

Directed Cycle Double Cover Conjecture: Fork Graphs

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Abstract

We explore the well-known Jaeger’s directed cycle double cover conjecture which is equivalent to the assertion that every cubic bridgeless graph has an embedding on a closed orientable surface with no dual loop. We associate each cubic graph G with a novel object H that we call a *hexagon graph*; perfect matchings of H describe all embeddings of G on closed orientable surfaces. The study of hexagon graphs leads us to define a new class of graphs that we call *lean fork-graphs*. Fork graphs are cubic bridgeless graphs obtained from a triangle by sequentially connecting fork-type graphs and performing $Y-\Delta$, $\Delta-Y$ transformations; lean fork-graphs are fork graphs fulfilling a connectivity property. We prove that Jaeger’s conjecture holds for the class of lean fork-graphs. The class of lean fork-graphs is rich; namely, for each cubic bridgeless graph G there is a lean fork-graph containing a subdivision of G as an induced subgraph. Our results establish for the first time, to the best of our knowledge, the validity of Jaeger’s conjecture in a broad inductively defined class of graphs.

1 Introduction

One of the most challenging open problems in graph theory is the cycle double cover conjecture which was independently posed by Szekeres [14] and Seymour [13] in the seventies. It states that every bridgeless graph has a cycle double cover, that is, a system \mathcal{C} of cycles such that each edge of the graph belongs to exactly two cycles of \mathcal{C} . Extensive attempts to prove the cycle double cover conjecture have led to many interesting concepts and conjectures. In particular, some of the stronger versions of the cycle double cover conjecture are related to embeddings of graphs on a surface. The Jaeger’s directed cycle double cover conjecture [6] states that every cubic graph with no bridge has a cycle double cover \mathcal{C} to which one can prescribe orientations in such a way that the orientations of each edge of the graph induced by the prescribed orientations of the cycles are opposite. Jaeger’s conjecture is equivalent to the statement that every cubic bridgeless graph has an embedding on a closed orientable surface with no dual loop.

In our previous work [8], we took a new approach to Jaeger’s conjecture, see Proposition 1, motivated by the notion of critical embeddings. Critical embeddings are used extensively as a discrete tool towards mathematical understanding of criticality of basic statistical physics models of Ising and dimer, and conformal quantum field theory of free fermions [3, 9, 11]. We formulated

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the existence of embeddings of cubic bridgeless graphs with no dual loops as the existence of special perfect matchings in a subclass of braces that we call hexagon graphs.

Main Contribution

In the current work, we introduce new key notions of *safe reductions* and *cut obstacles*. The main results of this work are summarized in Theorems 1, 2 and 3. We prove that the directed cycle double cover conjecture is valid for all lean fork-graphs. The class of all lean fork-graphs is natural and rich. On the one hand this class is inductively defined starting from a triangle by sequentially adding “ears”; ears are the so-called fork-type graphs. On the other hand for each cubic bridgeless graph G it is possible to construct a lean fork-graph that contains a subdivision of G as an induced subgraph.

Related Work on dcdc

Jaeger’s directed cycle double cover conjecture trivially holds in the class of cubic bridgeless planar graphs. However, little is known about its validity in other classes of graphs. Indeed, our results establish for the first time, to the best of our knowledge, the validity of Jaeger’s conjecture in a rich inductively defined class of graphs.

Related Work on cdc

Much more is known about the weaker cycle double cover conjecture. Jaeger [6] proved that any minimal counterexample to the cycle double cover conjecture is a snark; namely, a connected cubic graph which cannot be properly edge-colored with three colors. The famous snark is the Petersen graph. Alon and Tarsi [1] conjectured that the edge set of every bridgeless cubic graph with m edges has a cycle cover where the total sum of the length of the cycles is at most $7m/5$, and Jamshy and Tarsi [7] proved that this conjecture implies the cycle double cover conjecture.

Existence of a cycle double cover in the classes of 3-edge-colorable and 4-edge-connected cubic graphs have also been positively settled [5]. The cycle double cover conjecture also holds for all cubic bridgeless graphs that do not contain a subdivision of the Petersen’s graph [2] and for graphs which have Hamiltonian paths [4]. The important connection with the theory of nowhere-zero-flows is exploited in [15].

In the next section we present the main ideas on which our work is based and formally establish our contributions.

2 Main Ideas and Results

One natural way of starting the construction of a directed cycle double cover of a cubic bridgeless graph G with vertex set V and edge set E is to select a vertex $v \in V$ and to *wire* its incident edges $\{v, x\}, \{v, y\}, \{v, z\}$: this creates a directed triangle consisting of three new directed edges $(x, y), (y, z), (z, x)$. Once such a directed triangle is formed, the vertex v and the edges $\{v, x\}, \{v, y\}, \{v, z\}$ are deleted from G , resulting in a new *mixed* graph with vertex set $V - \{v\}$ and edge set $E - \{\{v, x\}, \{v, y\}, \{v, z\}\}$, together with a set $\{(x, y), (y, z), (z, x)\}$ of directed edges of the triangle. We can continue this procedure by sequentially selecting a vertex u in $V - \{v\}$ and wiring its incident edges and arcs. We note that for some pairs of the created directed edges, it is

forbidden to belong to the same cycle of a directed cycle double cover. *If* we could continue this procedure until every edge is wired, we *might* be able to show the existence of a directed cycle double cover. But, this naive approach leads to the following crucial questions: what do mixed graphs look like, in the middle of the wiring procedure? For which classes of graphs is it possible to apply the wiring procedure until we find a directed cycle double cover? What are *obstacles* that hinder the continuation of the wiring procedure? The first question leads to the definition of *mixed graphs*, and to the novel concept of *safe reductions*. The last two questions to concepts of *fork-collections*, *fork-graphs*, and *cut-obstacles*.

A *mixed graph* is a 4-tuple (V, E, A, R) where V is the vertex set, E is the edge set, A is the set of the directed edges (arcs), and R is a subset of $A \times A$, that is, a set of pairs of arcs. We require that in the graph induced by E , each vertex has degree at least two and at most three, and that each vertex of degree two is the tail of exactly one arc and the head of exactly one arc. Regarding the discussion in the previous paragraph, the set R contains those pairs of arcs, which cannot be together in the same directed cycle of the constructed directed cycle double cover.

Safe reductions

A *reduction* of a subset S of the vertices of a mixed graph is defined naturally as wiring the edges and directed edges incident with S , and updating R . However, R becomes complicated and actually our life would be easier if we could avoid it. It turns out that indeed updating R is not necessary if we perform *safe reductions*.

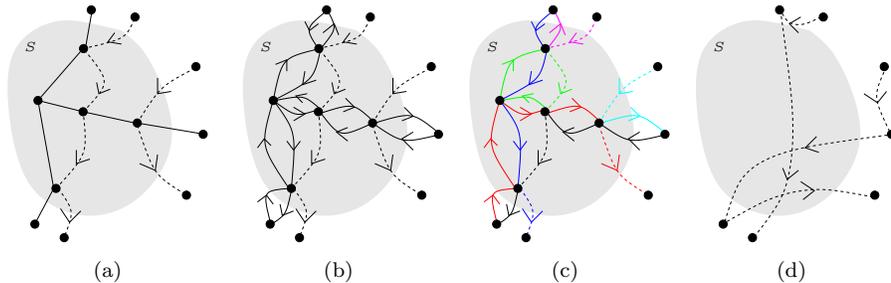


Figure 1: Safe reduction of a subset of vertices S : (a) elements from A are represented by dotted lines, (b) replace edges in E by two arcs oppositely directed, (c) partition of $A_S \cup A'$ into safe paths and cycles and (d) resulting structure.

Let (V, E, A, R) be a mixed graph and $S \subset V$. We say that a mixed graph is obtained from (V, E, A, R) by a *safe reduction* of S if it is constructed as follows. We replace each edge in E incident to a vertex of S by two arcs oppositely directed; let these new arcs form the set A' . Let A_S be the subset of A that contains all directed edges incident to a vertex of S . Then we partition $A_S \cup A'$ into safe directed cycles and safe directed paths with both end vertices in $V - S$. A cycle or a path is *safe* if it has at most one edge from A_S , and if it is not a 2-cycle composed of only edges from A' . Finally we replace each chosen safe path by the directed edge between its end vertices and then, we delete the vertices of S . In Figure 1, we show an example of a safe reduction.

Consecutive safe reductions

In order to construct directed cycle double covers, we decide to perform only safe reductions. Hence, the set R introduced in the definition of mixed graphs is not needed. We observe that to get a directed cycle double cover of a cubic graph G it is sufficient to consecutively perform safe reductions, starting by safely reducing a subset, say S , of the vertex set of G . In other words, let (V, E, A) be the mixed graph obtained from G by a safe reduction of S . Then this safe reduction of S along with consecutive safe reductions of subsets V_1, \dots, V_k that partition V construct a directed cycle double cover of G .

Obstacles

Which are the configurations that do not allow us to perform safe reductions? We refer to them as *obstacles*. We first observe that if $S \subset S' \subset V$, where V is the vertex set of a mixed graph, and no safe reduction of S exists, then no safe reduction of S' exists. For instance, a bridge is an obstacle: if a mixed graph with vertex set V has an edge whose deletion separates V into two sets with no edge or arc between them, then there is no safe reduction of V . Another basic notion of our reasoning is a generalization of a bridge which we call *cut-obstacle*. Let S be a subset of vertices of a mixed graph, and let there E_S and A_S denote the sets of edges and arcs, respectively, with exactly one end-vertex in S . We say that S is a *cut-obstacle* if there is no set P of pairs in $E_S \cup A_S$ such that each edge of E_S is in exactly two pairs of P , each arc of A_S is in exactly one pair of P and no pair of P contains two arcs.

We note that bridges are cut obstacles. In addition, an important example of a cut obstacle is formed by a subset S of vertices such that the number of edges with exactly one end in S is strictly less than twice the number of directed edges with exactly one end in S .

