Chapter 5

Turán-type problems

5.1 Disjoint edges in geometric graphs

Theorem 5.1 (Hopf–Pannwitz, 1934). *For every $n \geq 3$, the maximum number of times the diameter (the largest distance) can occur among n points in the plane is n.*

Proof. Let S be the given set of n points. Let G be a graph with vertex set S such that two vertices form an edge if and only if the corresponding two points form a diameter of S. If the degree of every vertex in G is at most 2, then we are done. If G has a vertex v whose degree is at least 3, then let u be one of its neighbors that is not the leftmost nor the rightmost one. A simple geometric observation shows that the degree of u is 1; see Figure 5.1. Indeed, all vertices of S must lie in the region that is an intersection of the unit discs centered in u, v, and the leftmost and the rightmost neighbor of v. The only point in the intersection region that has distance 1 from u is the point v. By induction, $G - u$ has at most $n - 1$ edges, thus G has at most n edges.

On the other hand, the vertices of a unit triangle and a set of $n-3$ points on a unit circle centered in one of its vertices show that n diameters can be achieved. □

Now comes the definition that gave this course its name.

Definition 5.2. A **geometric graph** is a graph whose vertices are represented by distinct points in general position in the plane and whose edges are drawn as straight-line segments, possibly with crossings.

Using the triangle inequality, it is an easy exercise to show that in the geometric graph formed by diameters of a finite point set in the plane there are no two disjoint edges. Theorem 5.1 can be generalized as follows.

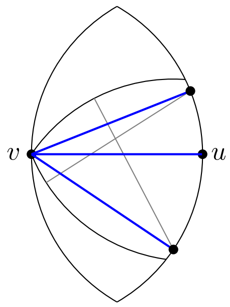

Figure 5.1: A vertex of degree at least 3 in the diameter graph has a neighbor of degree 1.

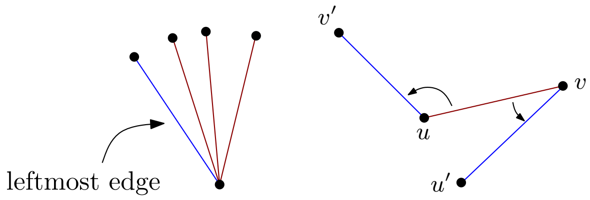

Figure 5.2: Perles' argument

Theorem 5.3. *Let G be a geometric graph with no two disjoint edges. Then* $|E(G)| \leq |V(G)|$.

Proof. (Perles) We call a vertex $v \in V(G)$ *pointed* if there is a line ℓ passing through v such that all edges incident to v lie in a halfplane bounded by ℓ. At each pointed vertex v, a chicken lays an egg on the "leftmost" edge incident to v, that is, the first edge in the clockwise order of edges around v, starting from the line ℓ. Now we observe that every edge of G has an egg on it, which proves the theorem. Indeed, suppose that there is no egg on an edge $uv \in E(G)$. Thus, originally G contained two edges, uv' and $u'v$, with clockwise angles $v'uv$ and $u'vu$ smaller than $180°$. These two edges must lie on opposite sides of the line uv; see Figure 5.2. Hence, they are disjoint, contradicting the assumption. □

In other words, Theorem 5.3 says that the number of edges in a straight-line thrackle is at most n.

Problem 5.4 (Avital, Hanani, 1966; Kupitz, 1979; Perles, Erdős)**.** *Fix a $k \geq 2$. What is the maximum number $f_k(n)$ of edges that a geometric graph on n vertices can have without containing k pairwise disjoint edges?*

The following table summarizes the current knowledge.

k	$f_k(n)$	
2	$= n$	Hopf–Pannwitz, 1934
3	$= 2.5n + O(1)$	Černý, 2005 [17]
4	$\leq 10n$	Goddard, Katchalski and Kleitman, 1996 [30]
> 4	$\leq 2^9(k-1)^2 n$	Tóth, 2000 [65]

Definition 5.5. A geometric graph is **convex** if its vertices are in convex position; that is, they form the vertex set of a convex polygon.

