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Binary linear codes, dimers and hypermatrices

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Abstract

We show that the weight enumerator of any binary linear code is equal to the permanent of a 3-dimensional hypermatrix (3-matrix). We also show that each permanent is a determinant of a 3-matrix. As an application we write the dimer partition function of a finite 3-dimensional cubic lattice as the determinant of the vertex-adjacency 3-matrix of a 2-dimensional simplicial complex which preserves the natural embedding of the cubic lattice.

Keywords: Ising problem, dimer problem, partition function, perfect matching, Kasteleyn matrix, linear binary code, permanent, 3-dimensional cubic lattice, triangular configuration, 2-dimensional simplicial complex.

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1 Introduction

The *Kasteleyn method* is a way to calculate the Ising partition function on a finite graph G, which can be described as follows. We first realize that the Ising partition function is equivalent to a multivariable weight enumerator of the cut space of G. We modify G to obtain a new graph G' so that this weight enumerator is equal to the generating function of the perfect matchings of G' (the dimer partition function of G). Such generating functions are hard to calculate. In particular, if H is a bipartite graph then the generating function of the perfect matchings of H is equal to the permanent of the biadjacency matrix of H. If however this permanent may be turned into the determinant of a modified matrix, then the calculation can be successfully carried over since the determinants may be calculated efficiently. In 1913 Polya asked for a characterization of those non-negative matrices M for which we can change the signs of the entries so that, denoting by M' the resulting matrix, we have $\mathrm{per}(M) = \det(M')$. We call these matrices *Kasteleyn matrices*, after the physicist Kasteleyn who invented the Kasteleyn method. In the 1960's, Kasteleyn proved that all the biadjacency matrices of the planar bipartite graphs are Kasteleyn. We say that a bipartite graph is Pfaffian if its biadjacency matrix is Kasteleyn. The problem of characterizing Kasteleyn matrices (or equivalently Pfaffian bipartite graphs) remained open until 1993, when Robertson, Seymour and Thomas [4] found a polynomial recognition method and a structural description of Kasteleyn matrices. They showed that the class of Kasteleyn matrices is rather small and extends only moderately beyond the class of biadjacency matrices of planar bipartite graphs.

Some years ago, the first author suggested to (1) extend the Pfaffian method to weight enumerators of general binary linear codes, and (2) to use hypermatrices instead of matrices to gain new insight into the Ising and dimer problems for the cubic lattice.

In this paper we show that the weight enumerator of any binary linear code is equal to the permanent of the triadjacency 3-matrix of a 2-dimensional simplicial complex. In analogy to the standard (2-dimensional) matrices, we say that a 3-dimensional non-negative matrix A is *Kasteleyn* if the signs of its entries may be changed so that, denoting by A' the resulting 3-dimensional matrix, we have $\mathrm{per}(A) = \det(A')$. We show that, in contrast with the 2-dimensional case, the class of Kasteleyn 3-dimensional matrices is rich; namely, for each 2-dimensional non-negative matrix M there is a 3-dimensional non-negative Kasteleyn matrix A so that $\mathrm{per}(M) = \mathrm{per}(A)$.

Summarising, we have the following applications for the basic 3-dimensional statistical physics models.

- We write the partition function of the dimer problem in the cubic lattice as a 3-determinant.
- We write the partition function of the Ising problem in the cubic lattice as a 3-permanent.

Using some results of the present paper, the first author showed in [3] how to write both these partition functions as a single formal product.

1.1 Basic definitions

A linear code $\mathcal{C}$ of length n and dimension d over a field $\mathbb{F}$ is a linear subspace of dimension d of the vector space $\mathbb{F}^n$. Each vector in $\mathcal{C}$ is called a codeword. The weight $w(c)$ of a codeword c is the number of non-zero entries of c; if we are given $w(i)$, $i = 1, \ldots, n$, then $w(c) = \sum_{i:c_i=1} w(i)$. The weight enumerator of a finite code $\mathcal{C}$ is defined by the formula

$$W_{\mathcal{C}}(x, w) := \sum_{c \in \mathcal{C}} x^{w(c)}.$$

A simplex X is the convex hull of an affine independent set V in $\mathbb{R}^d$. The dimension of X is $|V| - 1$, denoted by $\dim X$. The convex hull of any non-empty subset of V that defines a simplex is called a face of the simplex. A simplicial complex Δ is a set of simplices fulfilling the following conditions: every face of a simplex of Δ belongs to Δ and the intersection of every two simplices of Δ is a face of both. The dimension of Δ is $\max\{\dim X | X \in \Delta\}$. Let Δ be a d-dimensional simplicial complex. We define the incidence matrix $I = (I_{ij})$ as follows: the rows are indexed by $(d-1)$-dimensional simplices and the columns are indexed by d-dimensional simplices. We set $I_{ij} := 1$ if the $(d-1)$-simplex i belongs to the d-simplex j, and $I_{ij} := 0$ otherwise.