Aside of cut obstacles, there is a wide variety of other concrete obstacles to the existence of safe reductions. However, it appears hopeless to analyze and keep track of all of them. This leads to a natural question:

Is there a class \mathcal{C} of cubic bridgeless graphs such that: (1) the dcdc conjecture may be reduced to the dcdc conjecture for \mathcal{C} , and (2) for each graph G of \mathcal{C} , the existence of consecutive safe reductions which do not create cut obstacles leads to the existence of a directed cycle double cover?

In this work we propose the class of lean fork-graphs as a candidate for such class \mathcal{C} .

Fork graphs

The basic structures for the construction of *fork graphs* are contained in the *fork-collection*. The fork-collection, denoted by \mathcal{F} , consists of the *i-big-forks* for every $i \geq 1$, the *p-fork*, the *fork*, the *star fork*, the *subfork* and the *dot*. The p-fork, the fork and the star fork are depicted in Figure 2, while a subfork is simply a pair of vertices connected by an edge and a dot is an isolated vertex. The 1-big-forks are depicted in Figure 3. If B is 1-big-fork, we let $C(B) = \{x, a, y, b'\}$ be the *connecting set* of B . For $i \geq 2$, each *i-big-fork* B is obtained from a $(i-1)$ -big-fork B' and a star fork T by connecting two leaves of T to two vertices of degree at most two of B' with the following restriction: if we connect to a vertex of degree 2, then it cannot belong to $C(B')$. We let $C(B) = C(B') \cup \{v\}$, where v is the remaining leaf of T . This operation is well explained in Figure 3, since each 1-big-fork is obtained from a fork and a star fork in exactly the same way. We

refer to 1-big-forks simply as big-forks. Furthermore, the *exclusive fork-collection* \mathcal{E} is a subset of \mathcal{F} that contains all members of \mathcal{F} but the fork.

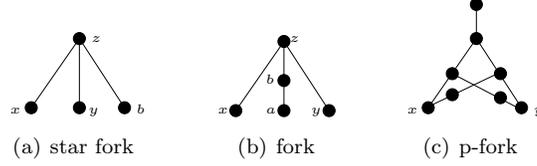


Figure 2: fork-type graphs

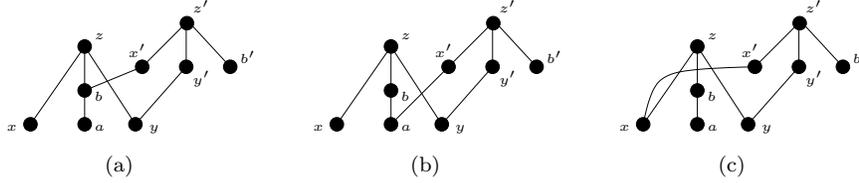


Figure 3: Three kinds of big-forks. Note that a big-fork consists of the union of a fork and a star fork by means of the addition of two new edges.

Given a member L of the fork-collection, a *bold* L is obtained from L by adding several extra half-edges and edges following the next four rules. (i) If L is the fork, the star fork, or the subfork we obtain a bold L by adding one half-edge to each leaf of L . (ii) If L is the p-fork, a bold L is obtained by adding one half-edge to the leaf of L and to each vertex x, y (see Figure 2(c)). (ii) Add two or three half-edges to the dot to obtain a bold dot. (iii) For each $j \geq 1$, if L is a j -big-fork, we obtain a bold L by adding a half-edge to each vertex from the connecting set of L and add a set, possibly empty, of disjoint edges and half-edges so that the degrees of the vertices of the bold L are at most 3.

Definition of fork graphs

We say that a cubic graph G is a *fork graph* if there is a sequence G_0, \dots, G_n of 2-connected graphs so that G_0 is a triangle, $G_n = G$ and G_i is obtained from G_{i-1} by connecting to its vertices of degree 2 the half-edges of a bold L_i , where L_i is from the exclusive fork-collection; therefore, half edges of a bold L_i become edges of G_i . In addition, we allow that for at most one j , L_j is the fork; the fork is depicted in Figure 2(b). Moreover, we can perform several $Y-\Delta$, $\Delta-Y$ transformations; that is, replacement of a vertex by a triangle, and vice-versa. We say, in the situations described above, that G_i is obtained from G_{i-1} by *addition* of a bold L_i .

Example 1 (Petersen's graph). *Let G_1 be the graph obtained from a triangle by addition of a bold p-fork such that G_1 has exactly 3 vertices of degree two. Let G_2 be the cubic graph obtained from G_1 by addition of a bold dot. The graph obtained from G_2 by performing one $\Delta-Y$ transformation at the initial triangle is the Petersen's graph.*

Our first main contribution concerns cut-type sufficient conditions for the existence of a safe reduction and therefore the existence of a directed cycle double cover conjecture.

Theorem 1. *Let G be a fork graph and G_0, \dots, G_n be its building sequence. Let $i \leq n$ and G_i be obtained from G_{i-1} by addition of a bold L_i . If $V(G) - V(G_i)$ can be safely reduced in G and G'_i denotes the obtained mixed graph, then the following two statements hold.*

1. *If L_i is not a j -big-fork for all $j \geq 1$, then $V(L_i)$ can be safely reduced in G'_i .*
2. *If L_i is a j -big-fork for some $j \geq 1$ and L_i is not a cut-obstacle in G'_i , then $V(L_i)$ can be safely reduced in G'_i .*

Lean fork-graphs

Let G be a fork graph and G_0, \dots, G_n be its building sequence. For $i \leq n$ and $j \geq 1$, let G_i be obtained from G_{i-1} by addition of a bold j -big-fork L_i . Since bold L_i has at most $j + 4$ vertices of degree 2, we observe that there are at most $j + 4$ vertex-disjoint paths from $V(L_i)$ to $V(G_{i-1})$ using only edges from $E(G) - E(G_i)$. The connectivity property of a lean fork graph is that instead of $j + 4$, at most $j + 3$ vertex-disjoint paths are allowed. Namely, G is said to be *lean* if for each i such that G_i is obtained from G_{i-1} by addition of a bold j -big-fork L_i for some $j \geq 1$, there are at most $j + 3$ vertex-disjoint paths from $V(L_i)$ to $V(G_{i-1})$ using only edges from $E(G) - E(G_i)$.

Our next main contribution is confirmation of Jaeger's conjecture for the class of all lean fork-graphs.

Theorem 2. *The directed cycle double cover conjecture holds for all lean fork-graphs.*

Finally, we show that each cubic bridgeless graph G is naturally embedded in a lean fork-graph.

Theorem 3. *For every cubic bridgeless graph G there exists a lean fork graph that contains a subdivision of G as an induced subgraph.*

In the following section, we explain basic technical tool of our reasoning, namely the hexagon graphs, introduced in [8].

3 Hexagon graphs

We refer to the complete bipartite graph $K_{3,3}$ as a *hexagon* and say that a bipartite graph H has a hexagon h if h is a subgraph of H . For a graph G and a vertex v of G , let $N_G(v)$ denote the set of neighbors of v in G .

Let G be a cubic graph with vertex set V and edge set E . A hexagon graph of G is a graph H obtained from G following the rules:

1. We replace each vertex v in V by a hexagon h_v so that for every pair $u, v \in V$, if $u \neq v$, then h_u and h_v are vertex disjoint. Let $\{V(h_v) : v \in V\}$ be the vertex set of H .
2. For each vertex $v \in V$, let $\{v_i : i \in \mathbb{Z}_6\}$ denote the vertex set of h_v and $\{v_i v_{i+1}, v_i v_{i+3} : i \in \mathbb{Z}_6\}$ its edge set. With each neighbor u of v in G , we associate an index $i_{v(u)}$ from the set $\{0, 1, 2\} \subset \mathbb{Z}_6$ so that if $N_G(v) = \{u, w, z\}$, then $i_{v(u)}, i_{v(w)}, i_{v(z)}$ are pairwise distinct.
3. (See Figure 4). Let $X = \cup_{v \in V} \{v_{2i} : i \in \mathbb{Z}_6\}$ and $Y = \cup_{v \in V} \{v_{2i+1} : i \in \mathbb{Z}_6\}$. We replace each edge uv in E by two vertex disjoint edges e_{uv}, \bar{e}_{uv} so that if both $v_{i_{v(u)}}, u_{i_{u(v)}}$ belong to either X or Y , then $e_{uv} = v_{i_{v(u)}} u_{i_{u(v)+3}, \bar{e}_{uv} = v_{i_{v(u)+3}} u_{i_{u(v)}}$. Otherwise, $e_{uv} = v_{i_{v(u)}} u_{i_{u(v)}}$, $\bar{e}_{uv} = v_{i_{v(u)+3}} u_{i_{u(v)+3}$. Let $E(H) = \{E(h_v) : v \in V\} \cup \{e_{uv}, \bar{e}_{uv} : uv \in E\}$.

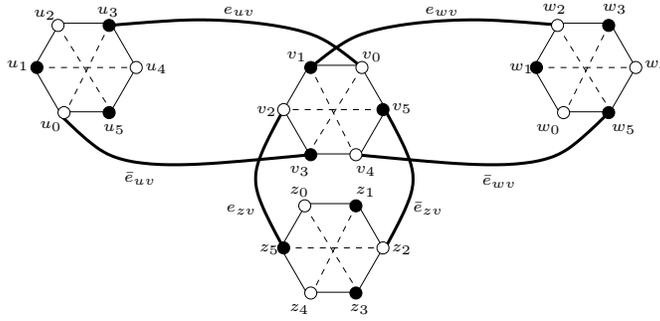


Figure 4: Local representation of a hexagon graph H of a cubic graph G . The hexagon h_v is associated with vertex v , where $N_G(v) = \{u, w, z\}$. Red edges are depicted as red lines, blue edges are depicted as blue lines and white edges as black lines. The set X is represented by white vertices and the set Y by black vertices.

We say that h_v is the hexagon of H associated with the vertex v of G and that $\{h_v : v \in V\}$ is the *set of hexagons of H* . We shall refer to the set of edges $\bigcup_{v \in V} \{v_i v_{i+3} : i \in \mathbb{Z}_6\}$ as the set of *red edges of H* , to the set of edges $\{e_{uv}, \bar{e}_{uv} : uv \in E\}$ as the set of *white edges of H* , and finally to the set of edges $\bigcup_{v \in V} \{v_i v_{i+1} : i \in \mathbb{Z}_6\}$ as the set of *blue edges of H* (see Figure 4). Moreover, we shall say that a perfect matching of H containing only blue edges is a *blue perfect matching*. In the rest of this work, if x is a vertex, say v_i , of a hexagon, then \bar{x} denotes v_{i+3} .

