THE ODD CASE OF ROTA'S BASES CONJECTURE

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Abstract. The paper links four conjectures:

1. (Rota's bases conjecture): For any system $\mathcal{A} = ({}^1A, \ldots, {}^nA)$ of non-singular real valued matrices the multiset of all columns of matrices in $\mathcal{A}$ can be decomposed into n independent systems of representatives of $\mathcal{A}$.
2. (Alon-Tarsi): For even n, the number of even $n \times n$ Latin squares differs from the number of odd $n \times n$ Latin squares.
3. (Stones-Wanless, Kotlar): For all n, the number of even $n \times n$ Latin squares with the identity permutation as first row and first column differs from the number of odd $n \times n$ Latin squares of this type.
4. (Aharoni-Berger): Let $\mathcal{M}$ and $\mathcal{N}$ be two matroids on the same vertex set, and let $A_1, \ldots, A_n$ be sets of size $n+1$ belonging to $\mathcal{M} \cap \mathcal{N}$. Then there exists a set belonging to $\mathcal{M} \cap \mathcal{N}$ meeting all A_i.

Huang and Rota [9] and independently Onn [14] proved that for any n (2) implies (1). We prove equivalence between (2) and (3). Using this, and a special case of (4), we prove the Huang-Rota-Onn theorem for n odd and a restricted class of input matrices: assuming the Alon-Tarsi conjecture for $n-1$, Rota's conjecture is true for any system of non-singular real valued matrices where one of them is non-negative and the remaining have non-negative inverses.

Key words and phrases: Rota's bases conjecture, rainbow matchings, hyperdeterminants.

1. Introduction

Given a system $\mathcal{F} = (F_1, F_2, \ldots, F_n)$ of not necessarily disjoint sets, a *rainbow (multi)set* is a multiset consisting of one element from each F_i. A famous conjecture of Rota [9] is that if $\mathcal{M}$ is a matroid and $\mathcal{F}$ is a system of independent sets of size n, then $\bigcup \mathcal{F}$, viewed as a multiset, can be decomposed into n rainbow independent sets belonging to $\mathcal{M}$ (in particular, having no repeating elements). Here is the linear case of the conjecture, over the reals, in an equivalent formulation:

Conjecture 1.1. *The sets of columns of a system of n non-singular $n \times n$ real valued matrices have n disjoint rainbow bases of $\mathbb{R}^n$.*

Given a Latin square L we write $\mathrm{sign}(L)$ for the product of all signs of its rows and columns (where the sign of a permutation is 1 if the permutation is even, and -1 if it is odd). Let Λ be the set of Latin squares of order n (we suppress the dependence on n), and write

$$L(n) = \sum_{L \in \Lambda} \mathrm{sign}(L).$$

The Alon-Tarsi conjecture [4] is:

Conjecture 1.2. *If n is even then $L(n) \neq 0$.*

Since exchanging two rows in a Latin square of odd order reverses its sign, for n odd $L(n) = 0$. Huang and Rota [9] and independently Onn [14] proved the following reduction:

Theorem 1.3. *If $L(n) \neq 0$ then Conjecture 1.1 is true for n.*

In [5] an intriguing online version of this theorem is proved.

In this paper we show:

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Theorem 1.4. *If n is odd and $L(n-1) \neq 0$ then the sets of columns of a system of n non-singular $n \times n$ real valued matrices have $n-1$ disjoint rainbow bases of $\mathbb{R}^n$.*

and:

Theorem 1.5. *If $L(n-1) \neq 0$ then Conjecture 1.1 is true for any system of n non-singular $n \times n$ real matrices such that ${}^1U \geq 0$ and ${}^iU^{-1} \geq 0$ for all $i > 1$.*

Here ≥ 0 is taken in the meaning of elementwise inequality.

Both results are based on showing that $L(n-1) \neq 0$ if and only if $\ell(n) \neq 0$ for an Alon-Tarsi like parameter $\ell(n)$ defined in [17, 10] (see the beginning of Section 3 for its definition). Theorem 1.4 follows directly from this fact, and a result proved in [3]. Theorem 1.4 demands the proof of an identity, similar to that proved by Onn [14] (see (1) below).

Drisko [7] proved Conjecture 1.2 for $n = p+1$ and Glynn [8] proved it for $n = p-1$ (here p is prime). It follows that the "$n-1$ disjoint rainbow bases" conclusion in Theorem 1.4 is true for $n = p$ and $n = p+2$. Similarly, the conclusion of Theorem 1.5 is true for these values of n.

1.1. **Notation.** The ith component of a vector θ is sometimes denoted by $\theta(i)$ and sometimes by θ_i, depending on whether the expression includes already many parentheses or many subscripts. By $\tilde{\theta}$ we denote the multiset consisting of the entries of θ.

The ith row of a matrix A will be denoted by A_i, and the j-th column by A^j. The i,j-th element of A is sometimes denoted by $A(i,j)$ and sometimes by A_i^j. Let $A_{(\setminus i)}$ (respectively $A^{(\setminus j)}$) be the matrix obtained from A be removing the ith row (respectively the j-th column). Following common notation (see e.g. [13]) we write $A(i \mid j)$ for $A_{(\setminus i)}^{(\setminus j)}$. Given a sequence ζ of length m of indices we write A_ζ for the matrix having as rows $A_{\zeta(k)}$, $1 \leq k \leq m$. When reading the notation A_C care should be applied to notice whether C is a single index, in which case A_C is a row vector, or C is a sequence of indices, in which case A_C is a matrix. Given a sequence $\mathcal{A} = ({}^1A, \ldots, {}^mA)$ of matrices, we write $\mathcal{A}^T$ for the sequence $({}^1A^T, \ldots, {}^mA^T)$ and if the matrices ${}^iA$ are non-singular then we write $\mathcal{A}^{-1} = ({}^1A^{-1}, \ldots, {}^mA^{-1})$. We denote by $DET(\mathcal{A})$ the product $\prod_{i \leq m} \det({}^iA)$.

2. Permutation systems, decompositions and hyperdeterminants

For natural numbers n and k we denote by $\Gamma^{n,k}$ the set of all sequences $\bar{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of permutations of the set $[k] := \{1, \ldots, k\}$. By $\Gamma_r^{n,k}$ (the subscript r standing for "restricted") we denote the set of all permutation systems $\bar{\gamma} \in \Gamma^{n,k}$ with γ_1 the identity permutation. The *sign* of $\bar{\gamma}$, denoted $\mathrm{sign}(\bar{\gamma})$, is defined as $\prod_{i \geq 1} \mathrm{sign}(\gamma_i)$.