This paper studies 2-dimensional simplicial complexes where each maximal simplex is a triangle. We call them triangular configurations. We denote the set of vertices of Δ by $V(\Delta)$, the set of edges by $E(\Delta)$ and the set of triangles by $T(\Delta)$. The cycle space of Δ over a field $\mathbb{F}$, denoted $\ker_{\mathbb{F}} \Delta$, is the kernel of the incidence matrix A of Δ over $\mathbb{F}$, that is $\{x | Ax =_{\mathbb{F}} 0\}$.

Let Δ be a triangular configuration. A matching of Δ is defined with respect to edges; hence, a matching of Δ is a subconfiguration M of Δ such that $t_1 \cap t_2$ does not contain an edge for every distinct $t_1, t_2 \in T(M)$. Let Δ be a triangular configuration. Let M be a matching of Δ. Then the defect of M is the set $E(T) \setminus E(M)$. A perfect matching of Δ is a matching with empty defect.

We denote the set of all perfect matchings of Δ by $\mathcal{P}(\Delta)$. Let $w : T(\Delta) \mapsto \mathbb{R}$ be the map giving the weights of the triangles of Δ. The generating function of perfect matchings in Δ is defined to be $P_\Delta(x, w) = \sum_{P\in\mathcal{P}(\Delta)} x^{w(P)}$, where $w(P) := \sum_{t\in P} w(t)$.

A triangular configuration Δ is tripartite if the set of the edges of Δ can be partitioned into three disjoint sets E_1, E_2, E_3 such that every triangle of Δ contains edges from all sets E_1, E_2, E_3. We call the sets E_1, E_2, E_3 a tripartition of Δ.

We recall that the biadjacency matrix of a bipartite graph $G = (V, W, E)$ is the $|V| \times |W|$ matrix $A(x) = (a_{ij})$ defined as follows: we set $a_{ij} := x^{w(e)}$ if $v_i \in V, v_j \in W$ form an edge e with weight $w(e)$, and $a_{ij} := 0$ otherwise.

The triadjacency 3-matrix of a tripartite triangular configuration Δ with tripartition E_1, E_2, E_3 is the $|E_1| \times |E_2| \times |E_3|$ three dimensional array of numbers $A_\Delta(x, w) = A(x, w) = (a_{ijk})$ defined as follows: we set

$$a_{ijk} := \begin{cases} x^{w(t)} & \text{if } e_i \in E_1, e_j \in E_2, e_k \in E_3 \text{ form a triangle } t \text{ with weight } w(t), \\ 0 & \text{otherwise.} \end{cases}$$

The permanent of a $n \times n \times n$ 3-matrix A is defined to be

$$\mathrm{per}(A) = \sum_{\sigma_1,\sigma_2\in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

The determinant of a $n \times n \times n$ 3-matrix A is defined to be

$$\det(A) = \sum_{\sigma_1,\sigma_2\in S_n} \mathrm{sign}(\sigma_1)\mathrm{sign}(\sigma_2) \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

1.2 Main results

Theorem 1.1 *Let $\mathcal{C}$ be a binary linear code of length n and let $w(i), i = 1, \ldots, n$, be given. Then there exists a tripartite triangular configuration Δ and a map of the weights $w' : T(\Delta) \mapsto \mathbb{R}$ such that, if $A_\Delta(x, w')$ is the triadjacency matrix of Δ, then*

$$\mathrm{per}(A_\Delta(x, w')) = W_\mathcal{C}(x, w).$$

Proof. Follows from Theorems 1.2, 1.3 and 1.4 below. □

Theorem 1.2 *Let $\mathcal{C}$ be a binary linear code of length n and let $w(i), i = 1, \ldots, n$, be given. Then there exists a triangular configuration Δ and a map of the weights $w' : T(\Delta) \mapsto \mathbb{R}$ such that $W_{\mathcal{C}}(x, w)$ is equal to the weight enumerator $W_K(x, w')$ of $K = \ker_{\mathbb{F}_2} \Delta$.*

Proof. This follows from Theorem 6 of [5] by setting the weights of all the auxiliary triangles to zero. □

This theorem is extended to linear codes over $GF(p)$, where p is a prime, in [6].

Theorem 1.3 (Rytíř [5]) *Let Δ be a triangular configuration with map of the weights $w : T(\Delta) \mapsto \mathbb{R}$. Let $K = \ker_{\mathbb{F}_2} \Delta$. Then there exists a triangular configuration Δ' and map of the weights $w' : T(\Delta') \mapsto \mathbb{R}$ such that $W_K(x, w) = P_{\Delta'}(x, w')$.*

Theorem 1.4 *Let Δ be a triangular configuration with map of the weights $w : T(\Delta) \mapsto \mathbb{R}$. Then there exists a tripartite triangular configuration Δ' and a map of the weights $w' : T(\Delta') \mapsto \mathbb{R}$ such that $P_\Delta(x, w) = \text{per}(A_{\Delta'}(x, w'))$, where $A_{\Delta'}(x, w')$ is the triadjacency matrix of Δ'.*

Proof. Follows directly from Proposition 2.10 and Proposition 2.11 of Section 2. □

Definition 1.5 We say that an $n \times n \times n$ 3-matrix A is *Kasteleyn* if there is a 3-matrix A' obtained from A by changing signs to some of the entries so that $\text{per}(A) = \det(A')$.