Proposition 5.6 (Kupitz, 1982)**.** *For any $n \geq 2k+1$, the maximum number of edges that a convex geometric graph with n vertices can have without containing $k+1$ pairwise disjoint edges is kn.*

Proof. Let G be a convex geometric graph with n vertices. Without loss of generality we can assume that the vertex set of G is the set of vertices of a regular n-gon. Partition the set of edges of G into n classes so that two segments belong to the same class if and only if they are parallel. If G has no $k+1$ pairwise disjoint edges, then each class contains at most k elements of $E(G)$. Thus, $|E(G)| \leq kn$.

To show that the bound can be attained, take a graph G with vertices $V = \{x_0, \dots, x_{n-1}\}$ that appear clockwise in this order and with the edges $x_i x_{i+\lfloor n/2 \rfloor + j}$, $0 \leq i \leq n-1$, $1 \leq j \leq k$, where the index $i + \lfloor n/2 \rfloor + j$ is computed modulo n. □

5.2 Partial orders and Dilworth's theorem

Definition 5.7. A binary relation $\preceq$ on a set X is a **partial order** on X if $\preceq$ is reflexive, antisymmetric and transitive. That is, for every $x, y, z \in X$, we have $x \preceq x$, $(x \preceq y) \wedge (y \preceq x) \Rightarrow (x = y)$, and $(x \preceq y) \wedge (y \preceq z) \Rightarrow (y \preceq z)$. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. Two elements of X are **comparable** by $\preceq$ if $x \preceq y$ or $y \preceq x$, otherwise they are **incomparable**. A partial order $\preceq$ on X is a **total order** if every two elements of X are comparable. A pair $(X, \preceq)$ where X is a set and $\preceq$ is a partial order on X is called a **partially ordered set** or also a **poset**. The **comparability graph** $G(P)$ of a partially ordered set $P = (X, \preceq)$ is the graph with vertex set X such that for every two distinct elements $x, y \in X$, xy is an edge of $G(P)$ if and only if x and y are comparable by $\preceq$. A **chain** in a poset is a totally ordered subset, that is, a subset whose elements are pairwise comparable. An **antichain** in a poset is a subset of elements that are pairwise incomparable. See Figure 5.3 for an illustration of the power set of a four-element set ordered by inclusion.

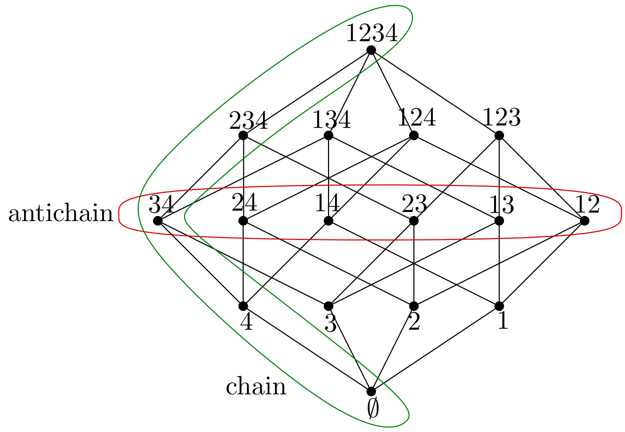

Figure 5.3: A *Hasse diagram* of the set of subsets of $\{1, 2, 3, 4\}$ ordered by inclusion. The comparability graph is obtained by joining all pairs of vertices connected by a vertically monotone path in the diagram. A chain and an antichain are highlighted.

Dilworth's theorem and Mirsky's theorem are important results about partially ordered sets. Notice they are "dual" to each other.

Theorem 5.8 (Dilworth, 1950 [20]). *If the maximum size of an antichain in a partially ordered set P is k, then P is a union of k chains.*

See e.g. [48, Theorem 14.9] for the proof. Dilworth's theorem is closely related (in fact, easily shown to be equivalent) to Hall's marriage theorem, König's theorem about vertex covers in bipartite graphs, and can be also derived from the max-flow min-cut theorem.

Theorem 5.9 (Mirsky, 1971 [46]). *If the maximum length of a chain in a partially ordered set P is k, then P is a union of k antichains.*

Although Mirsky's theorem was published later than Dilworth's theorem, its proof is significantly easier. We warn the reader that some authors include Mirsky's theorem as a part of Dilworth's theorem.