Assuming that n is fixed and known, we write Γ for $\Gamma^{n,n}$.

Given a system of matrices $\mathcal{U} = ({}^1U,\ {}^2U, \ldots, {}^nU)$, a decomposition into rainbow sets of columns can be represented by a system $\bar{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Gamma$, where γ_i dictates how to distribute the columns of ${}^iU$ among the rainbow sets ${}^jR$: the ith column of ${}^jR$ is ${}^iU^{\gamma_i(j)}$. We denote this decomposition (namely, the sequence of matrices obtained) by $D^{\bar{\gamma}}(\mathcal{U})$.

Rota's conjecture is tantamount to the claim that $DET(D^{\bar{\gamma}}(\mathcal{U})) \neq 0$ for some $\bar{\gamma} \in \Gamma$. Assuming the Alon-Tarsi conjecture, for even n this follows from an identity which is the crux of Onn's proof that Alon-Tarsi implies Rota:

$$\sum_{\bar{\gamma} \in \Gamma} \mathrm{sign}(\bar{\gamma}) DET(D^{\bar{\gamma}}(\mathcal{U})) = L(n) DET(\mathcal{U}). \tag{1}$$

Onn's elegant proof of this identity is based on the double role played by permutation systems, as determining the decompositions $D^{\bar{\gamma}}$ and in the calculation of determinants, and on reversing the order of these two roles. With the aim of putting the identity in a more general context, we give here a proof based on an identity on *hyperdeterminants*.

A *tensor* T of dimension n and size k is a $k\times k\times\ldots\times k$ (n-fold product) array of numbers. The $(i_1,\ldots,i_n)$ element of T (where $i_j\le k$) is denoted by $T_{i_1,\ldots,i_n}$. The *hyperdeterminant* of T, denoted by hypdet(T), is defined by:

$$\mathrm{hypdet}(T)=\sum_{\bar\gamma\in\Gamma^{n,k}}\mathrm{sign}(\bar\gamma)\prod_{1\le i\le k}T_{\gamma_1(i),\gamma_2(i),\ldots,\gamma_n(i)}$$

The *restricted* hyperdeterminant of T, denoted by hypdet$_r(T)$, is

$$\mathrm{hypdet}_r(T)=\sum_{\bar\gamma\in\Gamma^{n,k}_r}\mathrm{sign}(\bar\gamma)\prod_{1\le i\le k}T_{\gamma_1(i),\gamma_2(i),\ldots,\gamma_n(i)}$$

Remark 2.1. For every permutation σ, it is true that

$$\mathrm{hypdet}(T)=\sum_{\bar\gamma\in\Gamma^{n,k}}\mathrm{sign}(\bar\gamma)\prod_{1\le i\le k}T_{\sigma\gamma_1(i),\sigma\gamma_2(i),\ldots,\sigma\gamma_n(i)}\tag{2}$$

Since for n odd and σ odd $\mathrm{sign}(\sigma\gamma_1,\sigma\gamma_2,\ldots,\sigma\gamma_n)=-\mathrm{sign}(\gamma_1,\gamma_2,\ldots,\gamma_n)$, it follows that for n odd $\mathrm{hypdet}(T)=0$. Therefore hypdet$_r$ is customarily used in the odd case. For n even, (2) implies that $\mathrm{hypdet}(T)=n!\mathrm{hypdet}_r(T)$, and hence it is possible to use the restricted hyperdeterminant also there. This is actually the common definition of the determinant for $n=2$.

Hyperdeterminants have the same properties as determinants:

Observation 2.2.

1. *Let $i\le n, j\le k$. If T' is obtained by multiplying all entries in T having j as their ith coordinate by a constant α then* $\mathrm{hypdet}(T')=\alpha\mathrm{hypdet}(T)$
2. *Let $p\le n$, let $j_1,j_2\le k$, and let β be a real number. Let T' be obtained from T by adding $\beta T_{i_1,\ldots,j_1,\ldots i_n}$ (where j_1 appears in the pth coordinate) to $T_{i_1,\ldots,j_2,\ldots i_n}$ for all sequences of indices $(i_t\mid t\le n, t\ne p)$, where again j_2 appears in the pth coordinate. Then* $\mathrm{hypdet}(T')=\mathrm{hypdet}(T)$.

For a system $\mathcal{U}=({}^1U,\ {}^2U,\ldots,{}^nU)$ of matrices let $T=T(\mathcal{U})$ be defined by

$$T_{i_1,\ldots,i_n}=\det({}^1U^{i_1},{}^2U^{i_2},\ldots,{}^nU^{i_n})$$

Observation 2.3. *[Zappa]* $\mathrm{hypdet}(T(I,I,\ldots,I))=L(n)$.

Proof. Write $T=T(I,I,\ldots,I)$. Let $\bar\gamma$ be any element of Γ. If $p\ne q$ and $\gamma_p(i)=\gamma_q(i)$ for some i, then $T_{\gamma_1(i),\ldots,\gamma_n(i)}=0$ since it is a determinant of a matrix with two identical columns. So, the only systems $\bar\gamma$ contributing to the hyperdeterminant satisfy $\gamma_p(i)\ne\gamma_q(i)$ whenever $p\ne q$, meaning that the matrix whose columns are the permutations γ_i is a Latin square $L=L(\bar\gamma)$, and clearly $\mathrm{sign}(\bar\gamma)\prod_{i\le n}T_{\gamma_1(i),\ldots,\gamma_n(i)}=\mathrm{sign}(L)$. □

Given two systems of matrices $\mathcal{U}=({}^1U,\ {}^2U,\ldots,{}^nU)$ and $\mathcal{V}=({}^1V,\ {}^2V,\ldots,{}^nV)$ we write $\mathcal{V}\cdot\mathcal{U}$ for the system $({}^1V^1U,\ {}^2V^2U,\ldots,{}^nV^nU)$.

Observation 2.4. $\mathrm{hypdet}(T(\mathcal{V}\cdot\mathcal{U}))=\mathrm{hypdet}(T(\mathcal{V}))DET(\mathcal{U})$.

Proof. It suffices to show the equality when ${}^iU=I$ for all but one i. Since every matrix is the product of elementary matrices of column operations, it suffices to assume that for this i the matrix ${}^iU$ is elementary. For this case, the observation follows from Observation 2.2, since an elementary column operation on ${}^iV$ translates into a corresponding elementary column operation on $T(\mathcal{U})$.

□

Applying Observation 2.4 to the systems $\mathcal{V}=(I,I,\ldots,I)$ and the given system $\mathcal{U}$ and using Observation 2.3 yields (1).