The following Theorem 1.6 is proved in Section 3.

Theorem 1.6 *Let M be a $n \times n$ matrix. Then one can construct a $m \times m \times m$ Kasteleyn 3-matrix A with $m \leq n^2 + 2n$ and such that $\text{per}(M) = \text{per}(A)$. Moreover, the Kasteleyn signing is trivial, i.e., $\text{per}(A) = \det(A)$, and if M is non-negative then A is non-negative.*

In the last section, applying Theorem 1.6, we write the dimer partition function of a finite 3-dimensional cubic lattice as the determinant of the vertex-adjacency 3-matrix of a 2-dimensional simplicial complex which preserves the natural embedding of the cubic lattice. We also include the Binet-Cauchy formula for the determinant of a 3-matrix.

2 Triangular configurations and permanents

In this section we prove Theorem 1.4. We use basic building blocks as in Rytíř [5]. However, the use is novel and we need to stress the tripartiteness of basic blocks. Hence we briefly describe them again.

2.1 Triangular tunnel

Triangular tunnel is a triangular configuration consisting of six triangles, best defined by Figure 1 where thin dotted lines depict identification. Let us denote the triangular tunnel of Figure 1 by R. An *empty triangle* is a set of three edges forming a boundary of a triangle. For example, sets $\{a, b, c\}$ and $\{a', b', c'\}$ of edges of R are empty triangles. We say that empty triangle $\{a, b, c\}$ is $\{a, b, c\}$ ending of R, and analogously empty triangle $\{a', b', c'\}$ is $\{a', b', c'\}$ ending of R.

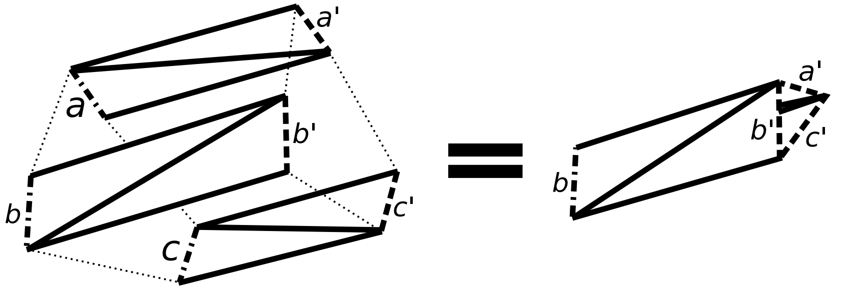

Fig. 1. Triangular tunnel

Proposition 2.1 *The triangular tunnel has exactly one matching M^L with defect $\{a, b, c\}$ and exactly one matching M^R with defect $\{a', b', c'\}$.*

Proposition 2.2 *The triangular tunnel is tripartite.*

Proof. See Figure 2: one partity is $\{a, b, c, a', b', c'\}$ and another partity is formed by the three 'diagonal' edges. □

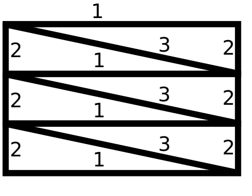

Fig. 2. Tunnel tripartition

2.2 Triangular configuration S^5

The triangular configuration S^5 is depicted in Figure 3. The letter "X" denotes an empty triangle. These empty triangles will be called ending empty triangles.

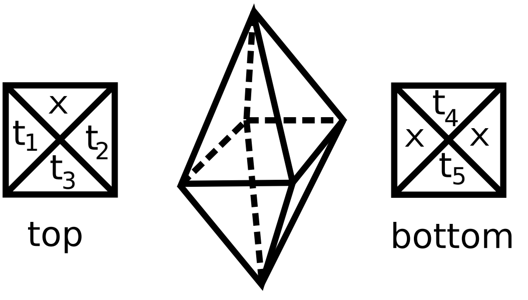

Fig. 3. Triangular configuration S^5

Proposition 2.3 *The triangular configuration S^5 has exactly one perfect matching and exactly one matching whose defect is the set of the edges of all empty triangles.*

Proof. The unique perfect matching is $\{t_1, t_2, t_4, t_5\}$. We denote it by $M^1(S^5)$. The unique matching whose defect is the set of the edges of all empty triangles is $\{t_3\}$. We denote it by $M^0(S^5)$. □