Let G be a cubic graph and H be a hexagon graph of G . We observe two important properties: (i) H is bipartite, and (ii) if H' is another hexagon graph of G , then H and H' are isomorphic.

In the next paragraphs we briefly recall a combinatorial representation of embedding of graphs on closed orientable surfaces, namely *rotation systems*, and describe the embeddings encoded by the blue perfect matchings.

Let G be a graph. For each $v \in V(G)$, let π_v be a cyclic permutation of the edges incident with v . A collection $\pi = \{\pi_v : v \in V(G)\}$ is called a *rotation system* of G . Edmonds [12, §3.2] proved that each such a π encodes an embedding of G on a closed orientable surface with face boundaries $e_1 e_2 \cdots e_k$ such that $e_i = v^i v^{i+1} \in E(G)$, $\pi_{v^{i+1}}(e_i) = e_{i+1}$, $e_{k+1} = e_1$ and k minimal.

Let M be a blue perfect matching of H and let W be the set of white edges of H . Each cycle C in $M \Delta W$ induces a subgraph in G defined by the set of edges $\{uv \in E(G) : e_{uv} \in C \text{ or } \bar{e}_{uv} \in C\}$. The following lemma follows via a natural bijection between blue perfect matchings and rotation systems.

Lemma 1. *Let G be a cubic graph, H the hexagon graph of G , and W the set of white edges of H . Each blue perfect matching M of H encodes an embedding of G on a closed orientable surface with set of face boundaries the set of subgraphs of G induced by the cycles in $M \Delta W$. Moreover, the converse also holds.*

In [8], we establish the following approach to the directed cycle double cover conjecture.

Proposition 1. *Let G be a cubic graph, H the hexagon graph of G , M a blue perfect matching of H , and W the set of white edges of H . The embedding of G encoded by M has a dual loop if and only if there is a cycle in $M \Delta W$ that contains both end-vertices of a red edge.*

In the same work [8], we prove the following structural result regarding hexagon graphs. We

recall that braces, along with bricks, form the basic building blocks of the perfect matching decomposition theory [10].

Theorem 4. *Let G be a cubic graph. Then the hexagon graph H of G is a brace if and only if G is bridgeless.*

The following section (Section 4) is divided into two parts: the first one shows how Theorem 1 implies Theorem 2 and in the second part we prove the statement of Theorem 3. Finally, in Section 5 we present the proof of Theorem 1 in the context of hexagon graphs.

4 DCDC and richness of lean fork graphs

In this section we show how Theorem 1 implies Theorem 2 and discuss the proof of Theorem 3.

Proof of Theorem 2

Let G be a lean fork-graph. Hence, there is a sequence G_0, \dots, G_n of 2-connected graphs such that G_0 is a triangle, $G_n = G$, and for $i \leq n$, G_i is constructed from G_{i-1} by adding a bold L_i , where L_i is a member of the exclusive fork-collection and it is the fork at most once.

Using Theorem 1, we only need to show that in subsequent safe reductions of vertex sets of bold L_i 's, we do not create a cut-obstacle formed by the vertex set of an added bold j -big-fork, for some $j \geq 1$.

We assume, for the sake of contradiction, that for some $i \leq n$ and $j \geq 1$, G_i is obtained from G_{i-1} by addition of a bold j -big-fork B and $V(B)$ is a cut-obstacle in G'_i , where G'_i denotes a mixed graph obtained by safely reducing $V(G) - V(G_i)$. This implies that the number of directed edges in G'_i with exactly one end-vertex in $V(B)$ is more than $2(j+3)$. Hence, this number is exactly $2(j+4)$, because a bold j -big-fork has at most $j+4$ vertices of degree 2 in G_i (see Figure 3 for $j = 1$).

These $2(j+4)$ directed edges are obtained by the reduction of $V(G) - V(G_i)$. Let G' be the graph obtained from G by deleting all edges in $E(G_i)$. By definition of safe reductions, the digraph D obtained from G' by replacing each edge by two oppositely directed edges, has $2(j+4)$ directed edge-disjoint paths between $V(B)$ and $V(G_{i-1})$. Hence, D has no set of strictly less than $2(j+4)$ directed edges which completely separates $V(B)$ from $V(G_{i-1})$. Consequently, G' has no set of at most $j+3$ edges which completely separates $V(B)$ from $V(G_{i-1})$. But this means, by the Menger's theorem, that G' has at least $j+4$ edge-disjoint paths between $V(B)$ and $V(G_{i-1})$. Since each vertex of G' has degree at most three, these paths are also vertex disjoint. This contradicts the assumption that G is lean. \square

Proof of Theorem 3

Given a cubic bridgeless graph G , the construction of the lean fork-graph \tilde{G} that contains a subdivision of G as an induced subgraph can be split into two steps:

Step 1: We create a lean fork-graph \tilde{G}^1 with arbitrarily many vertices of degree 2: we perform this task by constructing the lean fork-graph with defining sequence $\tilde{G}_0^1, \tilde{G}_1^1, \dots, \tilde{G}_m^1$, where \tilde{G}_0^1 is the triangle, \tilde{G}_1^1 is obtained from \tilde{G}_0^1 by adding a bold fork and for each $i > 1$, \tilde{G}_i^1 is

obtained from \tilde{G}_{i-1}^1 by adding a bold $(i-1)$ -big-fork. We note that \tilde{G}^1 is a lean fork-graph with exactly $m + 3$ vertices of degree two.

Step 2: In the second step we obtain \tilde{G} from \tilde{G}^1 by sequentially adding bold subforks and bold dots following an ear-decomposition of G . Let $(G_0, G_1, \dots, G_l; P_1, \dots, P_l)$ be an ear decomposition of G , where $G = G_l$, G_0 is a cycle of G and G_i is obtained from G_{i-1} connecting two vertices of $V(G_{i-1})$ by a path P_i such that $E(P_i) \cap E(G_{i-1}) = \emptyset$ and $|V(P_i) \cap V(G_{i-1})| = 2$, for each $i \in \{1, \dots, l\}$. In order to obtain \tilde{G} from \tilde{G}^1 with the property that \tilde{G} has a subdivision of G as an induce subgraph, we first obtain \tilde{G}_0 from \tilde{G}^1 with the property that \tilde{G}_0 contains a subdivision of G_0 and then for each $i \in \{1, \dots, l\}$, we obtain \tilde{G}_i from \tilde{G}_{i-1} with the property that \tilde{G}_i contains a subdivision of G_i . This procedure is best explained by means of an example, see Figure 5. Since \tilde{G}^1 is lean and by the construction of \tilde{G} from \tilde{G}^1 , we have that \tilde{G} is a lean fork-graph. \square

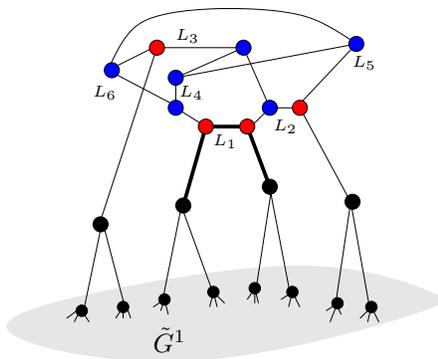


Figure 5: Construction of \tilde{G} from \tilde{G}^1 in the case that G is the complete bipartite graph on 6 vertices. The vertex set of G is colored by blue and the vertices produced by the subdivisions are colored by red. Moreover, note that L_1, L_2, L_3, L_4 are subforks and L_5, L_6 are dots.

5 Proof of Theorem 1

In this work, we perform safe reductions of subsets of vertices of cubic graphs. Recall that this leads us to the notion of mixed graphs (see Section 2). We suggest to study these reductions in the context of hexagon graphs. Then it is necessary to describe mixed graphs in terms of hexagon graphs. We refer to these new structures as *pseudohexes*, and introduce them and the corresponding concept of *safe reductions of pseudohexes* in the following paragraphs.

The statement of Theorem 1 follows from Lemma 2 and Theorems 7, 8, 11, 9.

5.1 Pseudohexes

Let K be a bipartite graph with vertex set $V(K)$ and edge set $E(K)$ such that edges are colored blue, red and white, and there may be parallel edges but no loops. We let $B(K)$ denote the set of its blue edges, $R(K)$ the set of its red edges, and $W(K)$ the set of its white edges. We call K *pseudohex* if it is empty or the following three properties are satisfied:

- $B(K)$ induces disjoint hexagons covering all the vertices of K ; we call the set of hexagons induced by $B(K)$ the hexagons of K .
- K contains all the edges $\{x, \bar{x}\}$ for each vertex x of K ; all these edges are red. Moreover, K may have additional red edges.
- $W(K) = E(K) - (B(K) \cup R(K))$ is a perfect matching of K .

We still need to introduce some extra definitions. A white edge $e = \{x, y\} \in W(K)$ is said to be *real* if K contains also the white edge $\bar{e} = \{\bar{x}, \bar{y}\}$ and the only red edges adjacent to e or \bar{e} are $\{x, \bar{x}\}$ and $\{y, \bar{y}\}$. Clearly e is real if and only if \bar{e} is real. We denote by $W_r(K)$ the set of the real white edges of K . In addition, we say that the white edges in $W_d(K) = W(K) - W_r(K)$ are *derived* and that two white edges are *red-connected* if there is a red edge adjacent to both of them.

We remark that pseudohexes correspond to mixed graphs; the set of hexagons of a pseudohex corresponds to the vertex set of the mixed graph, the sets of the real white edges and the derived white edges correspond to the sets of the edges and the arcs, respectively, of the mixed graph. Moreover, red-connected pairs of derived white edges correspond to pairs of arcs in the set R introduced in the definition of mixed graphs.