Notation 2.5. Given a sequence $\mathcal{A}=({}^1A,\ldots,{}^nA)$ of $n\times n$ matrices and a sequence $\bar j=(j_1,\ldots,j_n)$ of not necessarily distinct indices, we write $\mathcal{A}[\bar j]$ for the $n\times n$ matrix whose ith column is ${}^iA^{j_i}$.

3. Alon-Tarsi like parameters

In this section we introduce some Alon-Tarsi like parameters, and prove links between them.

Define:

$$\Lambda' = \{L \in \Lambda,\ L_1 = (L^1)^T = (1, 2, \ldots, n)\}$$

Namely, a Latin square belongs to Λ' if its first row and first column are the identity permutation. Let

$$\ell(n) = \sum_{L \in \Lambda'} \operatorname{sign}(L).$$

This parameter was defined and studied in [17] and in [10]. In [17] the following conjecture was proposed:

Conjecture 3.1. *$\ell(n) \neq 0$ for all $n \geq 1$.*

In [3] the following was proved, where as usual the collection of columns of the system is considered as a multiset.

Theorem 3.2. *If $\ell(n) \neq 0$ then any system of n non-singular $n \times n$ matrices has a system of $n-1$ disjoint rainbow bases of $\mathbb{R}^n$.*

For n even, permuting columns in a Latin square does not change its sign. Since the number of possible first row and first column configurations in a Latin square is $n!(n-1)!$, it follows that $L(n) = n!(n-1)!\ell(n)$. So, Conjectures 1.2 and 3.1 are equivalent for even n. We shall show that in fact they are equivalent in general, by proving:

Theorem 3.3. *If n is odd then $L(n-1) = (n-1)!\ell(n)$*

which obviously yields:

Corollary 3.4. *If n is odd then $\ell(n) = 0$ if and only if $L(n-1) = 0$.*

This, together with Theorem 3.2, yields Theorem 1.4.

The rest of this section is devoted to the proof of Theorem 3.3.

Given an $n \times n$ matrix M in which all rows $M_i,\ i > 1$, and all columns $M^j,\ j > 1$, are permutations of $[n]$, we write $\operatorname{sign}_r(M)$ for $\prod_{i>1} \operatorname{sign}(M_i) \times \prod_{j>1} \operatorname{sign}(M^j)$ (as before, the r subscript is for "restricted").

Notation 3.5. Denote by $\Omega = \Omega(n)$ the set of $(n-1) \times (n-1)$ matrices indexed by $(2, \ldots, n) \times (2, \ldots, n)$, having entries in $[n]$, and not having repeating entries in any row or in any column.

For a matrix $W \in \Omega$ define vectors $\theta = \theta(W)$ and $\psi = \psi(W)$ on the indices $2, \ldots, n$ by the rule that $\theta(i)$ is the element of $[n]$ missing in row i of W, and $\psi(j)$ is the element of $[n]$ missing in column j of W. Recalling that $\tilde{\theta}(W)$ (respectively $\tilde{\psi}(W)$) is defined as the multiset composing $\theta(W)$ (respectively $\psi(W)$), the elements of $\tilde{\theta}(W)$ are precisely those elements of $[n]$ that appear fewer than $n-1$ times in W, and the same goes for $\tilde{\psi}(W)$. Hence:

Observation 3.6. *$\tilde{\theta}(W) = \tilde{\psi}(W)$.*

Call a matrix $W \in \Omega$ *standard* if the 1 entries in $\theta(W)$ form an initial segment.

Let Υ be the set of those matrices $W \in \Omega$ for which $\tilde{\theta}(W)$ contains at most one copy of every element $i > 1$. For $m \leq n-1$ let Υ_m the set of those matrices $W \in \Upsilon$ for which $\tilde{\theta}(W)$ contains precisely m copies of the element 1. Write Υ_m^S for the set of standard elements of Υ_m, namely those matrices $W \in \Upsilon$ for which $\theta(M)$ has 1s precisely in its first m coordinates.

For a matrix $W \in \Omega$ let $Aug(W)$ be the $n \times n$ matrix obtained from W by attaching to it $\psi(W)^T$ as row 1, and $\theta(W)$ as column 1. Note that the $(1,1)$ entry in $Aug(W)$ is not defined, but we do not need it. Define $\operatorname{sign}(W)$ as $\operatorname{sign}_r(Aug(W))$. Let

$$L_r(n, m) = \sum_{W \in \Upsilon_m} \operatorname{sign}(W).$$

Examples:

(1) $L(3, 0) = 4,\ L(3, 1) = 8,\ L(3, 2) = 2.$

(2) A matrix W belonging to Υ_0 means that W contains 1 in each row and column. For any $k \in \{2, \ldots, n\}$, replacing all 1s in W by k and vice versa results in a matrix belonging to Υ_1 (remember that $\theta(W)$ contains only one copy of k). This map is injective. It is also sign preserving since all signs of all rows and all columns are reversed. Since every matrix in Υ_1 is obtained from some $W \in \Upsilon_0$ by performing this operation for some $k > 1$, there exists an $n-1$ to 1 sign preserving function between Υ_0 and Υ_1. Hence $L_r(n,1) = (n-1)L_r(n,0)$.

(3) A matrix W belonging to Υ_{n-1} means that W is a Latin square with elements in $\{2, \ldots, n\}$. Hence $L_r(n, n-1) = L(n-1)$.

(4) We have $L_r(n, n-2) = (n-1)^2 L(n-1)$. To see this, note that a matrix $W \in \Upsilon_{n-2}$ has a single 1 entry, say $W(i,j)$. Replacing this entry by $\theta(W)_i$ results in a Latin square W' of order $n-1$ with symbols $2, \ldots, n$. Note also that $\text{sign}(Aug(W')) = \text{sign}(Aug(W))$, since $Aug(W')$ is obtained from $Aug(W)$ by two transpositions, one in row i and one in column j. There are $(n-1)^2$ ways of choosing an entry in a Latin square of order $n-1$ with symbols $2, \ldots, n$ and replacing it by 1, hence the identity above.

(5) If n is even, then permuting any fixed pair of rows of W is a sign reversing involution. Hence $L_r(n,m) = 0$.

The following observation ensues from the fact that the signs of the first column and first row of $Aug(W)$ are not taken into account in the calculation of $\text{sign}(W)$.

Observation 3.7. *For n odd, permuting rows or columns in a matrix $W \in \Omega$ does not change* $\text{sign}(W)$.