Proof. Let $P = (X, \preceq)$. For $x \in P$, let $r(x)$ denote the maximum length of an increasing chain starting at x. By the assumption, we have $1 \le r(x) \le k$ for every x. Let $X_i = \{x \in X : r(x) = i\}$. Since $X = \bigcup_i X_i$, it suffices to prove that X_i is an antichain for every i. Take $x, y \in X_i$. Suppose for contradiction that $x \prec y$. We have

$$x \prec y = y_1 \prec y_2 \prec \cdots \prec y_i$$

for certain $y_2, \dots, y_i \in X$, which implies that $x \notin X_i$; a contradiction. $\square$

Both Dilworth's theorem and Mirsky's theorem (separately) imply the following interesting corollaries.

Corollary 5.10. *Every partially ordered set of size at least $kl+1$ has either a chain of length $k+1$ or an antichain of size $l+1$.*

In particular, every partially ordered set of size n has either a chain or an antichain of size at least $\sqrt{n}$. Notice that by a straightforward application of Ramsey's theorem to the corresponding comparability graph we get only a chain or an antichain of size $\log n$.

Corollary 5.11 (Gallai, Hajós)**.** *Every system of at least $kl+1$ intervals on a line has either $k+1$ disjoint members or $l+1$ intersecting members.*

Corollary 5.12 (Erdős–Szekeres lemma)**.** *Every sequence of at least $kl+1$ distinct real numbers has an increasing subsequence of length $k+1$ or a decreasing subsequence of length $l+1$.*

We now use Mirsky's theorem to obtain a generalization of Theorem 5.3 to geometric graphs with at most k pairwise disjoint edges.

Theorem 5.13 (Pach–Törőcsik, 1994)**.** *If the maximum number of pairwise disjoint edges in a geometric graph G is k, then $|E(G)| \le k^4|V(G)|$.*

Proof. We start by defining four strict partial orders $\prec_1, \prec_2, \prec_3, \prec_4$ on the family of segments in the plane. Analogous partial orders can also be defined for compact convex sets [40]. Hopefully the reader will be satisfied with a pictorial definition (Figure 5.4). For each of the orders, only disjoint segments are comparable, and every pair of disjoint segments is comparable by at least one of the four orders. The four orders are distinguished by the relative ordering of the x-coordinates of the endpoints of the segments; there are exactly six such possible orderings. If two segments e, f comparable by $\prec_i$ intersect a common vertical line ℓ, then $e \prec_i f$ if e intersects ℓ below f. We invite the reader to verify that each of the four orders is indeed a strict partial order. Note that if e lies completely to the left of f, then $e \prec_1 f$ and $f \prec_2 e$ simultaneously.

Interpret $E(G)$ as the set of closed segments representing the edges of G. Since G has no $k+1$ disjoint edges, there is no chain of length $k+1$ in either of the posets $(E_G, \prec_i)$. By Mirsky's theorem, we can divide $E(G)$ into k subsets $E_1 \cup \cdots \cup E_k$ that are antichains $(E_G, \prec_1)$. Now pick the largest of the subsets E_i, which is of size at least $\frac{|E(G)|}{k}$, and further divide it into

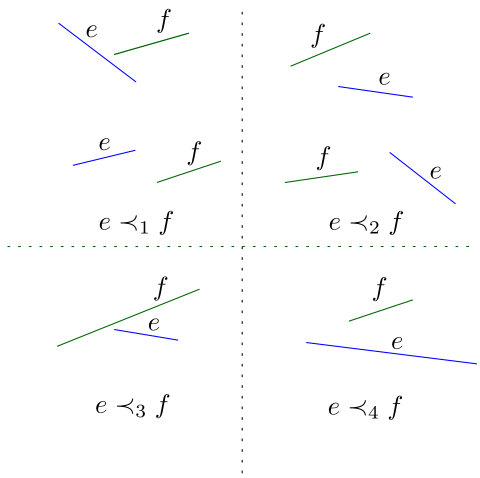

Figure 5.4: A schematic definition of the four partial orders of segments.

