Directed Cycle Double Covers: Hexagon Graphs

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Abstract

Jaeger's directed cycle double cover conjecture can be formulated as a problem of existence of special perfect matchings in a class of graphs that we call hexagon graphs. A hexagon graph can be associated with any cubic graph. We show that the hexagon graphs of cubic *bridgeless graphs* are braces that can be generated from the ladder on 8 vertices using two types of McCuaig's augmentations.

1 Introduction

The long-standing Jaeger's directed cycle double cover conjecture [1] (DCDC conjecture in short) is broadly considered to be among the most important open problems in graph theory. A typical formulation asks whether every 2-connected graph admits a family of cycles such that one may prescribe an orientation on each cycle of the family in such a way that each edge e of the graph belongs to exactly two cycles and these cycles induce opposite orientations on e. In order to prove the DCDC conjecture, a wide variety of approaches have arisen [1, 7], including a topological approach. The topological formulation of the DCDC conjecture is as follows: every cubic bridgeless graph admits an embedding in a closed Riemann surface such that every edge belongs to exactly two distinct face boundaries defined by the embedding; that is, with no dual loop.

In this work, we formulate the DCDC conjecture as a problem of existence of special perfect matchings in a class of graphs that we call hexagon graphs. Our initial motivation for the formulation of the DCDC conjecture using hexagon graphs are critical embeddings [2, 5], which are embeddings with no dual loop, in particular.

The main goal of this work is to discuss recent progress on the study of the structure and generation of hexagon graphs. The class of the hexagon graphs of the cubic bridgeless graphs turns out to be a subclass of braces. The class of braces, along with bricks, are a fundamental class of graphs in matching theory, mainly because they are building blocks of a perfect matching decomposition procedure, namely of the tight cut decomposition procedure [3]. In [4], McCuaig introduced a method for generating all braces starting from a large base set of graphs and recursively making use of 4 distinct types of operations. In this paper, we show that the hexagon graphs arising from cubic bridgeless graphs are braces that can be generated from the ladder on 8 vertices using 2 types of McCuaig's operations.

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2 Preliminaries

2.1 Braces and McCuaig's operations

A *brace* is a simple (i.e. no loops and no multiple edges), connected, bipartite graph on at least six vertices such that for every pair of non-adjacent edges, there is a perfect matching containing the pair of edges. In [4], McCuaig presented a method for generating braces. He showed that all braces can be constructed from a base set using four operations. In the following we sketch McCuaig's method for generating braces.

Let H be a bipartite graph and x be a vertex of H of degree at least 4. Let N_1, N_2 be a partition of $N_H(x)$ such that $|N_1|, |N_2| \geq 2$. Let $\{x^1, v, x^2\}$ be a set of vertices such that $\{x^1, v, x^2\} \cap V(H) = \emptyset$. The *expansion of x to x^1vx^2* is the operation composed of the following three steps: (i) delete x, (ii) add the new path x^1vx^2, and (3) connect every vertex of N_1 (N_2, respectively) to the vertex x^1 (x^2, respectively). Note that if H' is a graph obtained from H by the expansion of a vertex, then H' is also bipartite.

Augmentations. If H' is a bipartite graph obtained from H by adding a new edge, then we say that H' is obtained from H by a *type-1 augmentation.* Let x and w be two vertices in the same partition class of H such that x has degree at least 4. If H' is obtained from H expanding x to x^1vx^2 and adding the new edge vw, then we say that H' is obtained from H by a *type-2 augmentation.* Let x and y be two vertices of H of distinct partition classes such that $d_H(x), d_H(y) \geq 4$. Let H' be the bipartite graph obtained from H expanding x and y to x^1vx^2 and y^1uy^2 respectively, and adding the new edge vu. If x and y are not connected in H, the operation for obtaining H' from H is called a *type-3 augmentation*, otherwise it is called a *type-4 augmentation.*

If H' is obtained from H by a type-i augmentation for some $i \in \{1, 2, 3, 4\}$, then we say that H' is obtained from H by an *augmentation.* If $i \in \{1, 2\}$, then we say that H' is obtained from H by a *simple augmentation* (see Figure 1).