Observation 3.8. *For n odd $L_r(n,0) = (n-1)!^2 \ell(n)$.*

Proof. Every Latin square $L \in \Lambda'$ gives rise by permuting its columns and its rows to $(n-1)!^2$ matrices of the form $Aug(W)$, $W \in \Upsilon_0$. By Observation 3.7 each of these matrices has the same restricted sign as L. $\square$

For the convenience of reference, here is Example (3) above as an observation:

Observation 3.9. $L_r(n, n-1) = L(n-1)$.

Theorem 3.10. *Let n be odd and $0 \leq m < n-1$. Then*

$$L_r(n,m) = \frac{(m+1)^2}{n-1-m} L_r(n, m+1).$$

Note that this is consistent with the examples above ((2) is obtained by taking $m = 0$, and the case $m = n-2$ follows from (4)).

Theorem 3.10 will follow from:

Observation 3.11. *There is a partition $\Upsilon_m^S = Y \cup Z$ so that*
(♦) $\sum_{W \in Y}$ sign$(W) = 0$, and
(♦♦) There is $f : Z \to \Upsilon_{m+1}^S$ so that for each $W \in \Upsilon_{m+1}^S$, $|f^{-1}(W)| = m+1$ and sign(W) = sign(W') for each $W' \in f^{-1}(W)$.

Proof. We construct the partition into Y, Z, the function f, and a sign inverting function $g : Y \to Y$, together, by the following algorithm.

Let $W \in \Upsilon_m^S$, and let a be the first entry of $\theta(W)$ that is different from 1, namely $a = \theta(W)_{m+1}$.

Let j_1 be such that $\psi(W)_{j_1} = a$, meaning that the j_1 column of W does not contain a. We start constructing an $a \leftrightarrow 1$ alternating path in W, namely a path of entries that alternate between a and 1. Since $\psi(W)_{j_1} \neq 1$ there exists i_1 such that $W(i_1, j_1) = 1$. If $\theta(W)_{i_1} = a$ (meaning, in fact, that $i_1 = m+1$) then we stop the process, having obtained the alternating path $(0, j_1) - (i_1, j_1) - (i_1, 0)$. If $\theta(W)_{i_1} \neq a$, then there exists j_2 such that $W(i_1, j_2) = a$. We continue alternating this way between a and 1, until one of the following happens:

(1) $\theta(W)_{i_k} = a$ for some k (meaning that $i_k = m+1$ - remember that the rows of W are indexed by $2, \ldots, n$), or
(2) $\psi(W)_{j_k} = 1$.

In both cases we terminate the process, and in both we apply the alternating path obtained to W, meaning that each a entry of the path is replaced by 1 and vice versa. Let W' be the resulting matrix. In case (2), $W' \in \Upsilon^S_m$, and $\text{sign}(W') = -\text{sign}(W)$ since W' is obtained from W by applying an even number of transpositions in total to permutations of the set consisting of the rows $2, \dots, n$ and the columns $2, \dots, n$ of $Aug(W)$. (Remember that the transposition of 1 and a in the first row of $Aug(W')$ is not taken into account in the calculation of the sign). Define then $g(W) = W'$. Inverting the alternating path takes W' to W, showing that g is injective. Let Y be the set of all $W \in \Upsilon_m$ for which case (2) happens. Then g is a sign inverting involution on Y with no fixed points (the latter is clear by the definition of g or by the sign inversion), implying (♦).

In case (1) $W' \in \Upsilon^S_{m+1}$, and $\text{sign}(W') = \text{sign}(W)$ since W' is obtained from W by applying an even number of transpositions in total to permutations of the set consisting of the rows $2, \dots, n$ and the columns $2, \dots, n$ of $Aug(W)$. Define then $f(W) = W'$.

The proof will be complete if we show that for every $U \in \Upsilon^S_{m+1}$ there exist precisely $m+1$ matrices $W \in \Upsilon^S_m$ such that $f(W) = U$. For each $a \in [n] \setminus (\tilde{\theta}(W) \cup \{1\})$ there exists an entry $(m+2, j)$ for which $W(m+2, j) = a$. Construct a $1 \leftrightarrow a$ alternating path in $Aug(W)$, starting at the 1 in position $m+1$ of $\theta(W)$, going on to the a in entry $(m+2, j)$, and so on. Such a path must end, and this must happen when a column j_k is reached that does not contain 1, and then the next and last entry in the path is the j_k entry of $\psi(W)$, which is 1. Let P be the alternating path obtained this way. Then applying P to U results in a matrix $W \in \Upsilon^S_m$ such that $f(W) = U$, and this is the unique matrix satisfying this condition for which $\theta(W)_{m+2} = a$.

Since $[n] \setminus (\tilde{\theta}(W) \cup \{1\})$ contains $m+1$ elements, this argument shows that $|f^{-1}(W)| = m+1$, proving (♦♦).

□

Corollary 3.12. $\sum_{W \in \Upsilon^S_m} \text{sign}(W) = (m+1) \sum_{W \in \Upsilon^S_{m+1}} \text{sign}(W)$.

Together with Observation 3.7 this implies:

Observation 3.13. *For every $k < n-1$ we have $L_r(n,k) = \binom{n-1}{k} \sum_{W \in \Upsilon^S_k} \text{sign}(W)$.*

Theorem 3.10 now follows upon applying the observation to $k = m$ and to $k = m+1$, and using Corollary 3.12.

Recalling Observation 3.9, that stated that $L_r(n, n-1) = L(n-1)$, we obtain:

Corollary 3.14. $L_r(n,m) = L(n-1) \prod_{r=1}^{n-m-1} \frac{(n-r)^2}{r}$.

In particular we have:

Corollary 3.15. $L_r(n,0) = (n-1)!L(n-1)$.

Combining this with Observation 3.8 yields:

$$\ell(n) = \frac{L_r(n,0)}{(n-1)!^2} = \frac{L(n-1)}{(n-1)!}.$$

This completes the proof of Theorem 3.3.

4. Joint independent systems of representatives

A famous conjecture of Ryser-Brualdi-Stein [15], [6], [16] is that every $n \times n$ Latin square has a transversal (set of entries with distinct symbols, distinct rows and distinct columns) of size $n-1$. See [18] for a survey on this conjecture and related results. In [1] (Conjecture 2.3) this was strengthened as follows:

Conjecture 4.1. *A set of n matchings in a bipartite graph, each of size $n+1$, has a rainbow matching (a matching consisting of one edge from each of the given matchings).*

See [2, 11] for partial results.