Definition 1 (Reduction of a hexagon). *Let K be a pseudohex, h a hexagon of K and N a perfect matching of h . We define the reduction of h by N as follows. For each of the three paths $P_i, i = 1, 2, 3$, consisting of one edge of N and the white edges adjacent to this edge we introduce a new white edge e_i between the end-vertices of P_i whenever P_i is not a cycle of length 2. If P_i is not a cycle of length 2, then for each red edge $\{u, w\}$ attached to an interior vertex w of P_i we introduce new red edge $\{u, w'\}$ where w' is the vertex of e_i of the same bipartition class of K as w . If P_i is a cycle of length 2, then for each red edge $\{u, w\}$ attached to an interior vertex w of P_i we introduce a new red edge $\{u, \bar{u}\}$. Finally, we delete the vertices of h . The paths $P_i, i = 1, 2, 3$, are said to be contracted.*

We observe that a reduction never creates new real white edges. We say that a set of hexagons $\{h_1, \dots, h_l\}$ can be *safely reduced* (or the reduction is *safe*) if there are perfect matchings N_1, \dots, N_l of h_1, \dots, h_l , respectively, such that each contracted path of the joint reduction of h_1, \dots, h_l by N_1, \dots, N_l , respectively, has at most one white edge which is not real, in other words at most one derived white edge. The next statement follows from the facts that the set of derived white edges in pseudohexes is equal to the set of the arcs in mixed graphs and that the reduction of a hexagon by a perfect matching in a pseudohex corresponds to wiring a vertex in a mixed graph.

Lemma 2. *A safe reduction in a pseudohex corresponds to safe reduction in a mixed graph.*

Let K be a pseudohex. It is natural to associate a graph G^K with K . Recall that K corresponds to a mixed graph. The vertex set of G^K is the set of the hexagons of K . For u, v vertices of G^K , the set $\{u, v\}$ is an edge of G^K if and only if there is a pair of real white edges connecting the hexagons h_u, h_v in K . We know by definition of a pseudohex that each vertex of G^K has degree at most three. In other words, G^K is the graph induced by the mixed graph that corresponds to the pseudohex K . A subgraph of K is called *end* if it consists of a red edge parallel to a white edge. If K contains an end as a subgraph we say that K has an end. We say that K is *proper* if K has no end and G^K is a 2-connected graph without cycles of length 2.

We observe that any mixed graph obtained from a fork graph by a sequence of safe reductions of set of vertices of bold members of the fork-collection can be represented by a proper pseudohex.

Therefore, in order to prove Theorem 1, we are interested in obtaining safe reductions of set of hexagons of proper pseudohexes, where the set of hexagons correspond to the vertex-set of the fork-type graphs. In Subsections 5.2, 5.3 and 5.5 we study the aforementioned reductions.

5.2 Reduction of forks and 3-ears on pseudohexes

The aim of this section is to study safe reductions of forks on pseudohexes. However, we also establish results that are used to handle many of the proofs included in later sections. For this sake, we introduce a new fork-type graph that we call the *3-ear*. The 3-ear consist of a path on three vertices. We refer to the union of the fork-collection and the 3-ear as the *extended* fork-collection. The definition of a bold 3-ear is analogous to the one of bold fork, star fork and subfork.

Let K, K' be proper pseudohexes such that G^K is obtained from $G^{K'}$ by addition of a bold L , where L is a member of the extended fork-collection. We denote by L_K the subset of the hexagons of K corresponding to the vertices of L ; consequently, $V(L_K)$ denotes the set of the vertices of K corresponding to the hexagons in L_K . We say that K is obtained from K' by *L -addition*.

We refer to the subset of real white edges (derived white edges, respectively) of K with at least one end vertex in $V(L_K)$ as the *L -edges* (*L -no-edges*, respectively) of K . A pair of *L -no-edges* is called *potential* if each edge of the pair is incident to exactly one vertex of $V(L_K)$, and these two vertices belong to different bipartition classes of K .

In the case that L is the fork or the 3-ear, we say that K has a *L -obstacle* if each *L -no-edge* of K is incident to exactly one vertex of $V(L_K)$. We remark that the notion of *L -obstacle* is analogous to the notion of cut-obstacle for mixed graphs. Observation 1 illustrates the concept of *L -obstacles*.

Observation 1. *If a proper pseudohex K has a L -obstacle and all potential pairs of L -no-edges are red-connected, then each reduction of the hexagons corresponding to the vertices of L creates an end.*

Proof. We prove the observation for a *F -obstacle*, where F is the fork. The proof of the statement in the case that L is a 3-ear. Hence, we assume that K has a *F -obstacle* and let \tilde{K} denote the pseudohex obtained from K by some reduction of F_K .

Each new derived white edge of \tilde{K} is obtained by contracting a path that contains two white edges from the set S of the white edges of K with exactly one end-vertex in $V(F_K)$. Moreover, the end-vertices (contained in $V(F_K)$) of these two white edges belong to different bipartition classes of K .

The set S is formed by all eight *F -no-edges* of K and a subset consisting of six *F -edges*. Hence, necessarily a potential pair of *F -no-edges* belong to the same contracted path of the reduction; but each potential pair of *F -no-edges* of K is red-connected, and thus \tilde{K} has an end. \square

We recall that the reduction of a hexagon is *safe* if each contracted path has at most one derived white edge (see Definition 1). We say that a pseudohex has a *correct reduction* if it is possible to reduce all its hexagons without creating an end.

The following converse of Observation 1 is not difficult to prove.

Observation 2. *Let K be a proper pseudohex with a L -obstacle, where L is either the fork, or the 3-ear. If there exists a potential pair e_1, e_2 of L -no-edges of K that are not red-connected then, there is a reduction of L_K that creates no end, and such that all but one of the contracted*

paths contain at most one L -no-edge. Moreover, the contracted path that contains more than one L -no-edge contains exactly two L -no-edges; namely, e_1, e_2 .

Proof. We prove the observation first for a P -obstacle, where P is the 3-ear. We assume that there exists a potential pair e_1, e_2 of P -no-edges of K that are not red-connected. Without loss of generality two cases arise: the case that e_1 and e_2 have end vertices in $V(h_x \cup h_y)$ and the case that e_1 has an end vertex in $V(h_x)$ and e_2 has an end vertex in $V(h_z)$. Both cases are worked out in Figure 6.

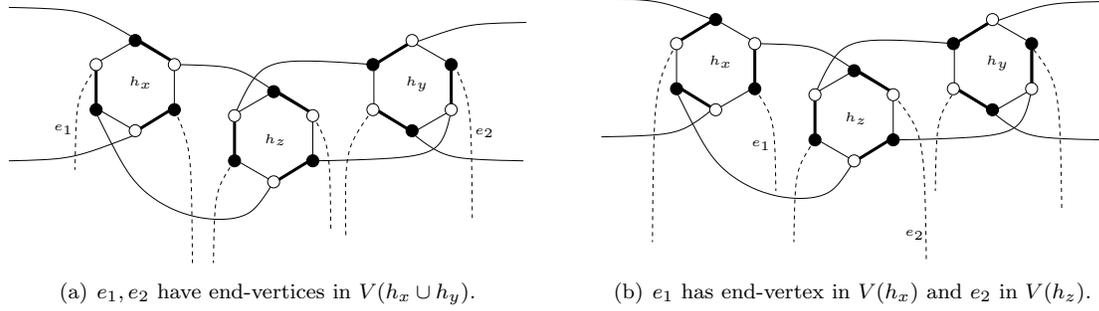


Figure 6: P -no-edges are represented by dotted lines. Perfect matchings leading to a reduction of $\{h_x, h_y, h_z\}$ without ends are depicted by thicker lines. In the case that $e_1 = e_2$, the reduction by the same perfect matchings is safe.

Secondly, we assume that K has a F -obstacle, where F is the fork. In this proof, we use names of vertices, edges and hexagons of F_K from Figure 10(a). Also, using Figure 10(a), we denote by M_a and M_b the perfect matchings indicated by thicker lines on hexagons h_a and h_b , respectively. Moreover, let N_a and N_b denote the perfect matchings of h_a and h_b that are complements of M_a and M_b , respectively.

In order to use the first part of this proof regarding the 3-ear, we observe that the reduction of h_a, h_b by M_a, N_b (by N_a, M_b , respectively) is safe and generates a P -obstacle in the resulting pseudohex, where P is the 3-ear induced by the set of vertices $\{x, z, y\}$ corresponding to the set of hexagons $\{h_x, h_z, h_y\}$.

Let e_1, e_2 be a potential pair of F -no-edges of K that is not red-connected. Without loss of generality, we distinguish three cases depending on the hexagons to which the end-vertices of the pair of edges e_1, e_2 belong to.

- (i) Both edges e_1, e_2 have an end-vertex in $V(h_x \cup h_y)$. Then we can safely reduce h_a, h_b by M_a, N_b . We get a P -obstacle, where the non red-connected pair e_1, e_2 of F -no-edges becomes a pair of P -no-edges that is not red-connected and we can use the first part of this proof to complete the argument.
- (ii) Edge e_1 is incident to a vertex in $V(h_x \cup h_y)$ and e_2 is incident to a vertex in $V(h_a \cup h_b)$. Then e_2 is incident to a vertex from the set $\{r_1, r'_1, r_2, r'_2\}$; see Figure 10(a). If e_2 is incident to r_1 then reduction of h_a, h_b by M_a, N_b generates a P -obstacle where e_1 and the derived white edge starting at w' (that is, the edge obtained by reduction of a path containing e_2) are not red-connected. Again, we use the first part of this proof to complete the argument. The remaining cases that e_2 is incident to r'_1, r_2 or r'_2 can be worked out in the same way.

(iii) Both edges e_1, e_2 have an end-vertex in $V(h_a \cup h_b)$. In the case that the pair e_1, e_2 is incident to r_1, r'_1 (r_2, r'_2 , respectively), we reduce h_a, h_b by M_a, M_b (N_a, N_b , respectively). In both cases a path consisting of real white edges and e_1, e_2 is contracted to a single derived white edge disjoint from $V(P_K)$. Moreover, the derived white edge incident to w (w' , respectively) is obtained by contracting a path consisting only of real white edges; therefore, this edge is not red-connected to the derived white edges incident to $V(h_x \cup h_y)$. Hence, both reductions generate a P -obstacle with a potential pair of P -no-edges that are not red-connected. Again, by the first part of this proof the result holds. □

The following statement basically claims that it is always possible to safely reduce the vertex set of the starting triangle of a fork-graph.

