our definition) had been forgotten and determinants became popular for solving equation systems in the years 1750-1900, later replaced by other methods. However, it turned out that a determinant is an important characteristic of a square matrix with many applications. The term "determinant" comes from Gauss (*Disquisitiones arithmeticae*, 1801), even thou Gauss had been using it with a slightly different meaning. Important contributions to the determinant theory also came from A. L. Cauchy and C. G. J. Jacobi.

Let us note that S_n denotes the set of all permutations of the set $\{1, \dots, n\}$, see Section 4.2).

Definition 9.1 (Determinant). Let $A \in \mathbb{C}^{n \times n}$. Then the *determinant* of the matrix A is the number

$$\det(A) = \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a_{i,p(i)} = \sum_{p \in S_n} \operatorname{sgn}(p) a_{1,p(1)} \cdots a_{n,p(n)}.$$

Denotation: $\det(A)$ or $|A|$.

What says the formula from the definition of a determinant? every term has the form $\operatorname{sgn}(p) a_{1,p(1)} \cdots a_{n,p(n)}$, which corresponds to selecting n elements of the matrix A so that we have selected exactly one element from each row and each column. Such elements are multiplied together and the term is associated with the plus or the minus sign depending on the sign of the permutation determining the elements.

Example 9.2 (Examples of determinants). A matrix of the order 2 has the determinant

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

A matrix of the order 4 has the determinant

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}.$$

Computing larger determinants according to the definition would be highly inefficient, because it would be necessary to consider $n!$ terms. The computation is simpler for special matrices only. For example, the determinant of a triangular matrix is the product of the diagonal elements, because in the definition of a determinent the only potentially non-zero term of the sum is for the premutation $p = id$ that corresponds to the choice of the diagonal elements. Especially, $\det(I_n) = 1$.

Theorem 9.3 (Determinant of transposition). Let $A \in \mathbb{C}^{n\times n}$. Then $\det(A^T) = det(A)$.

Proof.

$$\det(A^T) = \sum_{p\in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} A^T_{i,p(i)} = \sum_{p\in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a_{p(i),i} = \sum_{p\in S_n} \operatorname{sgn}(p^{-1}) \prod_{i=1}^{n} a_{i,p^{-1}(i)} =$$

$$= \sum_{p^{-1}\in S_n} \operatorname{sgn}(p^{-1}) \prod_{i=1}^{n} a_{i,p^{-1}(i)} = \sum_{q\in S_n} \operatorname{sgn}(q) \prod_{i=1}^{n} a_{i,q(i)} = \det(A). \qquad \clubsuit$$

In general, it is $\det(A+B) \neq \det(A) + \det(B)$ and there is no known simple formula for the determinant of the sum of matrices. The exception is the next special case of the row linearity.

Theorem 9.4 (Row linearity of determinant). Let $A \in \mathbb{C}^{n\times n}$, and $b \in \mathbb{C}^n$. Then for arbitrary $i = 1, \ldots, n$

$$\det(A + e_i b^T) = \det(A) + \det(A + e_i(b^T - A_{i*})).$$

In other words,

$$\det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1}+b_1 & \cdots & a_{in}+b_n \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ b_1 & \cdots & b_n \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Proof.

$$\det(A + e_i b^T) = \sum_{p\in S_n} \operatorname{sgn}(p) a_{1,p(1)} \cdots (a_{i,p(i)} + b_{p(i)}) \cdots a_{n,p(n)} =$$

$$= \sum_{p \in S_n} \operatorname{sgn}(p) a_{1,p(1)} \cdots a_{i,p(i)} \cdots a_{n,p(n)} + \sum_{p \in S_n} \operatorname{sgn}(p) a_{1,p(1)} \cdots b_{p(i)} \cdots a_{n,p(n)} =$$

$$= \det(A) + \det(A + e_i(b^T - A_{i*})) \qquad \clubsuit$$

In view of Theorem 9.3, the determinant is linear not only by rows, but by columns as well.