k subsets that are antichains in $(E_G, \prec_2)$. One of the parts will have size at least $\frac{|E(G)|}{k^2}$. Continue dividing in this fashion four times in total. In the end we obtain a set $H \subset E(G)$ such that $|H| \geq \frac{|E(G)|}{k^4}$, and H is an antichain in each of the four posets $(E_G, \prec_i)$. This implies that every two segments in H intersect. By Theorem 5.3, we have $|H| \leq n$. Therefore, $|E(G)| \leq k^4 n$. □

Definition 5.14. Let $f(n)$ denote the largest number such that any family of n convex sets in the plane has $f(n)$ disjoint or $f(n)$ pairwise intersecting members.

From the proof of Theorem 5.13 we have $f(n) \geq n^{1/5}$ [40].

The rest of this section was not presented during the lectures in 2014/2015.

We now show a trivial construction for an upper bound. Consider $\sqrt{n}$ groups of $\sqrt{n}$ segments, so that segments in each group are mutually intersecting, while the groups are pairwise disjoint. With this trivial construction we have a simple upper bound: $f(n) \leq \sqrt{n}$.

A less trivial construction tightens the upper bound to approximately $n^{0.431}$, where $0.431 \geq \frac{\log 2}{\log 5}$. The construction method is commonly used in combinatorics and graph theory in general: we find a small configuration that is good and we iterate it. We start with a pentagon formed by five segments. This configuration shows that $f(5) \leq 2 < \sqrt{5}$. Now we replace each segment with what would look like a very thin squeezed pentagon like

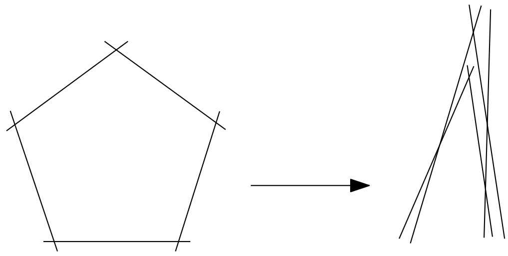

Figure 5.5: An iterative construction for an upper bound on $f(n)$.

in Figure 5.5. Now we have 5^2 segments with the maximum number of intersecting (or disjoint) segments being 2^2. Iterating in this fashion we get $n = 5^k$ segments, with at most 2^k pairwise disjoint or intersecting segments. Since $2^k = n^{\log 2/\log 5} \le n^{0.431}$, we have

$$n^{1/5} \le f(n) \le n^{0.431}.$$

As one can see, the bounds are not quite tight. Further small refinements can be made, but this is more difficult.

Definition 5.15. Let $F_k(n)$ be the largest number such that any graph G that is a union of k comparability graphs $G_1, \ldots, G_k$ contains $F_k(n)$ vertices that form a complete subgraph or $F_k(n)$ vertices that are independent.

Similarly as in Theorem 5.13, we get that

$$F_k(n) \ge n^{\frac{1}{k+1}},$$

as follows. Let $l = n^{1/(k+1)}$. Color the edges of G in k different colors corresponding to the k partial orders. If there is no complete subgraph of size l in the first color, say, red, we find a subset of at least n/l vertices with no red edge. We repeat this step for each of the colors. After k steps we will either find a complete subgraph with all edges of the same color or a set of at least $n/l^k = l$ vertices with no edges of any color, which means that it is an independent set.

An upper bound was constructed by Dumitrescu and Tóth, in a purely combinatorial way.

Theorem 5.16 (Dumitrescu and Tóth, 2002 [21])**.**

$$n^{1/(k+1)} \le F_k(n) \le n^{(1+\log k)/k}.$$

5.3 Crossing edges and bisection width

A graph that has a drawing in the plane in which there are no k pairwise crossing edges is called **k-quasiplanar**. Clearly, a graph is 2-quasiplanar if and only if it is planar, thus 2-quasiplanar grphs with $n \geq 3$ vertices have at most $3n - 6$ edges. The following table summarizes current known upper bounds on the number of edges of k-quasiplanar graphs with n vertices.