Observation 3. *If K is a proper pseudohex with G^K a cycle of length three, then the set of all hexagons of K has a correct (and safe) reduction.*

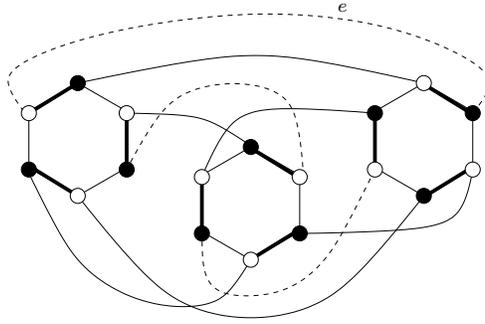


Figure 7: Correct and safe reduction of a proper triangle: reduction of each hexagon by the perfect matching represented by thicker lines. Derived white edges are depicted as dotted lines.

Proof. Without loss of generality, we assume that K has a derived white edge e as in Figure 7; there is no loss of generality, since G^K is symmetric. Then, since K is proper, the remaining two derived white edges have to be configured as in Figure 7. The perfect matchings indicated by thicker lines in Figure 7 lead to a correct and safe reduction of the set of all hexagons of K , since each contracted cycle (the edges of a contracted cycle alternate white edges and edges from the indicated perfect matchings) of this reduction has at most one derived white edge, and thus induces no red edge. □

In the next definition we present an extra obstacle for a safe reduction of a 3-ear.

Definition 2. *Let K, K' be proper pseudohexes and K obtained from K' by P -addition, where P is the 3-ear. Let $\{x, z, y\}$ be the vertex set of P with x, y its leaves. We say that K has a P -danger if one P -no-edge has its end vertices in $V(h_x \cup h_y)$, one P -no-edge has one end vertex in $V(h_x \cup h_y)$ and the other end vertex in $V(h_z)$, and the remaining two P -no-edges have exactly one end vertex in $V(P_K)$.*

Moreover, we say that K has a P -bad if it has a P -danger and the P -no-edge with both end vertices in $V(h_x \cup h_y)$ is red-connected to the P -no-edge that has exactly one end vertex in $V(P_K)$

and it belongs to $V(h_z)$. An example of a P -bad is in Figure 8, where the two edges which must be red-connected are e and γ .

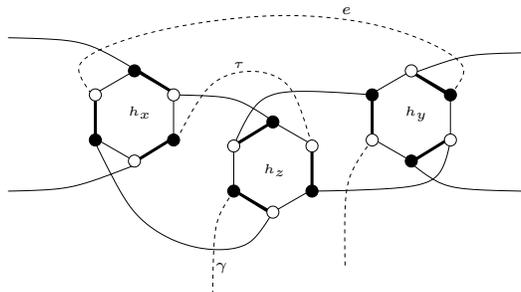


Figure 8: P -danger pseudo-hex; P -no-edges are depicted by dotted lines. If it is P -bad then edges e, γ are red-connected. If e, γ are not red-connected, then perfect matchings depicted by thicker lines lead to a reduction of $\{h_x, h_y, h_z\}$ that creates no end.

Theorem 5. *Let K, K' be proper pseudo-hexes and K obtained from K' by P -addition, where P is a 3-ear. If K has neither a P -obstacle nor a P -danger, then P_K can be safely reduced.*

Moreover, if K has a P -danger but not a P -bad then there is a reduction of P_K creating no end and such that each but one contracted path has at most one derived white edge, and the contracted path that has more than one derived white edge contains exactly two derived white edges, namely γ and e (see Figure 8 and Definition 2).

Proof. The second part of the theorem follows from Figure 8.

In order to prove the first part, without loss of generality we distinguish the following three cases.

- (i) There are no P -no-edges of K incident with exactly one vertex of $V(P_K)$; that is, all P -no-edges of K have both end vertices in $V(P_K)$. Then, the result of Theorem 5 follows from Observation 3.
- (ii) Exactly four P -no-edges of K are incident with exactly one vertex of $V(P_K)$. Hence, there is exactly one derived white edge with both end vertices in $V(P_K)$; let us denote it by e . Edge e connects either the two non-central hexagons h_x and h_y , or the central hexagon with one non-central hexagon; namely, h_x and h_z or h_y and h_z . Without loss of generality, these two cases are described in Figures 6(a) and 6(b), where $e = e_1 = e_2$. In both cases, the perfect matchings that lead to a safe reduction of $\{h_x, h_y, h_z\}$ are indicated by thicker lines.
- (iii) Exactly two P -no-edges of K are incident with exactly one vertex of $V(P_K)$. Hence, there are exactly two P -no-edges of K with both end vertices in $V(P_K)$. Without loss of generality, the two cases that arise when they both connect vertices between the same pair of hexagons are depicted in Figures 9(a) and 9(b). If they connect vertices of distinct pairs of hexagons then aside of the P -bad (P -danger in case that there is no red-connection) there is one more case, which is depicted in Figure 9(c). In all the three described cases, the perfect matchings that lead to a safe reduction of $\{h_x, h_y, h_z\}$ are indicated by thicker lines.

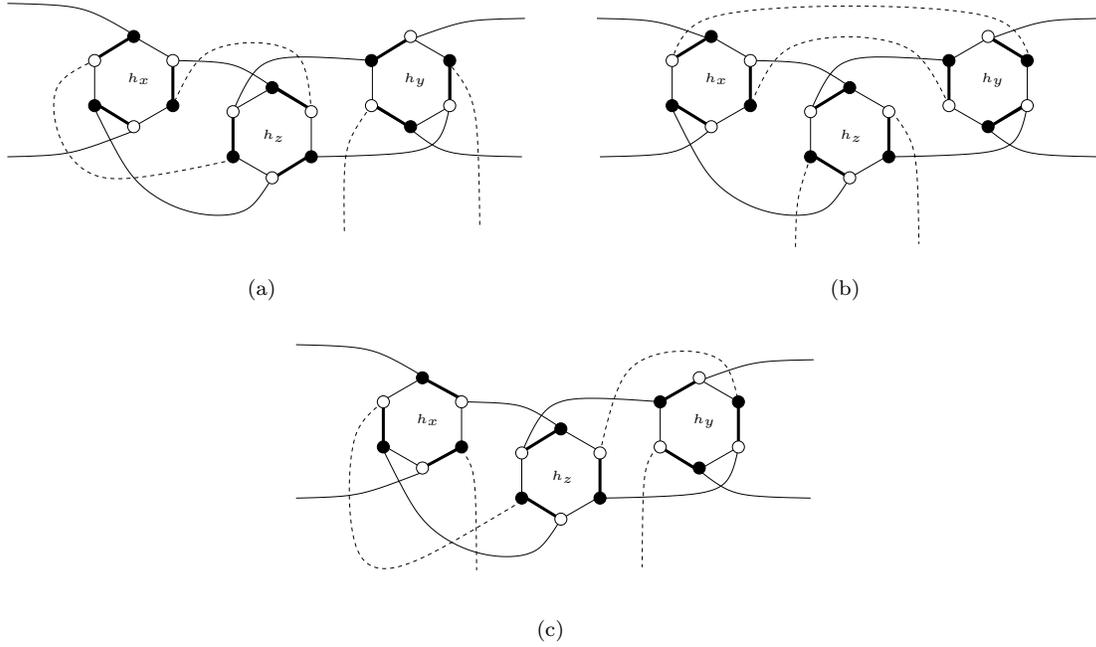


Figure 9: The distinct configurations of the case (iii) of the proof of Theorem 5: exactly two P -no-edges of K have both end vertices in $V(P_K)$. Again, P -no-edges are represented by dotted lines.

□

We define the remaining two obstacles for a correct reduction of a fork.

Definition 3. Let K and K' be proper pseudo-hexes such that K is obtained from K' by F -addition, where F is the fork. We say that K has a F -abad if F_K is configured as in Figure 10(a) and that K has a F -bbad if F_K is configured as in Figure 10(b).

Theorem 6. Let K, K' be proper pseudo-hexes such that K is obtained from K' by F -addition, where F is the fork. If K has neither a F -obstacle nor a F -abad nor a F -bbad, then there is a safe reduction of F_K .

Moreover, if K has a F -abad or a F -bbad, and the edge e (see Figure 10) is not red-connected to another F -no-edge e' of K , then there is a reduction of F_K so that all but one contracted path have at most one derived white edge and the contracted path that has more than one derived white edge contains exactly two derived white edges, namely e and e' .

Before presenting the proof of Theorem 6, we show that the following statement is a straightforward consequence of Theorem 6.

Theorem 7. Let K, K' be proper pseudo-hexes such that K is obtained from K' by F -addition, where F is a fork. If each F -no-edge of K is a subset of $V(F_K)$, then there is a safe reduction of F_K .

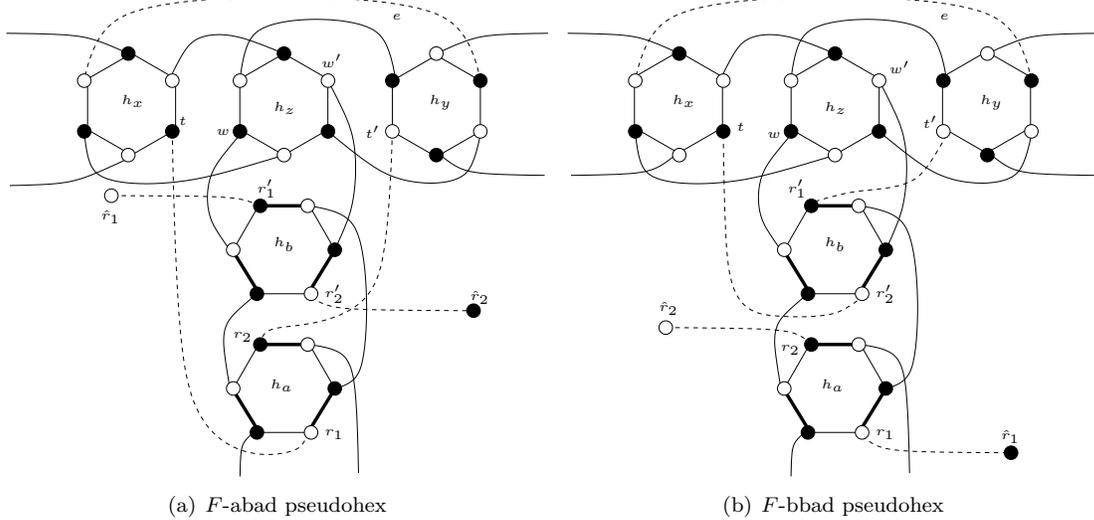


Figure 10: F -no-edges are represented by dotted lines.