9.1. Determinant and elementary operations

Our plan is to use Gaussian elimination for computing of a determinant. We must be first able to compute the determinant of a matrix in the row echelon form, and also to know, how the value of a determinant is influenced by elementary row operations. The answer to the first question is easy, because a matrix in the row chelon form is upper triangular, and hence its determinant is given by the product of the diagonal elements. The second question will be answered by the analysis of types of elementary row operations. Let a matrix A' is obtained by applying some elementary operation to A¿

1. Multiplying i-th row by a number $\alpha \in \mathbb{C}$: $\det(A') = \alpha \det(A)$.

   Proof.
   $$det(A') = \sum_{p \in S_n} \operatorname{sgn}(p) a'_{1,p(1)} \cdots a'_{i,p(i)} \cdots a'_{n,p(n)} =$$
   $$= \sum_{p \in S_n} \operatorname{sgn}(p) a_{1,p(1)} \cdots (\alpha a_{i,p(i)}) \cdots a_{n,p(n)} =$$
   $$= \alpha \sum_{p \in S_n} \operatorname{sgn}(p) a_{1,p(1)} \cdots a_{i,p(i)} \cdots a_{n,p(n)}. \qquad \clubsuit$$

2. Exchanging the i-th row and the j-th row: $\det(A') = -\det(A)$.

   Proof. Denote $t = (i, j)$ (the transposition that exchanges i and j). Then
   $$det(A') = \sum_{p \in S_n} \operatorname{sgn}(p) a'_{1,p(1)} \cdots a'_{i,p(i)} \cdots a'_{j,p(j)} \cdots a'_{n,p(n)},$$
   where $a'_{1,p(1)} = a_{1,p(1)} = a_{1,p\circ t(1)}$, $a'_{i,p(i)} = a_{j,p(i)} = a_{j,p\circ t(j)}$, etc. Therefore
   $$\det(A') = \sum_{p \in S_n} \operatorname{sgn}(p) a_{1,p\circ t(1)} \cdots a_{j,p\circ t(j)} \cdots a_{i,p\circ t(i)} \cdots a_{n,p\circ t(n)} =$$
   $$= \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a_{i,p\circ t(i)} = - \sum_{p\circ t \in S_n} \operatorname{sgn}(p \circ t) \prod_{i=1}^{n} a_{i,p\circ t(i)} =$$
   $$= - \sum_{q \in S_n} \operatorname{sgn}(q) \prod_{i=1}^{n} a_{i,q(i)} = -\det(A). \qquad \clubsuit$$

Consequence 9.5 If a matrix $A \in \mathbb{C}^{n \times n}$ has two identical rows, then $\det(A) = 0$.

Proof. By exchanging the two rows we get $\det(A) = -\det(A)$, which implies $\det(A) = 0$. ♣.

Let us note that the consequence is valid for any field $\mathbb{C}$, but for fields of the characteristic 2 we need an alternative proof, because, e.g., in $\mathbb{Z}_2$ it is $-1 = 1$.

Proof no. 2. Let us define $t := (i, j)$, where i, j are indices of the identical rows. Let S'_n be the set of all even permutations from S_n. Then S_n is the disjoint union of S'_n and $\{p \circ t \ : \ p \in S'_n\}$. Therefore

$$\det(A) = \sum_{p \in S'_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a_{i,p(i)} + \sum_{p \in S'_n} \operatorname{sgn}(p \circ t) \prod_{i=1}^{n} a_{i,p \circ t(i)} =$$

$$= \sum_{p \in S'_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a_{i,p(i)} - \sum_{p \in S'_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a_{i,p(i)} = 0. \qquad ♣$$

3. Adding the α multiplication of the j-th row to the i-the row, where $i \neq j$: $\det(A') = \det(A)$.

   Proof. We get from the row linearity of a determinant, Consequence 9.5, and the first elementary operation

$$\det(A') = \det \begin{pmatrix} A_{1*} \\ \vdots \\ A_{i*} + \alpha A_{j*} \\ \vdots \\ A_{n*} \end{pmatrix} = \det(A) + = \det \begin{pmatrix} A_{1*} \\ \vdots \\ \alpha A_{j*} \\ \vdots \\ A_{n*} \end{pmatrix} = \det(A) + \alpha \cdot 0 = \det(A).$$

♣

The above observations have several consequences¿ For an arbitrary matrix $A \in \mathbb{C}^{n \times n}$ it is $\det(\alpha A) = \alpha^n \det(A)$. Moreover, if A contains a zero row or column, than $\det(A) = 0$.

The main importance of the influence of elementary row operations to a determinant is that we can compute easily the determinant of a matrix using Gaussian elimination:

Algorithm 9.6 (Computation of a determinant using Gaussian elimination). Transform the matrix A into the row echelon form A' and remember changes of the determinant in the form of a coefficient c. Then $\det(A)$ is equal to the product of c^{-1} and the diagonal elements of A'.