$k = 2$	$3n - 6$	Euler
$k = 3$	$O(n)$	Pach, Radoičić and Tóth, 2006 [50]
$k = 3$	$8n - 20$	Ackerman and Tardos, 2007 [3]
$k = 4$	$72(n - 2)$	Ackerman, 2009 [1]
$k > 4$	$O(n \log^{4k-12} n)$	Pach, Radoičić and Tóth, 2006 [50]
$k > 4$	$O(n \log^{4k-16} n)$	Ackerman, 2009 [1]

A linear upper bound is conjectured.

Conjecture 5.17. *For every k there is a constant c_k such that every k-quasiplanar graph with n vertices has at most $c_k n$ edges.*

For geometric graphs with no k pairwise crossing edges, slightly better upper bounds are known.

$k = 3$	$O(n)$	Agarwal et al., 1997 [4]
$k = 3$	$6.5n - O(1)$	Ackerman and Tardos, 2007 [3]
$k \geq 2$	$O(n \log^{2k-4} n)$	Pach, Shahrokhi and Szegedy, 1996 [51]
$k \geq 4$	$O(n \log^{2k-6} n)$	Agarwal et al., 1997 [4]
$k \geq 4$	$O(n \log n)$	Valtr, 1998 [69]

For convex geometric graphs, a linear upper bound is known.

Theorem 5.18 (Capoyleas–Pach, 1992 [13])**.** *For any $n \geq 2k - 1$, the maximum number of edges in a convex geometric graph with n vertices and no k pairwise crossing edges is*

$$2(k-1)n - \binom{2k-1}{2}.$$

The upper bound in Theorem 5.18 is tight due to the following construction. Let $x_1, \ldots, x_n$ be the vertices of a convex n-gon in clockwise order. Connect x_i and x_j by an edge if and only if they are separated by fewer than k vertices along the boundary of the polygon, or $1 \leq i \leq k - 1$.

Now we prove a weaker upper bound on the number of edges in geometric graphs with no three pairwise crossing edges.

Theorem 5.19. *The maximum number of edges in a geometric graph with n vertices and no three pairwise crossing edges is $O(n^{3/2})$.*

Proof. Let G be a geometric graph with n vertices, m edges and no three pairwise crossing edges. By the crossing lemma, G has at least $\frac{1}{64} \cdot \frac{m^3}{n^2}$ crossings, so there is an edge e with at least $\frac{1}{32} \cdot \frac{m^2}{n^2}$ crossings. By our assumption, the edges that cross e do not cross each other, in other words, they form a plane graph, and thus there are at most $3n$ of them. In fact, since they also form a bipartite graph, there are at most $2n$ of them. Therefore, we have $\frac{1}{32} \cdot \frac{m^2}{n^2} \leq 2n$, which implies that $m \leq 8n^{3/2}$. □

Next we prove a general upper bound for geometric graphs with no k pairwise crossing edges.

Definition 5.20. Let G be a graph with n vertices. The **bisection width** of G, denoted by $b(G)$, is the minimum number of edges one has to remove from G so that the vertex set of the resulting graph G' can be divided into two parts, A and B, such that there is no edge between A and B in G' and $|A|, |B| \leq 2n/3$. Instead of the last inequality, we can equivalently require that $|A|, |B| \geq n/3$.

Clearly, $b(G) \leq 2n^2/9$ for every graph G with n vertices. This is of course tight if G is the complete graph. The bisection width of a planar graph with n vertices can be as large as $2n/3$; this contrasts with the separator theorem and shows that removing vertices might be much more powerful in disconnecting the graph. It is a simple exercise to show that the bisection width of an m times m grid is at least $m/3$ and at most m.

The following theorem may be considered as a variant of the crossing lemma, which gives a lower bound on the crossing number in terms of the bisection width.

Theorem 5.21 (Leighton [41]; Pach, Sharokhi and Szegedy [51])**.** *Let G be a graph with n vertices and degree sequence $d_1, d_2, \ldots, d_n$. For the bisection with of G, we have*

$$b(G) \leq 1.58\sqrt{16\mathrm{cr}(G) + \sum_{i=1}^{n} d_i^2}.$$

It is an easy exercise to see that if G is a random graph with n vertices where every edge is taken independently with probability $1/2$, then with probability more than 0.99, G has at least $n^2/10$ edges, and thus the crossing number of G is at least cn^4 for some constant c, by the crossing lemma. It is also an easy exercise to show that with probability at least $1/2$, the bisection width of G is at least $n^2/100$.