Proposition 2.4 *The triangular configuration S^5 is tripartite.*

Proof. Follows from the inspection of Figure 4. □

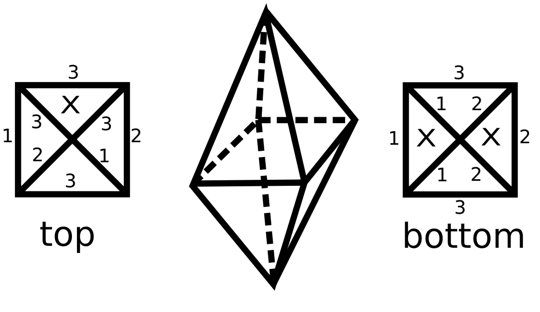

Fig. 4. Triangular configuration S^5 with partitioning

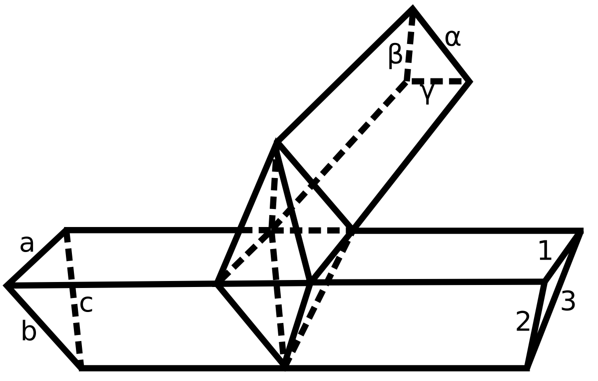

Fig. 5. Matching triangular triangle

2.3 Matching triangular triangle

A *matching triangular triangle* is obtained from the triangular configuration S^5 and three triangular tunnels in the following way. Let T_1, T_2 and T_3 be triangular tunnels. Let $t_1^{T_1}, p^{T_1}$, $t_1^{T_2}, q^{T_2}$ and $t_1^{T_3}, r^{T_3}$ be the ending empty triangles of T_1, T_2 and T_3, respectively. Let $t_1^{S^5}, t_2^{S^5}, t_3^{S^5}$ be the ending empty triangles of S^5. We identify $t_1^{T_1}$ with $t_1^{S^5}$, $t_1^{T_2}$ with $t_2^{S^5}$ and $t_1^{T_3}$ with $t_3^{S^5}$. The matching triangular triangle is defined as $T_1 \cup S^5 \cup T_2 \cup T_3$. Ending empty triangles of this matching triangular triangle are p^{T_1}, q^{T_2}, r^{T_3}. A matching triangular triangle is depicted in Figure 5, where ending empty triangles are $\{1, 2, 3\}$, $\{a, b, c\}$, $\{\alpha, \beta, \gamma\}$.

Proposition 2.5 *A matching triangular triangle has exactly one perfect matching M^1 and exactly one matching M^0 with defect $\{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$. It has no matching with defect E, where $\emptyset \neq E \subsetneq \{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$.*

Proof. The perfect matching is $M^1 := M^1(S^5) \cup M^L(T_1) \cup M^L(T_2) \cup M^L(T_3)$. The matching M^0 is $M^0(S^5) \cup M^R(T_1) \cup M^R(T_2) \cup M^R(T_3)$. Any matching of a matching triangular triangle with defect $E \subset \{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$ contains $M^1(S^5)$ or $M^0(S^5)$. This determines remaining triangles in a matching with defect $E \subseteq \{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$. Hence, there are just two matchings M^1 and M^0 with defect $E \subseteq \{1, 2, 3, a, b, c, \alpha, \beta, \gamma\}$. □

Proposition 2.6 *A matching triangular triangle T is tripartite and there is a tripartition of T such that $a, b, c \in E_1$, $1, 2, 3 \in E_2$, and $\alpha, \beta, \gamma \in E_3$.*

Proof. Follows by inspection of Figure 4 and Figure 2. □

2.4 Linking three triangles by matching triangular triangle

Let Δ be a triangular configuration. Let t_1, t_2 and t_3 be three edge-disjoint triangles of Δ.

The *link by matching triangular triangle* between t_1, t_2 and t_3 in Δ is the triangular configuration Δ' defined as follows. Let T be a matching triangular triangle as defined in Section 2.3. Let $\{a, b, c\}, \{1, 2, 3\}, \{\alpha, \beta, \gamma\}$ be the ending empty triangles of T. Let t_1^1, t_1^2, t_1^3 and t_2^1, t_2^2, t_2^3 and t_3^1, t_3^2, t_3^3 be the edges of t_1 and t_2 and t_3, respectively. We relabel the edges of T so that $\{a, b, c\} = \{t_1^1, t_1^2, t_1^3\}$ and $\{1, 2, 3\} = \{t_2^1, t_2^2, t_2^3\}$ and $\{\alpha, \beta, \gamma\} = \{t_3^1, t_3^2, t_3^3\}$. We set $\Delta' := \Delta \cup T$.

2.5 Construction

Let Δ be a triangular configuration and let $w : T(\Delta) \mapsto \mathbb{R}$ be the associated map of the weights. We construct a tripartite triangular configuration Δ' and a map of the weights $w' : T(\Delta') \mapsto \mathbb{R}$ in two steps.

First, we start with the triangular configuration

$$\Delta_1' := \Delta_1 \cup \Delta_2 \cup \Delta_3,$$

where $\Delta_1, \Delta_2, \Delta_3$ are disjoint copies of Δ. Let t be a triangle of Δ. We denote the corresponding copies of t in $\Delta_1, \Delta_2, \Delta_3$ with t_1, t_2, t_3, respectively.

Second, for every triangle t of Δ, we link t_1, t_2, t_3 in Δ_1' by matching triangular triangle that will be called T_t. Then we remove triangles t_1, t_2, t_3 from Δ_1'. We choose a triangle t' from $M^1(T_t)$ and set $w'(t') := w(t)$. We set $w'(t') := 0$ for $t' \in T(T_t) \setminus \{t\}$. The resulting configuration is the desired configuration Δ'.