9.2. Other properties of a determinant

Theorem 9.7 (Criterion of regularity). A matrix $A \in \mathbb{C}^{n\times n}$ is regular if and only if $\det(A) \neq 0$.
Proof. Transform A by means of elementary operations to the row echelon form A'. The operations can change the value of the determinant, but not it (un)equality to 0. Then A is regular if and only if A' has non-zero elements on its diagonal. ♣

Remark 9.8 Skipped

Theorem 9.9 (Multiplicativity of determinants). For each $A, B \in \mathbb{C}^{n\times n}$ we have $\det(AB) = \det(A)\det(B)$.
Proof.
(1) Let us consider first the case when A is the matrix of an elementary row operation:
1. $A = E_i(\alpha)$, the multiplication of the i-th row by a number α. Then $\det(AB) = \alpha\det(B)$ and $\det(A)\det(B) = \alpha\det(B)$.
2. $A = E_{ij}$, the exchange of the i-th row and the j-th row. Then $\det(AB) = -\det(B)$ and $\det(A)\det(B) = -1\cdot\det(B)$.
3. $A = E_{ij}(\alpha)$, adding the α multiple of the j-th row to the i-th row. Then $\det(AB) = \det(B)$ and $\det(A\det(B) = 1\cdot\det(B)$.

It follows that the equality holds in all cases.
(2) Now, consider a general case. If A is singular, then AB is singular as well (Theorem 3.26) and then, according to Theorem 9.7, $\det(AB) = 0 = 0\cdot\det(B) = \det(A)\det(B)$. If A is regular, it can be decomposed to the product of elementary matrices $A = E_1\cdots E_k$. Now, let us proceed by mathematical induction. The case $k = 1$ is solved in (1). The induction step: Using (1) and the induction hypothesis, we get

$$\det(AB) = \det((E_1\cdots E_k)B) = \det(E_1(E_2\cdots E_kB) = \det(E_1)\det((E_2\cdots E_k)B) =$$

$$= \det(E_1)\det(E_2\cdots E_k)\det(B) = \det(E_1E_2\cdots E_k)\det(B) = \det(A)\det(B).$$

♣

Consequence 9.10 (). If $A \in \mathbb{C}^{n\times n}$ is regular, then $\det(A^{-1}) = \det(A)^{-1}$.
Proof. $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$. ♣

Now we will show a recurrent formula for computing a determinant.

Theorem 9.11 (Laplace expansion of a determinant by the i-th row). Let $A \in \mathbb{C}^{n\times n}$, $n \geq 2$. Then for each $i = 1, \ldots, n$ it is

$$\det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(A^{ij}),$$

where A^{ij} is the matrix obtained from A by erasing the i-th row and the j-th column.

Remark. Similarly as by rows, we can expand a determinant by columns.

Proof. (1) Let us consider first the case $A_{i*} = e_j^T$, i.e., the i-th row of the matrix A is a unit vector. By subsequent exchanging rows $(i, i+1)$, $(i+1, i+2)$, $\ldots$, $(n-1, n)$ we move the unit vector to the last row. similarly with columns and the j-th row will be moved to the last position. Let us denote the resulting matrix by

$$A' := \begin{pmatrix} & A^{ij} & & \\ & & & \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

and the sign of the determinant is changed b multiplying by the coefficient $(-1)^{(n-1)+(n-j)} = (-1)^{i+j}$. Now,

$$\det(A) = (-1)i + j \det(A') = (-1)i + j \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a'_{i,p(i)} =$$

$$= (-1)i + j \sum_{p;p(n)=n} \operatorname{sgn}(p) \prod_{i=1}^{n-1} a'_{i,p(i)} = (-1)i + j \det(A^{ij}).$$

(2) Now the general case. It follows from the row linearity of a determinant and from the above that

$$\det(A) = \det \begin{pmatrix} \cdots & & & \\ a_{i1} & 0 & \cdots & 0 \\ \cdots & & & \end{pmatrix} + \cdots + det \begin{pmatrix} \cdots & & & \\ 0 & \cdots & 0 & a_{in} \\ \cdots & & & \end{pmatrix} =$$

$$a_{i1}(-1)^{i+1} \det(A^{i1}) + \cdots + a_{in}(-1)^{i+n} \det(A^{in}). \qquad \clubsuit$$

Example 9.12 Skipped

Theorem 9.13 Skipped

Example 9.14 Skipped

9.3. Adjugate matrix

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9.4. Applications

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