Proof. By Theorem 6, it is enough to prove that K has neither a F -obstacle, nor a F -abad and nor a F -bbad. We consider names of vertices, edges and hexagons of F_K from Figure 10. By the hypothesis, it is trivial the fact that K does not have a F -obstacle.

For the sake of contradiction, we assume that K has a F -abad (F -bbad, respectively). Then F_K is configured as in a F -abad (F -bbad, respectively). But since K has no ends (proper), the edges with end vertices r'_1, r'_2 (r_1, r_2 , respectively) are distinct and therefore, each of them has an end-vertex in $V(K) - V(F_K)$, a contradiction. The result follows. \square

5.2.1 Proof of Theorem 6

In the following, we recall that we consider names of vertices, edges and hexagons of F_K from Figure 10. Also, using Figure 10(a), we denote by M_a and M_b the perfect matchings indicated by thicker lines on hexagons h_a and h_b , respectively. Moreover, let N_a and N_b denote the perfect matchings of h_a and h_b that are complements of M_a and M_b , respectively.

We first prove the second part of Theorem 6. If K has a F -abad and $e, e' = \{r'_2, \hat{r}_2\}$ are not red-connected, then we reduce h_a, h_b by N_a, M_b . If K has a F -abad and $e, e' = \{r'_1, \hat{r}_1\}$ are not red-connected, then we reduce h_a, h_b by M_a, N_b . If K has a F -bbad and $e, e' = \{r_2, \hat{r}_2\}$ are not red-connected, then we reduce h_a, h_b by N_a, M_b . If K has a F -bbad and $e, e' = \{r_1, \hat{r}_1\}$ are not red-connected, then we reduce h_a, h_b by M_a, N_b . It is a routine to check that all these reductions of $\{h_a, h_b\}$ are safe, and that in the resulting pseudo-hex a P -danger is generated, where P is the 3-ear with vertex set $\{x, z, y\}$, but the P -danger is not a P -bad; since e is not red-connected to the P -no-edge obtained from contracting a path that contains e' , and e' is incident to a vertex in $V(h_z)$ and to a vertex in $V(h_x \cup h_y \cup h_z)$. Hence, the second statement of Theorem 6 follows from the second part of Theorem 5.

In order to prove the first part of the statement of Theorem 6, we assume that K has neither a F -obstacle, nor a F -abad, nor a F -bbad.

Without loss of generality, we distinguish three main cases; Case 1: $\{r_1, r'_1\}$ is a white edge of

K , Case 2: $\{r_2, r'_2\}$ is a white edge of K , and Case 3: Neither $\{r_1, r'_1\}$ nor $\{r_2, r'_2\}$ are white edges of K . In what follows, we deal with the analysis of these cases.

Case 1: If $\{r_1, r'_1\}$ is a white edge of K , then we reduce h_a, h_b by M_a, M_b .

Case 2: If $\{r_2, r'_2\}$ is a white edge of K , then we reduce h_a, h_b by N_a, N_b .

Clearly, the reductions indicated in Cases 1 and 2 are safe.

Further, in both cases a path consisting entirely of real white edges is contracted to a single new derived white edge; let us denote it by e_r . In Case 1, e_r is incident to vertex w and in Case 2, e_r is incident to vertex w' .

We assume that Case 1 holds (Case 2 is discussed analogously). If reduction of h_a, h_b by M_a, M_b generates a P -danger, where P is the 3-ear with vertex set $\{x, z, y\}$, then necessarily K has a F -no-edge with its end vertices in $V(h_x \cup h_y)$. We may assume it is the edge e of Figure 8. However, this edge e is not red-connected to the new derived white edge e_r ; which has been producing by contracting a path that consists only of real white edges of K . Hence, we have a P -danger but not P -bad and the result follows from the second part of Theorem 5.

Therefore, we need to assume that the indicated reduction do not generate a P -danger. If the indicated reductions do not generate a P -obstacle then we are done by Theorem 5. Hence, we assume that a P -obstacle is generated. Recall that there exist one P -no-edge, namely e_r , that is the result of the contraction of a path that consists only of real white edges of K . Therefore, by Observation 2, to have the desired result it is enough to prove that there is a potential pair of P -no-edges containing e_r that is not red-connected.

As we are in Case 1, the edge e_r is incident to w which belongs to the bipartition class represented by black vertices in Figure 10(a). For the sake of contradiction we assume that e_r is red-connected to both P -no-edges incident to two available white vertices, say x' and y' , in $V(h_x)$ and $V(h_y)$, respectively. Then, each of these two P -no-edges must be created by contracting paths that intersect at least one of the hexagons h_a, h_b . Hence, in K there are white edges connecting x', y' to r'_1, r_2 , which contradicts the assumption of Case 1 that $\{r_1, r'_1\}$ is a white edge of K .

Case 3: The vertices r_1, r'_1, r_2, r'_2 are not connected by white edges of K . We note that reductions of h_a, h_b by M_a, N_b and by N_a, M_b are safe. Without loss of generality, we consider two subcases.

Case 3.1: None of the four distinct F -no-edges incident to a vertex in $\{r_1, r'_1, r_2, r'_2\}$ is incident to a vertex in $V(h_x \cup h_z \cup h_y)$.

Since K is not a F -obstacle, at least one F -no-edge of K has its end vertices in $V(h_x \cup h_z \cup h_y)$. In this case no reduction of h_a, h_b can lead to a P -obstacle. Moreover, if in the pseudohex obtained by reducing h_a, h_b by M_a, N_b both new derived white edges incident to w, w' are not incident to a vertex in $V(h_x \cup h_z \cup h_y)$ and thus this reduction does not lead to a P -danger either. Hence, the statement of Theorem 6 follows from Theorem 5.

Case 3.2. At least one of the four F -no-edges incident to a vertex in $\{r_1, r'_1, r_2, r'_2\}$ is also incident to a vertex in $V(h_x \cup h_z \cup h_y)$.

Case 3.2.1. For each $i = 1, 2$, exactly one of the F -no-edges incident to a vertex in $\{r_i, r'_i\}$ is incident to a vertex in $V(h_x \cup h_z \cup h_y)$. Hence, each of the two reductions of h_a, h_b by M_a, N_b and M_b, N_a is safe and creates a derived white edge that has both end vertices either in $V(h_x \cup h_z)$ or in $V(h_y \cup h_z)$ and thus, in particular, none of these two reductions creates a P -obstacle, where P is the 3-ear on vertex set $\{x, y, z\}$. By Theorem 5, it is enough to show that for each possible case, at least one of these reductions does not generate a P -danger.

For the sake of contradiction, suppose that for some of the cases both reductions generate a P -danger. As we observed reductions of h_a, h_b by M_a, N_b and M_b, N_a create derived white edge with both end vertices either in $V(h_x \cup h_z)$ or in $V(h_y \cup h_z)$. As both reductions generate a P -danger, it implies that there exist a F -no-edge with its end-vertices in $V(h_x \cup h_y)$. Without loss of generality, we may assume it is the edge e of Figure 10(a).

Hence the two F -no-edges incident to a vertex in $\{r_1, r'_1, r_2, r'_2\}$ and to a vertex in $V(h_x \cup h_z \cup h_y)$ are incident to vertices t, t' in $V(h_x \cup h_z \cup h_y)$. These two vertices belong to different bipartition classes of K and so the F -no-edges incident to them are incident either to r_1, r_2 , or to r'_1, r'_2 . Hence, F_K is configured as a F -abad or a F -bbad, see Definition 3. A contraction to the hypothesis assumption.

Case 3.2.2. Suppose that Case 3.2.1. does not hold. If both F -no-edges incident to r_1, r'_1 are incident to vertices in $V(h_x \cup h_z \cup h_y)$ then reduction of h_a, h_b by M_a, N_b is safe and generates two distinct derived white edges, each of them with an end vertex in $\{w, w'\}$ and an end-vertex in $V(h_x \cup h_z \cup h_y)$. Thus, neither a P -obstacle nor a P -danger is created. Analogously we solve the case that the two F -no-edges incident to r_2, r'_2 are incident to $V(h_x \cup h_z \cup h_y)$.

Without loss of generality, we are left with the cases that there is exactly one F -no-edge with an end-vertex in $\{r_1, r'_1\}$, say r_1 , and one end-vertex in $V(h_x \cup h_z \cup h_y)$ and there are no F -no-edges incident to both subsets, $\{r_2, r'_2\}$ and $V(h_x \cup h_z \cup h_y)$.

By Theorem 5, it is enough to prove that there is a safe reduction of $\{h_a, h_b\}$ that does not generate either a P -obstacle, or a P -danger. Since, we have exactly three available vertices in $V(h_x \cup h_y)$ which are end-vertices of F -no-edges and are not incident to $V(h_a \cup h_b)$, the following two cases arise.

Case 3.2.2.2. Three F -no-edges incident to $V(h_x \cup h_z \cup h_y)$ have exactly one end vertex in $V(F_K)$. Then no reduction of h_a, h_b can create a P -danger. Moreover, by assumption, there is a F -no-edge incident to r_1 and to a vertex in $V(h_x \cup h_z \cup h_y)$, and thus, the safe reduction of h_a, h_b by M_a, N_b creates a P -no-edge with end vertices in $V(h_x \cup h_z \cup h_y)$ and hence this reduction does not lead to a P -obstacle. Consequently, in this case, Theorem 6 holds.