Proposition 2.7 *The triangular configuration Δ' is tripartite.*

Proof. The triangular configuration Δ' is constructed from three disjoint triangular configurations $\Delta_1, \Delta_2, \Delta_3$. From these configurations all triangles are removed. Hence, we put edges $E(\Delta_i)$ to set E_i for $i = 1, 2, 3$. The remainder of Δ' is formed by matching triangular triangles. Every matching triangular triangle connects edges of $\Delta_1, \Delta_2, \Delta_3$. By Proposition 2.6 the matching triangular triangle has a tripartition such that the edges of its ending empty triangles belong to different partities; Hence the constructed sets E_1, E_2, E_3 can be extended to a tripartition of Δ'. $\square$

We denote with 2^X the set of all subsets X. We define a mapping f :

$2^{T(\Delta)} \mapsto 2^{T(\Delta')}$ as follows. Let S be a subset of $T(\Delta)$; then

$$f(S) := \{M^1(T_t)|t \in S\} \cup \{M^0(T_t)|t \in T(\Delta) \setminus S\}.$$

Proposition 2.8 *The mapping f is a bijection between the set of perfect matchings of Δ and the set of perfect matchings of Δ' and $w(M) = w'(f(M))$ for every $M \subseteq T(\Delta)$.*

Proof. By definition, the mapping f is an injection. By Proposition 2.5, every inner edge of T_t, $t \in T(\Delta)$, is covered by $f(S)$ for any subset S of $T(\Delta)$. Let M be a perfect matching of Δ. We show that $f(M)$ is perfect matching of Δ'.

$$f(M) = \{M^1(T_t)|t \in M\} \cup \{M^0(T_t)|t \in T(\Delta) \setminus M\}.$$

Let e be an edge of Δ and let e_1, e_2, e_3 be corresponding copies in $\Delta_1, \Delta_2, \Delta_3$. Let $t_1, t_2, \dots, t_l$ be triangles incident with edge e in Δ. Let t_k be the triangle from perfect matching M incident with e. By definition of Δ', the edges e_1, e_2, e_3 are incident only with triangles of T_{t_i}, $i = 1, \dots, l$. The edges e_1, e_2, e_3 are covered by $M^1(t_k)$. The edges of T_{t_i}, $i = 1, \dots, l$, $i \neq k$ are covered by $M^0(T_{t_i})$. Hence $f(M)$ is a perfect matching of Δ'.

Let M' be a perfect matching of Δ'. By Proposition 2.5, $M' = \{M^1(T_t)|t \in S\} \cup \{M^0(T_t)|t \in T(\Delta) \setminus S\}$ for some set S. The set S is a perfect matching of Δ. Thus, the mapping f is a bijection. □

Corollary 2.9 $P_\Delta(x, w) = P_{\Delta'}(x, w')$.

Proposition 2.10 *Let Δ be a triangular configuration with map of the weights $w : T(\Delta) \mapsto \mathbb{R}$. Then there exist a tripartite triangular configuration Δ' and map of the weights $w' : T(\Delta') \mapsto \mathbb{R}$ such that there is a bijection f between the set of perfect matchings $\mathcal{P}(\Delta)$ and the set of perfect matchings of $\mathcal{P}(\Delta')$. Moreover, $w(M) = w'(f(M))$ for every $M \in \mathcal{P}(\Delta)$, and $P_\Delta(x, w) = P_{\Delta'}(x, w')$.*

Proof. Follows directly from Propositions 2.7 and 2.8 and Corollary 2.9. □

Proposition 2.11 *Let Δ be a tripartite triangular configuration with tripartition E_1, E_2, E_3 such that $|E_1| = |E_2| = |E_3|$ and with map of the weights $w : T(\Delta) \mapsto \mathbb{R}$. Let $A_\Delta(x, w)$ be its triadjacency matrix. Then $P_\Delta(x, w) =$ $\mathrm{per}(A_\Delta(x, w))$.*

Proof. We have

$$\mathrm{per}(A_\Delta(x)) = \sum_{\sigma_1, \sigma_2 \in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

Every perfect triangular matching between partities E_1, E_2, E_3 can be encoded by two permutations σ_1, σ_2 and vice versa. If matching M is a subset of $T(\Delta)$, then

$$\prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)} = \prod_{i=1}^{n} x^{w([i\sigma_1(i)\sigma_2(i)])} = x^{w(M)},$$

where $[ijk]$ denotes a triangle of Δ with edges i, j, k. If M is not a subset of $T(\Delta)$, then there is i such that $a_{i\sigma_1(i)\sigma_2(i)} = 0$. Hence $\prod_{i=1}^{n} x^{w([i\sigma_1(i)\sigma_2(i)])} = 0$. Therefore

$$\sum_{\sigma_1,\sigma_2 \in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)} = P_\Delta(x, w).$$