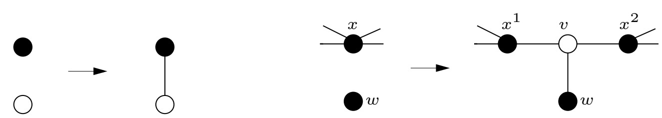

Figure 1: Simple augmentations

Let $\mathcal{B}$ be the infinite set consisting of all bipartite Möbius ladders, ladders, and biwheels. In [4, § 2], the set $\mathcal{B}$ is depicted.

Theorem 1 (McCuaig)**.** *Let H be a bipartite graph. Then H is a brace if and only if there exists a sequence $H_0, H_1, \ldots, H_k$ of bipartite graphs such that $H_0 \in \mathcal{B}$, H_i may be obtained from H_{i-1} by an augmentation for each $i \in \{1, \ldots, k\}$, and $H_k = H$.*

2.2 Rotation systems and embeddings without dual loops

We briefly recall a combinatorial representation of embedding of graphs on closed Riemann surfaces, namely *rotation systems.* Let G be a graph. For each $v \in V(G)$, let π_v be a cyclic permutation of the edges incident with v. A collection $\pi = \{\pi_v : v \in V(G)\}$ is called a *rotation system* of G. The proof of the following seminal theorem of Edmonds can be found in [6, §3.2].

Theorem 2 (Edmonds)**.** *Let π be a rotation system of a graph G. Then π encodes an embedding of G on a closed Riemann surfaces with set of face boundaries*

$$\{e_1e_2 \cdots e_k : e_i = v^iv^{i+1} \in E(G),\ \pi_{v^{i+1}}(e_i) = e_{i+1},\ e_{k+1} = e_1 \text{ and } k \text{ minimal}\}. \tag{1}$$

Moreover, the converse holds. That is, every embedding of G on a closed Riemann surface defines a rotation system π of G, where the set of face boundaries is given by the set described in (1)*.*

3 Hexagon graphs

In this section we introduce hexagon graphs associated with cubic graphs. We refer to the complete bipartite graph $K_{3,3}$ as a *hexagon* and say that a bipartite graph H has a hexagon h if h is a subgraph of H. For a graph G and a vertex v of G, let $N_G(v)$ denote the set of neighbors of v in G.

Definition 1. *Let G be a cubic graph with vertex set V and edge set E. A hexagon graph of G is a graph H obtained from G following the next rules:*

1. *We replace each vertex v in V by a hexagon h_v so that for every pair u, $v \in V$, if $u \neq v$, then h_u and h_v are vertex disjoint. Let $V(H) = \{V(h_v) : v \in V\}$.*
2. *For each vertex $v \in V$, let $\{v_i : i \in \mathbb{Z}_6\}$ denote the vertex set of h_v and $\{v_iv_{i+1}, v_iv_{i+3} : i \in \mathbb{Z}_6\}$ its edge set. With each neighbor u of v in G, we associate an index $i_{v(u)}$ from the set $\{0, 1, 2\} \subset \mathbb{Z}_6$ so that if $N_G(v) = \{u, w, z\}$, then $i_{v(u)}$, $i_{v(w)}$, $i_{v(z)}$ are pairwise distinct.*
3. *(See Figure 2). Let $X = \cup_{v\in V}\{v_{2i} : i \in \mathbb{Z}_6\}$ and $Y = \cup_{v\in V}\{v_{2i+1} : i \in \mathbb{Z}_6\}$. We replace each edge uv in E by two vertex disjoint edges e_{uv}, e'_{uv} so that if both $v_{i_{v(u)}}$, $u_{i_{u(v)}}$ belong to either X or Y, then $e_{uv} = v_{i_{v(u)}}u_{i_{u(v)}+3}$, $e'_{uv} = v_{i_{v(u)}+3}u_{i_{u(v)}}$. Otherwise, $e_{uv} = v_{i_{v(u)}}u_{i_{u(v)}}$, $e'_{uv} = v_{i_{v(u)}+3}u_{i_{u(v)}+3}$. Let $E(H) = \{E(h_v) : v \in V\} \cup \{e_{uv}, e'_{uv} : uv \in E\}$.*