Case 3.2.2.1. One F -no-edge has both end vertices in $V(h_x \cup h_z \cup h_y)$. Then, no reduction of h_a, h_b can create a P -obstacle. Furthermore, the safe reduction of h_a, h_b by N_a, M_b creates derived white edges incident to w, w' and to $V(h_x \cup h_z \cup h_y)$ and then, this reduction does not create a P -danger and hence, Theorem 6 holds.

5.3 Reduction of star forks, subforks and dots on pseudo-hexes

In this section we prove the following theorem.

Theorem 8. *Let L be the star fork, the subfork, or the dot. Let K, K' be proper pseudo-hexes and K obtained from K' by L -addition. Then L_K can be safely reduced.*

Proof. The proof in the case that L is the star fork is contained in Subsection 5.5.1. If L is the dot, then it is trivial that every reduction of L_K is safe.

Let L be the subfork. In the following, we shall use the names of vertices and hexagons of L_K from Figure 11. Two subcases arise: (2.1) at least one L -no-edge has both end-vertices in $V(L_K)$ and (2.2) all L -no-edges have exactly one end-vertex in $V(L_K)$. Let M_z and M_b denote the perfect matchings depicted by thicker lines in Figure 11 on h_z and h_b , respectively. Moreover, let there N_z and N_b denote the perfect matchings of h_z and h_b that are complements of M_z and M_b , respectively.

If K is such that subcase (2.1) holds, and without loss of generality $\{t, w\}$ (see Figure 11 for notation) is a derived white edge, then reduction of h_z, h_b by N_z, M_b is safe. In the subcase (2.2) the reduction of h_z, h_b by M_z, M_b is safe.

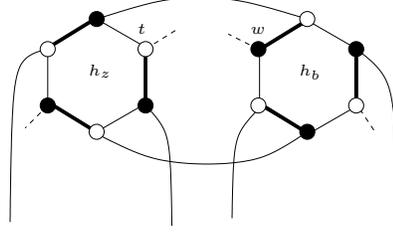


Figure 11: Subfork pseudohex. Perfect matchings M_z and M_b depicted in h_z and h_b by thicker lines. Derived white edges are represented by dotted lines.

□

5.4 Reduction of p-forks on pseudohexes

We prove that there is a safe reduction of the p-fork on pseudohexes.

Theorem 9. *Let L be a p-fork. Let K, K' be proper pseudohexes and K obtained from K' by L -addition. Then L_K has a safe reduction.*

Proof. We consider the notation from Figure 12.

In the case that the derived edges from the set $\{b_x, w_x, b_y, w_y, b_z, w_z\}$ are pairwise distinct, the reduction of L_K by $N_z, N'_z, N'_b, N'_a, N_b, M_a, M_x, M_y$ is safe. We suppose that there are exactly two derived white edges that are the same, but the rest are all pairwise distinct. Without loss of generality, two cases arise: (1) $b_z = w_x$, and (2) $b_x = w_y$. In the first case, we reduce L_K by $N_z, N'_z, M'_b, N'_a, M_b, N_a, N_x, N_y$, finding a safe reduction and in the second one by $M_z, N'_z, N'_b, M'_a, N_b, M_a, M_x, M_y$. If exactly two pairs of derived white edges are the same, again without loss of generality we have two cases: (1') $w_x = b_y$ and $w_y = b_z$, and (2') $w_z = b_x$ and $b_z = w_y$. The reductions of L_K by $N_z, M'_z, N'_b, M'_a, N_b, M_a, M_x, M_y$ for case (1') and by $N_z, M'_z, M'_b, N'_a, N_b, M_a, M_x, M_y$ for case (2') are safe reductions. We are left with the case that all derived white edges have both end-vertices in L_K . Then, the reduction of L_K by the perfect matchings $M_z, N'_z, N'_b, N'_a, N_b, M_a, N_x, M_y$ is safe. □

In the next Subsection we study safe reductions of j -big-forks on pseudohexes.

5.5 Reduction of j -big-forks on pseudohexes

We first need to introduce the concept of B -obstacles in the case that B a j -big-fork. This notion is equivalent to the notion of cut-obstacles.

Definition 4. *Let K, K' be proper pseudohexes such that K is obtained from K' by B -addition, where B is a j -big-fork. We say that K has a B -obstacle if it has $2(4 + j)$ distinct B -no-edges.*

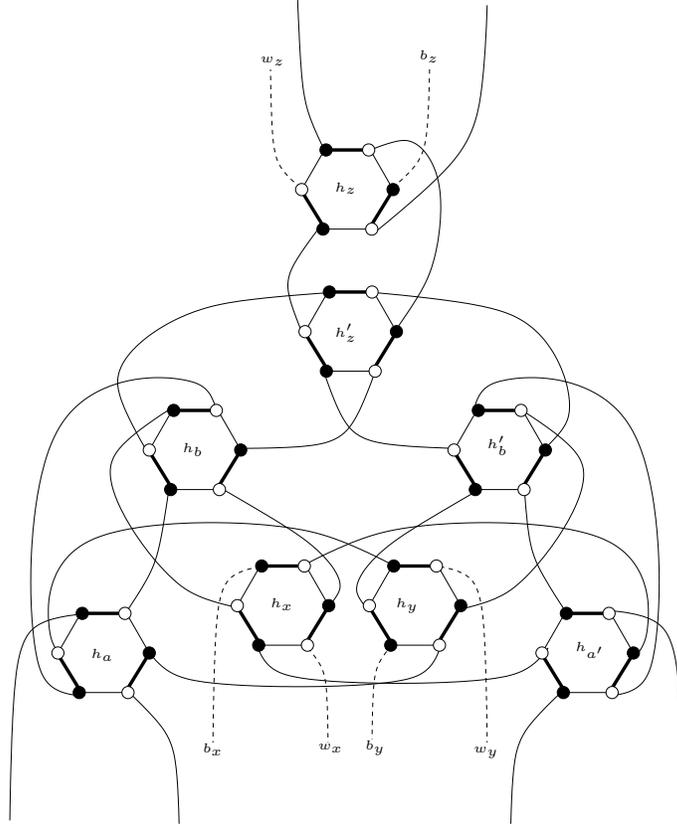


Figure 12: p-fork Pseudohex. Perfect matchings $M_a, M_b, M'_a, M'_b, M_z, M'_z, M_x, M_y$ depicted in $h_a, h_b, h'_a, h'_b, h_z, h'_z, h_x, h_y$ by thicker lines. Derived white edges are dotted.

In Theorem 10, we consider the case $j = 1$. We recall that 1-big-forks are simply called big-forks.

Theorem 10. *Let K, K' be proper pseudohexes such that K is obtained from K' by B -addition, where B is the big-fork. If K does not have a B -obstacle, then there is a safe reduction of B_K .*

5.5.1 Proof of Theorem 10

Let F and T denote the fork and the star fork respectively used to obtain B (as explained in Figure 3).

By Theorem 6 it suffices to show that there is a safe reduction of T_K that generates a pseudohex that has neither a F -obstacle, nor a F -abad, nor a F -bbad. We say that such a reduction of T_K is *fundamental*.

In this proof we use the names of vertices of the big-fork from Figure 3. Namely, the vertex set of T is $\{x', y', z', b'\}$ and the vertex set of F is $\{x, y, z, a, b\}$. In addition, we consider the bipartition classes of K as white and black vertices. We further denote by $b_{x'}$ the derived white edge incident to a black vertex of $h_{x'}$ and by $w_{x'}$ the derived white edge incident to a white vertex of $h_{x'}$. Analogously, we use this notation for the derived white edges incident to $h_{y'}, h_{b'}$. Let $M_{x'}, M_{y'}, M_{z'}, M_{b'}$ be the perfect matchings indicated in Figure 13 by thicker lines, and we denote

by $N_{x'}, N_{y'}, N_{z'}$ and $N_{b'}$ the perfect matchings that are complement of $M_{x'}, M_{y'}, M_{z'}$ and $M_{b'}$, respectively.

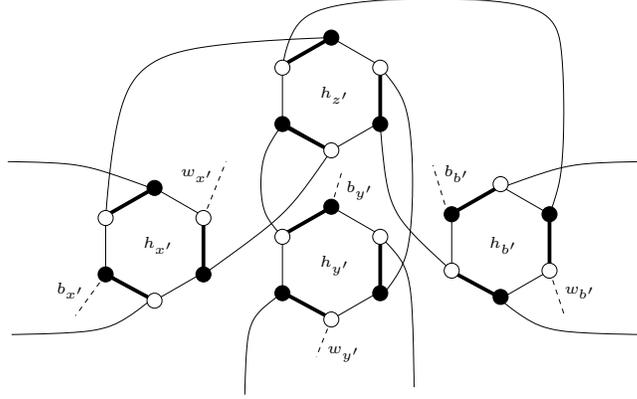


Figure 13: Star fork. Perfect matchings $M_{x'}$, $M_{y'}$, $M_{z'}$ and $M_{b'}$ are depicted in $h_{x'}$, $h_{y'}$, $h_{z'}$ and $h_{b'}$ by thicker lines. Derived white edges are dotted

We still introduce one more notation. The hexagons $h_{x'}, h_{y'}, h_{b'}$ are connected by three pairs of real white edges to three pairs of vertices in $V(K) \setminus V(T_K)$. Let us denote by u_1, v_1 the pair connected to $h_{x'}$, by u_2, v_2 the pair connected to $h_{y'}$ and by u_3, v_3 the pair connected to $h_{b'}$. We note that for each $i = 1, 2, 3$, the vertices u_i, v_i belong to the same hexagon of K . We further note that u_1, v_1, u_2, v_2 are in $V(F_K)$ and the u_3, v_3 belong to $V(K')$. Let us assume that u_i (v_i , respectively) belongs to the bipartition class of black vertices (white vertices) of K , for each $i = 1, 2, 3$.

By the hypothesis assumption, there exists a B -no-edge with both end-vertices in $V(B_K)$. We distinguish two cases.