This conjecture can be generalized to:

Conjecture 4.2 (Aharoni and Berger). *Let $\mathcal{M}$ and $\mathcal{N}$ be two matroids on the same vertex set, and let $A_1, \ldots, A_n$ be sets of size $n+1$ belonging to $\mathcal{M} \cap \mathcal{N}$. Then there exists a set belonging to $\mathcal{M} \cap \mathcal{N}$ meeting all A_i.*

In Conjecture 4.1 the two matroids are the partition matroids on the edge set of the graph, the parts in one being the stars in one side, and the parts in the other the stars in the other side. The case of linear matroids of Conjecture 4.1 is:

Conjecture 4.3. *Let $\mathcal{A} = ({}^1A, \ldots, {}^nA)$ and $\mathcal{B} = ({}^1B, \ldots, {}^nB)$ be two systems of non-singular $n \times n$ matrices. Then there exists an $n-1$ joint ISR, namely a pair $\bar{k} = (k_i \mid i \le n-1), \bar{j} = (j_i \mid i \le n-1)$ such that both sets $\{{}^{j_i}A^{k_i} \mid i \le n-1\}$ and $\{{}^{j_i}B^{k_i} \mid i \le n-1\}$ are linearly independent.*

We shall need a special case of this conjecture, in which ${}^iB = ({}^iA^{-1})^T$. In this case, it turns out that there is even a full (size n) joint ISR.

Theorem 4.4. *Let $\mathcal{A} = ({}^1A, \ldots, {}^nA)$ be a system of non-singular $n \times n$ matrices, and let $\mathcal{B} = (\mathcal{A}^{-1})^T$. Then $\mathcal{A}$ and $\mathcal{B}$ have a joint ISR, namely a sequence $\bar{j} = (j_i \mid i \le n)$ such that both $\mathcal{A}[\bar{j}]$ and $\mathcal{B}[\bar{j}]$ are linearly independent.*

The proof will require a notion concerning pairs of subspaces of $\mathbb{R}^n$:

Definition 4.5. Two subspaces K, L of $\mathbb{R}^n$ of the same dimension are called *bi-independent* if $K \cap L^\perp = \{0\}$.

The relation of bi-independence is symmetric, as can be realized for example from the following observation:

Lemma 4.6. *Let K, L be subspaces of $\mathbb{R}^n$ of dimension k, let C and D be respective bases of K and L, and let X and Y be matrices whose column sets are, respectively, C and D. Then K, L are bi-independent if and only if X^TY is non-singular.*

Proof. Suppose that X^TY is singular, and choose a non-zero vector $\vec{u} \in \mathbb{R}^m$ is a column vector such that $\vec{u}^T X^T Y = 0$. Then $X^T\vec{u}$ is a non zero vector in $K \cap L^\perp$. Conversely, a linear combination $X^T\vec{u}$ of the columns of X that belongs to $L^\perp$ satisfies $\vec{u}^T X^T Y = 0$. $\square$

Theorem 4.7. *Let e_i, $i \le n$ be the standard vector with 1 in the ith coordinate and 0 elsewhere. If K, L are bi-independent subspaces of $\mathbb{R}^n$ of dimension $k < n$ then there exists j such that $K + sp(e_j)$, $L + sp(e_j)$ are bi-independent of dimension $k+1$.*

For the proof of the theorem we shall need the following lemma. Here I_k is the identity $k \times k$ matrix:

Lemma 4.8. *If A is a $k \times k$ matrix of rank 1 and $I_k - A$ is singular, then $tr(A) = 1$.*

Proof. Since $rank(A) = 1$, there exist column vectors of length k, $\vec{x}$ and $\vec{y}$, so that $A = \vec{x}\vec{y}^T$. The vector $\vec{x}$ is proportionate to the columns of A, and the vector $\vec{y}^T$ is proportionate to the rows of A. Let $Z = support(\vec{x}) \cap support(\vec{y})$ (here $support(\vec{u}) = \{i : u_i \neq 0\}$ for any vector $\vec{u}$)). Define a matrix C by $C(i,j) = A(i,j)$ for $i, j \in Z$ and $C(i,j) = 0$ if $i \notin Z$ or $j \notin Z$. Then $C = \vec{u}\vec{v}^T$ where $\vec{u}$ and $\vec{v}$ are obtained from $\vec{x}$ and $\vec{y}$, respectively, by making all non-Z coordinates 0.

It is possible transform $I_k - A$ to the matrix $I_k - C$ by elementary row and column operations. To do this, for every $j \in support(\vec{x}) \setminus support(\vec{y})$ use the j-th column of $I_k - A$, which is identical to the j-th column of I_k, to annul all elements of $I_k - A$ in the j-th row, apart form the j-th element. Similarly, use each i-th row, for all $i \in support(\vec{y}) \setminus support(\vec{x})$, to annul all elements in the i-th column except for the i-th.

This implies that $I_k - C$ is also singular, which means that $I_k[Z \mid Z] - C[Z \mid Z]$ is singular. So, the matrix $C[Z \mid Z]$ satisfies the conditions of the lemma with k replaced by $|Z|$. Since $tr(C[Z \mid Z])) = tr(C) = tr(A)$, this means that without loss of generality we may assume that $support(\vec{x}) = support(\vec{y}) = [k]$.

Let D be the diagonal matrix whose (i,i)-th element is x_i, and let $A' = D^{-1}AD$. Then A' is of rank 1 and $I - A' = D^{-1}(I - A)D$ is singular. Thus A' can replace A in the theorem. But in A' every column is a constant vector (remember that in A every column is proportionate to $\vec{x}$). Hence may as well assume that $\vec{x} = \vec{1}$, the all 1 vector, and so $A = \vec{1}\vec{y}^T$. Since A is singular, there exists a non-zero column vector $\vec{z}$ such that $\vec{z}^T(I - \vec{1}\vec{y}^T) = \vec{0}$, namely

$$\vec{z} = (\vec{1}^T\vec{z})\vec{y} \tag{3}$$

This implies that $\vec{1}^T\vec{z} \neq 0$. Multiplying both sides of (3) by $\vec{1}$ we get:

$$\vec{z}^T\vec{1} = (\vec{1}^T\vec{z})(\vec{y}^T\vec{1})$$

Since as noted above $\vec{1}^T\vec{z} \neq 0$, this implies $\vec{y}^T\vec{1} = 1$. But $tr(A) = tr(\vec{1}\vec{y}^T) = tr(\vec{y}^T\vec{1})$, and thus $tr(A) = 1$. $\square$