□

3 Kasteleyn 3-matrices

We first introduce a necessary condition for a 3-matrix to be Kasteleyn. Let A be a $|V_0| \times |V_1| \times |V_2|$ non-negative 3-matrix, where $|V_i| = m, i = 1, 2, 3$. We first define two bipartite graphs G_1, G_2 as follows. We let, for $i = 1, 2$, $G_i^A = G_i = (V_0, V_i, E_i)$ where

$$E_1 = \{\{a, b\} | a \in V_0, b \in V_1 \text{ and } A_{abc} \neq 0 \text{ for some } c\},$$

and

$$E_2 = \{\{a, c\} | a \in V_0, c \in V_2 \text{ and } A_{abc} \neq 0 \text{ for some } b\}.$$

Theorem 3.1 *If A is such that both G_1^A, G_2^A are Pfaffian bipartite graphs then A is Kasteleyn.*

Proof. Let M_i be the biadjacency matrix of G_i and let $\text{sign}_i : E(G_i^A) \mapsto \{-1, 1\}$ be the signing of the entries of M_i which defines matrix M_i' such that $\text{per}(M_i) = \det(M_i')$. We define 3-matrix A' by

$$A'_{abc} = \text{sign}_1(\{a, b\})\text{sign}_2(\{a, c\})A_{abc}.$$

We have

$$\det(A') = \sum_{\sigma_1} \text{sign}(\sigma_1) \times \sum_{\sigma_2} \text{sign}(\sigma_2) \prod_j \text{sign}_2(\{j, \sigma_2(j)\})[\text{sign}_1(\{j, \sigma_1(j)\})A_{j\sigma_1(j)\sigma_2(j)}].$$

By the construction of sign_2 we have that for each σ_2 and each σ_1, if $\prod_j A_{j\sigma_1(j)\sigma_2(j)} \neq 0$ then

$$\text{sign}(\sigma_2)\prod_j \text{sign}_2(\{j,\sigma_2(j)\}) = 1.$$

Hence

$$\begin{aligned}\det(A') &= \sum_{\sigma_1}\text{sign}(\sigma_1)\times\sum_{\sigma_2}\prod_j[\text{sign}_1(\{j,\sigma_1(j)\})A_{j\sigma_1(j)\sigma_2(j)}]\\ &= \sum_{\sigma_2}\times\sum_{\sigma_1}\text{sign}(\sigma_1)\prod_j \text{sign}_1(\{j,\sigma_1(j)\})A_{j\sigma_1(j)\sigma_2(j)}.\end{aligned}$$

Analogously by the construction of sign_1 we have that for each σ_1 and each σ_2, if $\prod_j A_{j\sigma_1(j)\sigma_2(j)} \neq 0$ then

$$\text{sign}(\sigma_1)\prod_j \text{sign}_1(\{j,\sigma_1(j)\}) = 1.$$

Hence

$$\det(A') = \sum_{\sigma_1,\sigma_2}\prod_j A_{j\sigma_1(j)\sigma_2(j)} = \text{per}(A).$$

□

In the introduction we defined the triadjacency 3-matrix of a triangular configuration as the adjacency matrix of the *edges* of the triangles. We also defined a *matching* of a triangular configuration as a set of edge-disjoint triangles. In this section we concentrate on the vertices rather than on the edges.

A triangular configuration Δ is *vertex-tripartite* if vertices of Δ can be divided into three disjoint sets V_1, V_2, V_3 such that every triangle of Δ contains one vertex from each set V_1, V_2, V_3. We call the sets V_1, V_2, V_3 vertex-tripartition of Δ.

The *vertex-adjacency* 3-matrix $A(x) = (a_{ijk})$ of a vertex-tripartite triangular configuration Δ with vertex-tripartition V_1, V_2, V_3 is defined as follows: We set

$$a_{ijk} := \begin{cases} x^{w(t)} & \text{if } i\in V_1, j\in V_2, k\in V_3 \text{ forms a triangle } t \text{ with weight } w(t),\\ 0 & \text{otherwise.}\end{cases}$$

We will need the following modification of the notion of a matching. A set of triangles of a triangular configuration is called *strong matching* if its triangles are mutually vertex-disjoint.

Proof. [Proof of Theorem 1.6] Let M be a $n \times n$ matrix and let $G = (V_1, V_2, E)$ be the bipartite graph of its non-zero entries. We have $|V_1| = |V_2| = n$. We order vertices of each V_i, $i = 1, 2$ arbitrarily and let $V_i = \{v(i,1), \ldots, v(i,n)\}$. Let $V_i' = \{v'(i,1), \ldots, v'(i,n)\}$ be disjoint copy of V_i, $i = 1, 2$.