Case 1. There are distinct $p, q \in \{x', y', b'\}$ such that $w_p = b_q$ and $b_p = w_q$. Let $r \in \{x', y', b'\} - \{p, q\}$. Then both edges w_r, b_r have exactly one end-vertex in $V(T_K)$. We safely reduce hexagons $h_{x'}, h_{y'}, h_{z'}, h_{b'}$ by $M_{x'}, M_{y'}, M_{z'}, M_{b'}$. Two cases may happen: w_r, b_r have end-vertices either in $V(F_K)$, or in $V(K')$. We note that in both cases the suggested reduction creates one derived white edge e' subset of $V(F_K)$, and another derived white edge e'' incident to the same hexagon of F_K as e' that is incident to $V(F_K)$ and to $V(K')$. Hence, this reduction is fundamental.

Case 2. Case 1 does not happen. Then we can assume without loss of generality that the B -no-edges which are subsets of $V(T_K)$ belong to the set $\{w_{x'} = b_{y'}, w_{y'} = b_{b'}, w_{b'} = b_{x'}\}$.

We first make an observation that follows directly from the assumption of Case 2, Figure 13 and the symmetry of the star fork.

Observation 4. *The reduction of $h_{x'}, h_{y'}, h_{z'}, h_{b'}$ by $N_{x'}, M_{y'}, N_{z'}, N_{b'}$ is safe and generates a pseudohex so that the F -no-edge incident to u_1 (u_2, u_3, v_1, v_2 and v_3 , respectively) is obtained by contracting a path that contains $b_{b'}$ ($b_{x'}, b_{y'}, w_{x'}, w_{y'}$ and $w_{b'}$, respectively). By symmetry of the star-fork, there are safe reductions of T_K such that the F -no-edge incident to u_1 (u_2, u_3, v_1, v_2 and v_3 , respectively) is obtained by contracting a path that contains $b_{y'}$ ($b_{b'}, b_{x'}, w_{x'}, w_{y'}$ and $w_{b'}$, respectively), $b_{x'}$ ($b_{y'}, b_{b'}, w_{b'}, w_{x'}$ and $w_{y'}$, respectively), and $b_{x'}$ ($b_{y'}, b_{b'}, w_{y'}, w_{b'}$ and $w_{x'}$, respectively).*

We split up Case 2 into subcases.

Case 2.1 $w_{x'} = b_{y'}$, $w_{y'} = b_{b'}$, $w_{b'} = b_{x'}$. Then, we reduce $h_{x'}$, $h_{y'}$, $h_{z'}$, $h_{b'}$ by $N_{x'}$, $M_{y'}$, $N_{z'}$, $N_{b'}$. By Observation 4, this is a safe reduction and it creates derived white edges $\{u_1, v_2\}$, $\{v_1, u_3\}$ and $\{u_2, v_3\}$. Since, $\{u_1, v_2\}$ is a subset of $V(F_K)$ and each $\{v_1, u_3\}$, $\{u_2, v_3\}$ has exactly one end-vertex in $V(F_K)$ and these end-vertices belong to distinct hexagons of F_K , this reduction is fundamental.

Case 2.2 There are distinct $p, q, r \in \{x', y', b'\}$ so that $w_p = b_q$ and $w_r = b_p$. We first assume that either $w_{x'} = b_{y'}$ and $w_{y'} = b_{b'}$, or $w_{y'} = b_{b'}$ and $w_{b'} = b_{x'}$. By Observation 4, the reduction of $h_{x'}$, $h_{y'}$, $h_{z'}$, $h_{b'}$ by $N_{x'}$, $M_{y'}$, $N_{z'}$, $N_{b'}$ creates derived white edges either $\{v_1, u_3\}$ and $\{v_2, u_1\}$, or $\{u_2, v_3\}$ and $\{u_1, v_2\}$. We now assume that $w_x = b_y$ and $w_b = b_x$. By Observation 4, there is a the reduction of $h_{x'}$, $h_{y'}$, $h_{z'}$, $h_{b'}$ that creates derived white edges $\{u_2, v_3\}$ and $\{u_1, v_2\}$. Therefore, in all the cases there is a F -no-edge e with both end-vertices in $V(F_K)$ and a F -no-edge e' with exactly one end-vertex in $V(F_K)$ such that e and e' are incident to the same hexagon in F_K . Hence, for each case the indicated reduction is fundamental.

Case 2.3 There are distinct $p, q \in \{x', y', b'\}$ so that $w_p = b_q$.

If at least three derived white edges incident to vertices in $V(F_K)$ have an end vertex in $V(K')$ then, by Observation 4 there exists a safe reduction of T_K that creates a F -no-edge with both end-vertices in $V(F_K)$ (such a F -no-edge is the one created by the contraction of the path that contains $w_p = b_q$). Moreover, the resulting pseudo-hex still has at least three derived white edges incident to vertices in $V(F_K)$ and each with an end-vertex in $V(K')$. Hence, this reduction is fundamental.

If $\{u_1, u_2, v_1, v_2\} \cap V(F_K) \subset V(h_a \cup h_b)$ then, by Observation 4 there exists a safe reduction of T_K that creates a F -no-edge with both end-vertices in $V(h_a \cup h_b)$. Then, this reduction is fundamental; see Figures 10(a), 10(b).

The remaining case is that $\{u_1, u_2, v_1, v_2\} \cap V(h_x \cup h_y) \neq \emptyset$ and at most two derived white edges with an end-vertex in $V(F_K)$ have end-vertices in $V(K)$. By Observation 4 there exists a reduction of T_K that creates a derived white edge (denoted say by e' and obtained by contracting the path that contains $w_p = b_q$) with an end-vertex in $V(h_x \cup h_y)$ and an end-vertex in $V(K')$. Therefore, this reduction generates neither a F -bbad nor a F -abad, and hence, we only need to show that this reduction does not create a F -obstacle.

If $V(F_K)$ has a derived white edge as a subset then the last is trivial. Hence, let us assume that $V(F_K)$ have no derived white edges as a subset. There are 4 vertices of $V(F_K)$ contained in B -no-edges of K , and by assumption, at least two of them are in a B -no-edge with end-vertices in $V(T_K)$. One real white edge with end-vertices in $V(T_K)$ and in $V(K')$ is in the contracted path that creates e' , and hence at least one derived white edge with end-vertices in $V(T_K)$ and in $V(F_K)$ is not on a contracted path that creates a new derived white edge incident to a vertex in $V(K')$. Then, this reduction creates a derived white edge with both end-vertices in $V(F_K)$, and we thus, a F -obstacle is not generated.

Case 2.4 All derived white edges incident to a vertex in $V(T_K)$ are incident to a vertex in $V(K) \setminus V(T_K)$. We recall that K is not a B -obstacle, and so at least one derived white edge, say edge e' , has end-vertices in $V(T_K)$ and in $V(F_K)$. Now we proceed similarly as in Case 3. There are 6 distinct B -no-edges with one end-vertex in $V(T_K)$, and at most 4 of them have an end-vertex in $V(F_K)$. Hence at least two of them have an end-vertex in $V(K')$.

If $\{u_1, u_2, v_1, v_2\} \cap V(h_x \cup h_y) \neq \emptyset$ then by Observation 4 there exists a safe reduction of T_K so that in the resulting pseudo-hex there is a new derived white edge with end-vertices in $V(h_x \cup h_y)$ and in $V(K')$, and also a new derived white edge that has both end-vertices in $V(F_K)$. Hence, this reduction is fundamental, see Figures 10(a), 10(b).

Hence, let $\{u_1, u_2, v_1, v_2\} \cap V(F_K) \subset V(h_a \cup h_b)$. First let there be a B -no-edge with both end-vertices in $V(F_K)$. If there are two B -no-edges with end-vertices in $V(T_K)$ and in $V(K')$ and these two edges are incident to different hexagons of T_K , then by Observation 4 there is a safe reduction of T_K such that for both hexagons h_a, h_b there are new derived white edge incident to them and to $V(K')$. This reduction is fundamental by Figures 10(a), 10(b). Otherwise, necessarily exactly two B -no-edges have end-vertices in $V(T_K)$ and in $V(K')$, and four B -no-edges have end-vertices in $V(F_K)$. By Observation 4, there is a safe reduction of T_K such that there are new derived white edges with both end-vertices in $V(F_K)$ and incident to $V(h_a)$ and to $V(h_b)$ (not necessarily one derived white edge incident to both). Hence, this reduction is fundamental.

Let there be no B -no-edge subset of $V(F_K)$. If at least one B -no-edge have its end-vertices in $V(F_K)$ and in $V(K')$, then we note that by our assumptions such B -no-edge is incident to $V(h_x \cup h_y)$. By Observation 4, there exists a safe reduction of T_K so that the new derived white edge obtained by contracting the path that contains e' is incident to $V(h_a \cup h_b)$ and thus it is a subset of $V(F_K)$. Therefore, this reduction is fundamental.

Finally let all four B -no-edges have their end-vertices in $V(h_x \cup h_y)$ and in $V(T_K)$. Then by Observation 4, there exists a safe reduction of T_K so that there are new derived white edges incident to h_a and to h_b which are subsets of $V(F_K)$. Consequently, this reduction is fundamental. \square

Finally, we extend Theorem 10 from $j = 1$ to general $j \geq 1$.

Theorem 11. *Let K, K' be proper pseudohexes such that K is obtained from K' by B -addition, where B is the j -big-fork for $j \geq 1$. If K does not have a B -obstacle, then there is a safe reduction of B_K .*

Proof. By induction on j . By Theorem 10, the statement holds for $j=1$. Let $j > 1$ and B be a j -big-fork. Let there B' and T denote the $(j - 1)$ -big-fork and the star, respectively, from which B is obtained. Since K does not have a B -obstacle there is a derived white edge, say e , with both end vertices in $V(B_K)$. If such a derived white edge has both end-vertices in B'_K , then by the induction hypothesis and Theorem 8, we can conclude that there is a safe reduction of B_K . If e has one end-vertex in T_K and the other one in B'_K or both end-vertices in T_K , by the Observation 4 contained in the proof of Theorem 10, we have that there is a safe reduction of T_K that creates a derived white edge with both end-vertices in $K_{B'}$ and therefore, again by the induction hypothesis we can find a safe reduction B'_K . The result follows. \square

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