Corollary 4.9. *Let $n > k$ be two integers. If $B_1, B_2, \ldots, B_n$ are $k \times k$ matrices of rank 1 and $rank(\sum_{i\leq n} B_i) = k$ then there exists $j \leq n$ such that $rank(\sum_{i\leq m\ i\neq j} B_i) = k$.*

Proof. Applying simultaneous row operations to the B_is, we may assume that $\sum_{i\leq n} B_i = I$. Assume for contradiction that for every $j \leq n$ we have $rank(\sum_{i\leq n\ i\neq j} B_i) = k-1$. By the lemma, $tr(B_j) = 1$, but this means that $tr(\sum_{i\leq n} B_i) = n$, contradicting the assumptions that $\sum_{i\leq m} B_i = I_k$ and $n > k$. $\square$

Proof of Theorem 4.7

Let X, Y be $n \times k$ matrices whose column sets are bases of K and L, respectively, and let $\vec{x_i}$ $(i \leq n)$ and $\vec{y_i}$, $(i \leq n)$ be their rows, transposed so as to make them column vectors. Then $X^TY = \sum_{i\leq n} \vec{x}\vec{y}^T$, and by Lemma 4.6, $rank(X^TY) = k$. By Corollary 4.9 there exists $j \leq n$ such that $rank(\sum_{i\leq n,\ i\neq j} \vec{x}\vec{y}^T) = k$. We claim that this means that $K + sp(e_j)$, $L + sp(e_j)$ are bi-independent. To see this, add e_j as a column to X and to Y, and subtract its multiples from all columns of X and of Y, so as to reach in both a zero jth row. The two $n \times (k+1)$ matrices obtained, say X' and Y', have column sets that are bases for $K + sp(e_j)$, $L + sp(e_j)$, respectively. The matrix X'^TY' is obtained from $X^T_{(\setminus j)}Y_{(\setminus j)} = \sum_{i\leq n,\ i\neq j} \vec{x}\vec{y}^T$ by adding as row $k+1$ the vector $(0, \ldots, 0, 1)$, and as column $k+1$ the vector $(0, \ldots, 0, 1)^T$ (recall that $X_{(\setminus j)}$ is obtained from X by deleting row j, and similarly for Y.) Hence $rank(X'^TY') = k+1$, as desired.

We say that a sequence of pairs of vectors $(\vec{a}_i, \vec{b}_i)$, $i \leq p$ is bi-independent if for every $k \leq p$ the spaces $sp(\vec{a}_i \mid i \leq k)$, $sp(\vec{b}_i \mid i \leq k)$ is bi-independent of dimension k.

Theorem 4.4 will clearly follow from:

Theorem 4.10. *Let (A_i, B_i), $1 \leq i \leq n$ be pairs of $n \times n$ matrices, where $B_i = (A_i^{-1})^T$. Then there exists a rainbow bi-independent sequence $(\vec{a}_i, \vec{b}_i)$, $i \leq n$, where $\vec{a}_i$ is a column of A_i and $\vec{b}_i$ is a column of B_i.*

Proof. We show by induction on k that it is possible to choose bi-independent pairs $(\vec{a}_i, \vec{b}_i)$, $i \leq k$, representing A_i and B_i, respectively. Suppose that we have chosen $\vec{a}_i$, $\vec{b}_i$, for $i < k$. Let $P = A_k^{-1}$. By Lemma 4.6 the sequence $(P\vec{a}_i, (P^{-1})^T\vec{b}_i)$, $i < k$ is bi-independent. By Lemma 4.7 it follows that there exists an index j such that $\{(P\vec{a}_i, (P^{-1})^T\vec{b}_i) \mid i < k\} \cup \{(e_j, e_j)\}$ is bi-independent. Let $\vec{a}_k = A_k^j$, and $\vec{b}_k = B_k^j$. Since (a_i, b_i), $i \leq k$ is obtained from $\{(P\vec{a}_i, (P^{-1})^T\vec{b}_i) \mid i < k\} \cup \{(e_j, e_j)\}$ by multiplying the first vector in each pair by P^{-1} and the second vector by P^T, it follows that $(\vec{a}_i, \vec{b}_i)$, $i \leq k$ is bi-independent, as desired. $\square$

5. An identity on restricted permutations for odd n

Throughout this section we are assuming that n is an odd integer. Let $\bar{j}$ be a sequence of length n of indices, satisfying $j_1 = 1$ and let $\Gamma(\bar{j})$ be the set of permutation systems $\bar{\gamma}$ for which $\gamma_i(1) = j_i$ for each $1 \leq i \leq n$.

As recalled, for a given $\bar{\gamma}$, the k-th matrix in the sequence $D^{\bar{\gamma}}(\mathcal{U})$ has as columns ${}^iU^{\gamma_i(k)}$. By the definition of the determinant, the determinant of this matrix is $\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \prod_{i\leq n} {}^iU^{\gamma_i(k)}_{\sigma(i)}$. Hence:

$$DET(D^{\bar{\gamma}}\mathcal{U}) = \sum_{\bar{\sigma}\in\Gamma} \operatorname{sign}(\bar{\sigma}) \prod_{i,k} ({}^iU)^{\gamma_i(k)}_{\sigma_k(i)} \tag{4}$$

Here, as before, $\bar{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. Let:

$$\delta(\bar{j}, \mathcal{U}) = \sum_{\bar{\gamma}\in\Gamma(\bar{j})} (\prod_{i>1} \operatorname{sign}(\gamma_i)) DET(D^{\bar{\gamma}}(\mathcal{U}))$$

We wish to show, under special conditions on $\mathcal{U}$ and $\bar{j}$, that if $L(n-1) \neq 0$ then $\delta(\bar{j}, \mathcal{U}) \neq 0$. This will imply the conclusion of Rota's conjecture for this case.

For fixed $\bar{\sigma}$ write

$$V(\bar{\sigma},\bar{j},\mathcal{U}) = \sum_{\bar{\gamma}\in\Gamma(\bar{j})} \prod_{i>1} \text{sign}(\gamma_i) \prod_{1\leq i,k\leq n} ({}^iU)^{\gamma_i(k)}_{\sigma_k(i)} \tag{5}$$

Then, changing the order of summation

$$\delta(\bar{j},\mathcal{U}) = \sum_{\bar{\sigma}\in\Gamma} sign(\bar{\sigma}) V(\bar{\sigma},\bar{j},\mathcal{U}) \tag{6}$$

Given $\bar{\sigma} \in \Gamma$ we write $M(\bar{\sigma})$ for the $n \times n$ matrix whose ith column is the permutation σ_i. For $i > 1$ let $\tau_i = \tau_i(\bar{\sigma})$ be the ith row of $M(\bar{\sigma})$, namely $\tau_i(k) = \sigma_k(i)$. Note that τ_i is not necessarily a permutation. Note also that $(\tau_1(2), \ldots, \tau_1(n)) = \psi(M(\bar{\sigma})(1 \mid 1))$.