We next define three sets of vertices W_1, W_2, W_0 and system of triangles $\Delta(G) = \Delta$ so that each triangle intersects each W_i in exactly one vertex.

$$\begin{aligned}
W_1 &= V_1 \cup V_1' \cup \{w(1,e) | e \in E\}, \\
W_2 &= V_2 \cup V_2' \cup \{w(2,e) | e \in E\}, \\
W_0 &= \{w(0,e) | e \in E\} \cup \{w(0,i,j) | i = 1,2; j = 1, \ldots, n\},
\end{aligned}$$

$$\begin{aligned}
\Delta = \cup_{e=ab \in E} & \{(a, b, w(0,e)), (w(0,e), w(1,e), w(2,e))\} \cup \\
& \cup_{j=1}^{n} \{(w(0,1,j), v'(2,j), w(1,e)) | v(1,j) \in e\} \cup \\
& \cup_{j=1}^{n} \{(w(0,2,j), v'(1,j), w(2,e)) | v(2,j) \in e\}.
\end{aligned}$$

We let A be the vertex-adjacency 3-matrix of the triangular configuration $\mathcal{T}(G) = \mathcal{T} = (W_0, W_1, W_2, \Delta)$. We first observe that both bipartite graphs G_1, G_2 of A (introduced before Theorem 3.1) are planar; let us consider only G_1, the reasoning for G_2 is the same. First, vertices $v'(1,j)$ and $w(0,2,j)$ are connected only among themselves in G_1. Further, the component of G_1 containing vertex $v(1,j)$ contains also vertex $w(0,1,j)$ and consists of $deg_G(v(1,j))$ disjoint paths of length 3 between these two vertices. Here $deg_G(v(1,j))$ denotes the degree of $v(1,j)$ in graph G, i.e., the number of edges of G incident with $v(1,j)$. Thus, by Theorem 3.1, A is Kasteleyn.

We next observe that Kasteleyn signing is trivial. Let D_1 be the orientation of G_1 in which each edge is directed from W_0 to W_1. In each planar drawing of G_1, each inner face has an odd number of edges directed in D_1 clockwise. This means that D_1 is a *Pfaffian orientation* of G_1, and $\mathrm{per}(A) = \det(A)$ (see e.g. Loebl [2] for basic facts on Pfaffian orientations and Pfaffian signings).

Finally there is a bijection between the perfect matchings of G and the perfect strong matchings of $\mathcal{T}$: if $P \subset E$ is a perfect matching of G then let

$$P(\mathcal{T}) = \{(a, b, w(0,e)) | e = ab \in P\}.$$

We observe that $P(\mathcal{T})$ can be uniquely extended to a perfect strong matching of $\mathcal{T}$, namely by the set of triples $S_1 \cup S_2 \cup S_3$ where

$$S_1 = \cup_{e \in E \setminus P}\{(w(0,e), w(1,e), w(2,e))\},$$
$$S_2 = \cup_{j=1}^{n}\{(w(0,1,j), v'(2,j), w(1,e)) | v(1,j) \in e \in P\},$$
$$S_3 = \cup_{j=1}^{n}\{(w(0,2,j), v'(1,j), w(2,e)) | v(2,j) \in e \in P\}.$$

Set S_1 is inevitable in any perfect strong matching containing $P(\mathcal{T})$ since the vertices $w(0,e); e \notin P$ must be covered. This immediately implies that sets S_2, S_3 are inevitable as well.

On the other hand, if Q is a perfect strong matching of $\mathcal{T}$ then Q contains $P(\mathcal{T})$ for some perfect matching P of G. □

4 Application to 3D dimer problem

Let Q be cubic $n \times n \times n$ lattice. The dimer partition function of Q, which is equal to the generating function of the perfect matchings of Q, can be identified (by Theorem 1.6) with the determinant of the Kasteleyn vertex-adjacency matrix of triangular configuration $\mathcal{T}(Q)$. Natural question arises whether this observation can be used to study the 3D dimer problem.

We first observe that the natural embedding of Q in 3-space can be simply modified to yield an embedding of $\mathcal{T}(Q)$ in 3-space. This can perhaps best be understood by figures, see Figure 6; this figure depicts configuration $\mathcal{T}(Q)$ around vertex v of Q with neighbors $u_1, \ldots, u_6$.

Triangular configuration $\mathcal{T}(Q)$ is obtained by identification of vertices $v_i, i = 1, \ldots, 6$ in the left and right parts of Figure 6. Now assume that the embedding of left part of Figure 6 is such that for each vertex v of Q, the vertices $v_1, \ldots, v_6$ belong to the same plane and the convex closure of $v_1, \ldots, v_6$ intersects the rest of the configuration only in $v_1, \ldots, v_6$. Then we add the embedding of the right part, for each vertex v of Q, so that x_1 belongs to the plane of the v_i's and x_2 is very near to x_1 but outside of this plane.

Summarizing, the dimer partition function of a finite 3-dimensional cubic lattice Q may be written as the determinant of the vertex-adjacency 3-matrix of triangular configuration $\mathcal{T}(Q)$ which preserves the natural embedding of the cubic lattice.