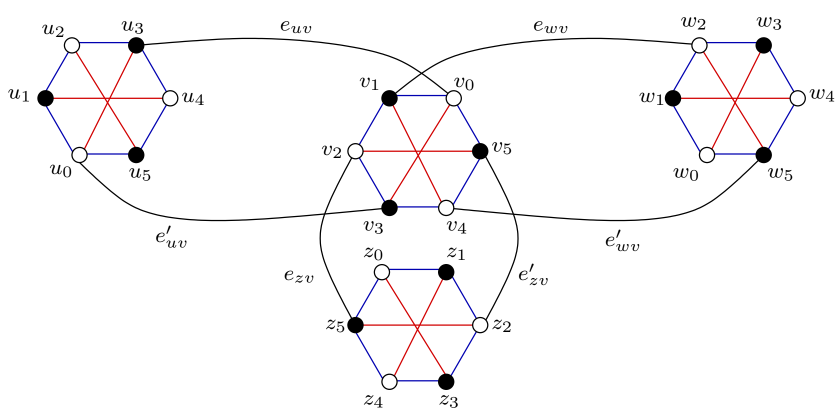

Figure 2: Local representation of a hexagon graph H of a cubic graph G. The hexagon h_v is associated with vertex v, where $N_G(v) = \{u, w, z\}$. Red edges are depicted as red lines, blue edges are depicted as blue lines and white edges as black lines. The set X is represented by white vertices and the set Y by black vertices.

We say that h_v is the hexagon of H associated with the vertex v of G and that $\{h_v : v \in V\}$ is the *set of hexagons of H*. We shall refer to the set of edges $\bigcup_{v\in V}\{v_iv_{i+3} : i \in \mathbb{Z}_6\}$ as the set of *red edges of H*, to the set of edges $\{e_{uv}, e'_{uv} : uv \in E\}$ as the set of *white edges of H*, and finally to the set of edges $\bigcup_{v\in V}\{v_iv_{i+1} : i \in \mathbb{Z}_6\}$ as the set of *blue edges of H* (see Figure 2). Moreover, we shall say that a perfect matching of H containing only blue edges is a *blue perfect matching*.

Let G be a cubic graph and H be a hexagon graph of G. We observe two important properties: (i) H is bipartite; and (ii) if H' is another hexagon graph of G, then H and H' are isomorphic.

Below we reformulate the topological statement of the DCDC conjecture, saying that every cubic bridgeless graph admits an embedding on a closed Riemann surface without dual loops, as the existence of special perfect matching in hexagon graphs.

Let M be a blue perfect matching of H and let W be the set of white edges of H. Each cycle C in $M\Delta W$ induces a subgraph in G defined by the set of edges $\{uv \in E(G) : e_{uv} \in C \text{ or } e'_{uv} \in C\}$. In the next theorem we state that each blue perfect matching of a hexagon graph of a cubic graph G defines an embedding of G on closed Riemann surfaces, and vice versa. The proof is based on a natural bijection between blue perfect matchings and rotation systems.

Theorem 3. *Let G be a cubic graph, H the hexagon graph of G, and W the set of white edges of H. Each blue perfect matching M of H encodes an embedding of G on a closed Riemann surface with set of face boundaries the set of subgraphs of G induced by the cycles in $M\Delta W$. Moreover, the converse holds. That is, each embedding of G on a closed Riemann surface defines a blue perfect matching M of H, where the set of subgraphs of G induced by all cycles in $M\Delta W$ coincides with the set of face boundaries of the embedding.*

The following result is crucial for our approach.

Proposition 4. *Let G be a cubic graph, H the hexagon graph of G, M a blue perfect matching of H, and W the set of white edges of H. The embedding of G encoded by M has a dual loop if and only if there is a cycle in $M\Delta W$ that contains both end vertices of a red edge.*

Proof. An embedding of G has a dual loop if and only if there is an edge $uv \in E(G)$ that belongs to exactly one face boundary, say C'. The face boundary C' is a subgraph of G induced by a cycle C of $M\Delta W$. We have C' is the only subgraph induced by a cycle of $M\Delta W$ that contains uv if and only if e_{uv} and e'_{uv} belong to C. The result follows. □

Motivated by Proposition 4, we shall say that a blue perfect matching M is *safe* if no cycle of $M\Delta W$ contains the end vertices of a red edge. As a direct consequence of Theorem 3 and Proposition 4, we establish the following formulation of the DCDC Conjecture on hexagon graphs.

Corollary 5. *A cubic graph G has a directed cycle double cover if and only if its hexagon graph H admits a safe perfect matching.*

4 Main Results

In this section we describe the structure and generation of hexagon graphs of cubic bridgeless graphs.