Studying the expression for $V(\bar{\sigma},\bar{j},\mathcal{U})$ we see that it is the product of the following terms:

- $\prod_{i\geq 1} {}^iU^{j_i}_{\sigma_1(i)}$, which comes from the elements in the first rows, namely $k = 1$ in (5), as chosen by γ_i.
- $(-1)^{1+j_i} \times \det({}^iU_{\tau_i}(1 \mid j_i))$ for $i > 1$. These appear since the permutations γ_i in the expression all satisfy $\gamma_i(1) = j_i$, and thus the permutation submatrix of ${}^iU$ determined by γ_i is in fact a permutation in the matrix ${}^iU_{\tau_i}(1 \mid j_i)$, and the sign of this permutation is $(-1)^{1+j_i} \times sign(\gamma_i)$.
- $\text{per}({}^1U_{\tau_1}(1 \mid j_1))$ The reasoning is similar, but here it is the permanent and not the determinant, because $\text{sign}(\gamma_1)$ does not appear in the expression for $V(\bar{\sigma},\bar{j},\mathcal{U})$.

(Recall that ${}^iU_{\tau_i}$ is the matrix having ${}^iU_{\tau_i(k)} ={}^i U_{\sigma_k(i)}$ as its k-th row).

Summarizing:

$$V(\bar{\sigma},\bar{j},\mathcal{U}) = (-1)^{\sum_{1<i\leq n}(1+j_i)} \text{per}({}^1U_{\tau_1}(1 \mid j_1)) \times \prod_{i\geq 1} {}^iU^{j_i}_{\sigma_1(i)} \times \prod_{i>1} \det({}^iU_{\tau_i}(1 \mid j_i)). \tag{7}$$

Notation 5.1. Let Φ be the set of those $\bar{\sigma} \in \Gamma$ for which $M(\bar{\sigma})(1 \mid 1) \in \Omega$ (where Ω is defined in Notation 3.5). We also denote $\theta(M(\bar{\sigma})(1 \mid 1))$ by $\theta(\bar{\sigma})$.

The presence of the determinant terms in (7) implies:

Corollary 5.2. *If $V(\bar{\sigma},\bar{j},\mathcal{U}) \neq 0$ then $\bar{\sigma} \in \Phi$. Hence $\delta(\bar{j},\mathcal{U}) = \sum_{\bar{\sigma}\in\Phi} \text{sign}(\bar{\sigma}) V(\bar{\sigma},\bar{j},\mathcal{U})$.*

Given $i > 1$ let ζ_i be the ith row of $Aug(M(\bar{\sigma})(1 \mid 1))$. Namely, ζ_i is obtained by replacing the first entry of τ_i by $\theta(\bar{\sigma})_i$. By a well known formula for the inverse of a matrix,

(8)

$$(-1)^{1+j_i} \times \det({}^iU_{\tau_i}(1 \mid j_i)) = (-1)^{1+j_i} \times \det({}^iU_{\zeta_i}(1 \mid j_i)) = (({}^iU^{-1}_{\zeta_i})^T)^{j_i}_1 \det({}^iU_{\zeta_i}) = (({}^iU^{-1}_{\zeta_i})^T)^{j_i}_1 \text{sign}(\zeta_i) \det({}^iU).$$

For an invertible matrix A and a permutation ζ we have $(A^{-1}_\zeta)^T = (A^{-1})^T_\zeta$ (to see this, write P for the permutation matrix representing ζ, and then $A_\zeta = PA$, and hence $(A^{-1}_\zeta)^T = P(A^{-1})^T = (A^{-1})^T_\zeta$.) Thus $({}^iU^{-1}_{\zeta_i})^T)^{j_i}_1 = (({}^iU^{-1})^T)^{j_i}_{\zeta_i(1)} = (({}^iU^{-1})^T)^{j_i}_{\theta(i)}$.

Implementing this observation in (8) yields, for $i > 1$:

$$(-1)^{1+j_i} \det({}^iU_{\zeta_i}(1|j_i)) = (({}^iU^{-1})^T)^{j_i}_{\theta(i)} \text{sign}(\zeta_i) \det({}^iU)$$

Combining this with (7), and writing θ for $\theta(\bar{\sigma})$, we get:

$$V(\bar{\sigma},\bar{j},\mathcal{U}) = \text{per}({}^1U_{\tau_1}(1|j_1)) \times \prod_{i\geq 1} {}^iU^{j_i}_{\sigma_1(i)} \times \prod_{i>1} (({}^iU^{-1})^T)^{j_i}_{\theta(i)} \times \prod_{i>1} sign(\zeta_i) \det({}^iU) \tag{9}$$

Our next step is to consider a special case, in which ${}^1U = I$.

Theorem 5.3. *Assuming ${}^1U = I$, we have:*

$$\delta(\bar{j},\mathcal{U}) = \ell(n) \times (n-1)! \times \text{per}((\mathcal{U}^{-1})^T[\bar{j}]) \times \det(\mathcal{U}[\bar{j}]) \times DET(\mathcal{U})$$

(See Notation 2.5 for the meaning of $\mathcal{U}[\bar{j}]$.) For the proof, note first:

Observation 5.4. *Suppose that $V(\bar{\sigma},\bar{j},\mathcal{U}) \neq 0$. Then:*

- *Since ${}^1U = I$, the fact that ${}^1U^{j_1}_{\sigma_1(1)} \neq 0$ implies that $\sigma_1(1) = \tau_1(1) = j_1 = 1$.*
- *Again, since ${}^1U = I$, the permanent in (9) being non-zero, together with the fact that $\tau_1(1) = j_1$, imply that τ_1 is a permutation.*

Write Φ_0 for the set of those permutation systems $\bar{\sigma}$ that besides belonging to Φ satisfy also the above conditions, namely (a) $\sigma_1(1) = j_1 = 1$ and (b) τ_1 is a permutation. Let us extend $\theta(\bar{\sigma})$ to a permutation by letting $\theta(\bar{\sigma})(1) = 1$. We again denote $\theta(\bar{\sigma})$ by θ if convenient. Since ${}^1U = I$ and $j_1 = 1 = \sigma_1(1)$, if $\bar{\sigma} \in \Phi_0$ then $\mathrm{per}({}^1U_{\tau_1}(1|j_1)) = 1$, and $(({}^1U^{-1})^T)^{j_1}_{\theta(1)} = 1$. Then:

Observation 5.5. *Since ${}^1U = I$, if $\bar{\sigma} \in \Phi_0$ then $\mathrm{per}({}^1U_{\tau_1}(1|j_1)) = 1$.*

By these observations,

$$(10)\quad \delta(\bar{j}, \mathcal{U}) = \sum_{\sigma \in \Phi_0} \mathrm{sign}(\bar{\sigma}) V(\bar{\sigma}, \bar{j}, \mathcal{U}) = \sum_{\sigma \in \Phi_0} DET(\mathcal{U}) \mathrm{sign}_r(M(\bar{\sigma})) \times \mathrm{sign}(\sigma_1) \prod_{i \geq 1} {}^iU^{j_i}_{\sigma_1(i)} \times \prod_{i \geq 1} (({}^iU^{-1})^T)^{j_i}_{\theta(i)}$$

For a permutation θ, let $\Lambda(\theta)$ be the set of $n \times n$ Latin squares having θ as their first column.

Lemma 5.6. $\sum_{L \in \Lambda(\theta)} sign_r(L) = (n-1)!\ell(n)$.

Proof. Let $\Lambda_d(\theta)$ be the set of Latin squares having θ as both first row and first column. Clearly, $\sum_{L \in \Lambda_d(\theta)} sign_r(L) = \ell(n)$. Since n is odd, permuting columns, other than the first, in $L \in \Lambda_d(\theta)$ does not change $sign_r(L)$, and every $L \in \Lambda_d(\theta)$ gives rise by permutations of columns $2, \ldots, n$ to $(n-1)!$ distinct Latin squares in $\Lambda(\theta)$. ☐

We let $\mathcal{L} = \{(L, \alpha); L^1_1 = \alpha(1) = 1,\ L \text{ latin square, } \alpha \text{ permutation}\}$.

Lemma 5.7. *There is a bijection $f : \Phi_0 \to \mathcal{L}$ such that, if $f(\bar{\sigma}) = (L, \alpha)$ then $L^1 = \theta(\bar{\sigma})$, $\alpha = \sigma_1$ and $\mathrm{sign}_r(M(\bar{\sigma})) = \mathrm{sign}_r(L)$.*

Proof. Given $\bar{\sigma}$, we let $\alpha = \sigma_1$ and obtain L from $M(\bar{\sigma})$ by replacing its first column (equal to σ_1) by $\theta(\bar{\sigma})$. ☐

Proof of Theorem 5.3 By (10), Lemma 5.6 and Lemma 5.7,

$$\delta(\bar{j}, \mathcal{U}) = \sum_{\sigma \in \Phi_0} DET(\mathcal{U}) \mathrm{sign}_r(M(\bar{\sigma})) \times \mathrm{sign}(\sigma_1) \prod_{i \geq 1} {}^iU^{j_i}_{\sigma_1(i)} \times \prod_{i \geq 1} (({}^iU^{-1})^T)^{j_i}_{\theta(i)} =$$
$$\sum_{(L, \sigma_1) \in \mathcal{L}} DET(\mathcal{U}) \mathrm{sign}_r(L) \times \mathrm{sign}(\sigma_1) \prod_{i \geq 1} {}^iU^{j_i}_{\sigma_1(i)} \times \prod_{i \geq 1} (({}^iU^{-1})^T)^{j_i}_{L^1(i)} =$$
$$DET(\mathcal{U}) [\sum_{\alpha \in S_n, \alpha(1) = 1} \mathrm{sign}(\alpha) \prod_{i \geq 1} {}^iU^{j_i}_{\alpha(i)}] \times [\sum_{\theta \in S_n, \theta(1) = 1} \prod_{i \geq 1} (({}^iU^{-1})^T)^{j_i}_{\theta(i)}] \times [\sum_{L; L^1 = \theta} \mathrm{sign}_r(L)] =$$
$$DET(\mathcal{U}) \times \det(\mathcal{U}[\bar{j}]) \times \mathrm{per}((\mathcal{U}^{-1})^T [\bar{j}]) \times \ell(n) \times (n-1)!$$

☐

Proof of Theorem 1.5. Since $L(n-1) = 0$ for n even, we may assume that n is odd. Since the validity of Rota's conjecture is invariant under simultaneous elementary row operations, we can multiply all matrices ${}^iU$ on the left by ${}^1U^{-1}$. This operation changes the matrices $({}^iU^{-1})^T, i \geq 1$, as well. Namely, each new $({}^iU^{-1})^T$ is obtained by left-multiplication of the old $({}^iU^{-1})^T$ by $({}^1U)^T$. By the assumption of Theorem 1.5, we obtain a sequence of matrices, which we still call ${}^iU$, such that ${}^1U = I$ and $({}^iU^{-1})^T \geq 0$ for all $i \geq 1$. By Theorem 4.4 there is a sequence of indices $\bar{j}$ so that $\det((\mathcal{U}^{-1})^T [\bar{j}]) \times \det(\mathcal{U}[\bar{j}]) \neq 0$. Since the matrices $({}^iU^{-1})^T, i \geq 1$, are non-negative, also $\mathrm{per}((\mathcal{U}^{-1})^T [\bar{j}]) \times \det(\mathcal{U}[\bar{j}]) \neq 0$. We can clearly assume $j_1 = 1$. By Theorem 5.3 and Corollary 3.4, $\delta(\bar{j}, \mathcal{U}) \neq 0$. Hence

$$0 \neq \delta(\bar{j}, \mathcal{U}) = \sum_{\bar{\gamma} \in \Gamma(\bar{j})} (\prod_{i > 1} \mathrm{sign}(\gamma_i)) DET(D^{\bar{\gamma}}(\mathcal{U}))$$

and thus there is $\bar{\gamma}$ so that $DET(D^{\bar{\gamma}}(\mathcal{U})) \neq 0$, meaning that $D^{\bar{\gamma}}(\mathcal{U})$ is a decomposition of the columns of $\mathcal{U}$ into non-singular rainbow sets, as desired.

☐

Remark 5.8. It is not true that it is possible to choose a sequence $\bar{j}$ of of indices so that both $\mathrm{per}((\mathcal{U}^{-1})^T[\bar{j}])$ and $\det(\mathcal{U}[\bar{j}])$ are non-zero. An example, taken from [19], is obtained by taking all ${}^iU$s to be the direct sum of $\frac{n}{2}$ copies of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

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