Calculating the determinant of a 3-matrix is hard, but perhaps formulas for the determinant of the particular vertex-adjacency 3-matrix of $\mathcal{T}(Q)$, illuminating the 3-dimensional dimer problem, may be found. An example of a formula valid for the determinant of a 3-matrix is shown in the next subsection. It is new as far as we know but its proof is basically identical to the

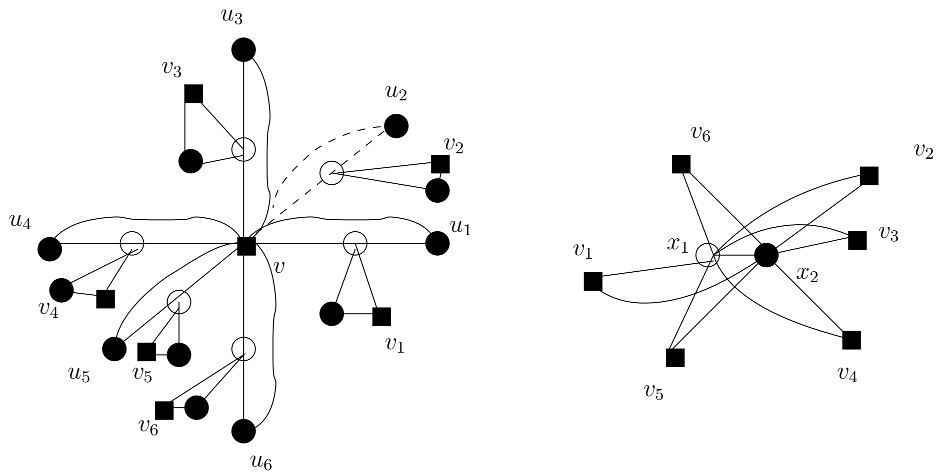

Fig. 6. Configuration $\mathcal{T}(Q)$ around vertex v of Q with neighbors $u_1, \ldots, u_6$ so that u_1, u_3, u_4, u_6 belong to the same plane in the 3-space, u_2 is 'behind' this plane and u_5 is 'in front of' this plane. Empty vertices belong to W_0, square vertices belong to W_1 and full vertices belong to W_2.

proof of Lemma 3.3 of Barvinok [1].

4.1 Binet-Cauchy formula for the determinant of 3-matrices

We recall from the introduction that the determinant of a $n \times n \times n$ 3-matrix A is defined to be

$$\det(A) = \sum_{\sigma_1, \sigma_2 \in S_n} \text{sign}(\sigma_1)\text{sign}(\sigma_2) \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}.$$

The next formula is a generalization of Binet-Cauchy formula (see the proof of Lemma 3.3 in Barvinok [1]).

Lemma 4.1 *Let A^1, A^2, A^3 be real $r \times n$ matrices, $r \leq n$. For a subset $I \subset \{1, \ldots, n\}$ of cardinality r we denote by A^s_I the $r \times r$ submatrix of the matrix A^s consisting of the columns of A^s indexed by the elements of the set I. Let C be the 3-matrix defined, for all i_1, i_2, i_3 by*

$$C_{i_1,i_2,i_3} = \sum_{j=1}^{n} A^1_{i_1,j} A^2_{i_2,j} A^3_{i_3,j}.$$

Then

$$\det(C) = \sum_I \operatorname{per}(A^1_I)\det(A^2_I)\det(A^3_I),$$

where the sum is over all subsets $I \subset \{1, \dots, n\}$ *of cardinality* r

Proof.

$$\begin{aligned}\det(C) &= \sum_{\sigma_1,\sigma_2 \in S_r} \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2) \prod_{i=1}^r \sum_{j=1}^n A^1_{i,j} A^2_{\sigma_1(i),j} A^3_{\sigma_2(i),j} \\ &= \sum_{\sigma_1,\sigma_2 \in S_r} \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2) \times \sum_{1 \le j_1,\dots,j_r \le n} \prod_{i=1}^r A^1_{i,j_i} A^2_{\sigma_1(i),j_i} A^3_{\sigma_2(i),j_i} \\ &= \sum_{1 \le j_1,\dots,j_r \le n} \sum_{\sigma_1,\sigma_2 \in S_r} \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2) \times \prod_{i=1}^r A^1_{i,j_i} A^2_{\sigma_1(i),j_i} A^3_{\sigma_2(i),j_i}.\end{aligned}$$

Now, for all $J = (j_1, \dots, j_r)$ we have

$$\begin{aligned}&\sum_{\sigma_1,\sigma_2 \in S_r} \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2) \times \prod_{i=1}^r A^1_{i,j_i} A^2_{\sigma_1(i),j_i} A^3_{\sigma_2(i),j_i} \\ =&(\prod_{i=1}^r A^1_{i,j_i})(\sum_{\sigma_1} \operatorname{sign}(\sigma_1) \prod_{i=1}^r A^2_{\sigma_1(i),j_i})(\sum_{\sigma_2} \operatorname{sign}(\sigma_2) \prod_{i=1}^r A^3_{\sigma_2(i),j_i}) \\ =&(\prod_{i=1}^r A^1_{i,j_i}) \det(\tilde{A}^2_J)\det(\tilde{A}^3_J),\end{aligned}$$

where $\tilde{A}^s_J$ denotes the $r \times r$ matrix whose ith column is the j_ith column of matrix A^s.

If sequence J contains a pair of equal numbers then the corresponding summand is zero, since $\det(\tilde{A}^2_J)$ is zero. Moreover, if J is a permutation, and J' is obtained from J by a transposition, then

$$\det(\tilde{A}^2_J)\det(\tilde{A}^3_J) = \det(\tilde{A}^2_{J'})\det(\tilde{A}^3_{J'}).$$

Therefore Lemma 4.1 follows. □

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