Theorem 6. *Let G be a cubic graph. Then the hexagon graph H of G is a brace if and only if G is bridgeless.*

Proof. Let B, W, and R denote the set of blue, white, and red edges, respectively. Moreover, a blue edge is denoted by b, a white edge by w, and a red edge by r. Each pair of disjoint edges, $\{b, b'\}$, $\{r, r'\}$, or $\{b, r\}$, can be simply extended to a perfect matching of H.

We note that each component of $W \cup R$ is a cycle on four vertices, a *square*. Let w, w' be a pair of disjoint white edges. The edges w, w' belong to the same square of $W \cup R$, or to two different squares of $W \cup R$. In either case w, w' can be naturally extended to a perfect matching of H. Similarly, each edge of a pair w, r of disjoint white and red edge belongs to different square of $W \cup R$, and therefore it can be completed into a perfect matching of H.

Finally we consider a pair b, w of disjoint white and blue edge. If the hexagon with b does not contain an end vertex of w, then it is not difficult to extend b, w to a perfect matching of H. Hence, let h_u be the hexagon that contains b and an end vertex of w, and let h_v be the hexagon that contains the other end vertex of w. Let $b = u_i u_{i+1}$, $w = u_k v_j$, where $i, j, k \in \mathbb{Z}_6$.

If $k \notin \{i+3, i+4\}$, then b, w can be completed into a perfect matching of H that contains the edges b, w, and $u_{i+3}u_{i+4}$.

Hence, without loss of generality we can assume that $k = i+3$. Let $e_{uv} = u_i v_{j+3}$ and $e_{uz} = u_{i+1} z_l$ (notation as in Definition 1.*3*), where z is the neighbor of v in G such that the white edge with an end vertex u_{i+1} has an end vertex in h_z, and $l \in \mathbb{Z}_6$. Given that in G, edges uv, uz have a common end vertex u represented by hexagon h_u, edge $b = u_i u_{i+1}$ can be seen as *the transition* between uv, uz, while $u_k u_{k+1}$ can be seen as this transition reversed.

Now let G be bridgeless. We observe that two adjacent edges in a cubic bridgeless graph belong to a common cycle. Let C be such a cycle for uv, uz.

The two possible orientations of C correspond to two disjoint cycles C_b, C_w in H, where $b \in C_b$ and $w \in C_w$; they contain the transition and transition reversed (between uv, uz), respectively. Let M_b be the perfect matching of C_b consisting of all blue edges and M_w be the perfect matching of C_w consisting of all white edges. In particular, $b \in M_b$ and $w \in M_w$. Since each hexagon of H is intersected by $C_b \cup C_w$ either in a pair of disjoint blue edges, or in the empty set, $M_b \cup M_w$ can be extended to a perfect matching of H.

On the other hand, if G has a bridge $e = \{u, v\}$, then let V_1 be the component of $G - e$ containing u. Any perfect matching of G extending b, w must induce a perfect matching of $\cup_{x \in V_1} h_x \setminus \{u_{i+3}\}$, but this set consists of an odd number of vertices and thus no perfect matching containing b, w can exist. □

The following is the main result of this work.

Theorem 7. *Let G be a cubic bridgeless graph and L_8 denote the ladder on 8 vertices. There is a sequence $H_0, H_1, \ldots, H_k$ of braces such that $H_0 = L_8$, H_i can be obtained from H_{i-1} by a simple augmentation for each $i \in \{1, \ldots, k\}$, and H_k is the hexagon graph of G.*

The crucial ingredients in the proof of Theorem 7 are ear decompositions of cubic bridgeless graphs. We give only a rough sketch of the proof, due to space limitation. Let G be a cubic bridgeless graph, H be its hexagon graph, and $G_i = \{x\} \bigcup_{j=0}^{i} P_j$, $i \in \{0, \ldots, p\}$ be an ear decomposition of G. With each intermediate subgraph G_i of the ear decomposition of G we associate an auxiliary graph H'_i. In particular, with (the cycle) G_0 we associate the ladder L_8. For each $i \in \{1, \ldots, p\}$, the auxiliary graph H'_i contains the hexagons h_v of H such that v has degree 3 in G_i. Hence, H'_p contains all hexagons of H and indeed (by construction) it turns out to be isomorphic to H. The proof is based on the fact that for each $i \in \{1, \ldots, p\}$, it is possible to generate H'_i from H'_{i-1} by a sequence of simple augmentations.

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