For the proof of Theorem 5.21, we use the following variant of the weighted separator theorem, where we remove edges instead of vertices.

Theorem 5.22 (Gazit–Miller, 1990 [29])**.** *Let G be a planar graph with n vertices. Let $f : V(G) \to [0, 2/3]$ be a weight function assigning a nonnegative real weight to each vertex of G. Suppose that $\sum_{v \in V(G)} f(v) = 1$. Then the vertex set of G can be partitioned into two sets A, B such that each of A, B has total weight at most $\frac{2}{3}$, and the number of edges between A and B in G is at most $1.58\sqrt{\sum_{i=1}^{n} d_i^2}$.*

Proof of Theorem 5.21. Let D be a drawing of G with $\mathrm{cr}(G)$ crossings. Construct a drawing D' of a graph G' by replacing each crossing in D by a *new* vertex of degree 4, subdividing the two edges participating in the crossing. The vertices of G are called the *old* vertices in G'. The new graph G' is planar and satisfies $|V(G')| = |V(G)| + \mathrm{cr}(G)$. Note that when the number of edges of G grows asymptotically faster than the number of vertices, the crossing lemma implies that the new vertices significantly outnumber the old vertices in G'.

We assign weight 0 to every new vertex in G', and weight $1/n$ to every old vertex. Now we apply Theorem 5.22 to G'. We get a set S' of at most $1.58\sqrt{16\mathrm{cr}(G) + \sum_{i=1}^{n} d_i^2}$ edges separating G' into two parts, each containing at most $2n/3$ old vertices. From S', we create a corresponding set S of edges in G by copying every edge of S' between two old vertices, and for every edge $e' \in S'$ incident to a new vertex, we take the edge e of G that extends e'. Now S separates G into two parts of size at most $2n/3$, and contains at most $1.58\sqrt{16\mathrm{cr}(G) + \sum_{i=1}^{n} d_i^2}$ edges. This gives an upper bound on the bisection width of G. $\square$

Sketch of proof of Theorem 5.22. (by Pach, Spencer and Tóth [52]). We consider only the case when G has two types of vertices, one type with weight 0 and the other type with weight $1/m$. Let G'' be a graph obtained from G by replacing each vertex v_i of weight $1/m$ of degree d_i by a $d_i \times d_i$ grid, and by connecting the edges that were incident to v_i to the vertices on one side of the grid (called *special* vertices), so that each vertex in the grid has degree at most 4. See Figure 5.6. Then all vertices of G'' have degree at most 4. The number of vertices in G'' is $|V(G'')| = \sum_{i=1}^{n} d_i^2$.

For every i, we assign weight $1/(md_i)$ to every special vertex in the $d_i \times d_i$ grid constructed from v_i, and weight 0 to all other vertices.

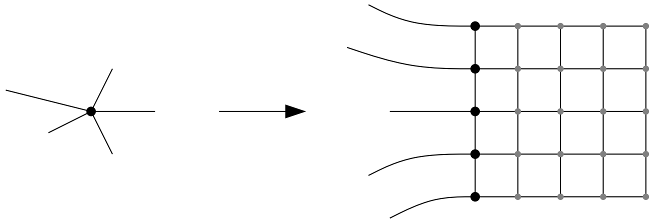

Figure 5.6: Replacing a vertex by a grid. Special vertices form the left column of the grid.

By the weighted separator theorem (Theorem 3.5), we can split G'' into two parts A, B with total weight at most $2/3$ by removing a set S of at most $2\sqrt{|V(G'')|} = 2\sqrt{\sum_{i=1}^{n} d_i^2}$ vertices of G''.

Using the partition A, B, S of G'', we want to define a partition of G into two parts, separated by few edges. The idea is to put the vertices v_i such that the corresponding $d_i \times d_i$ grid has many points in A to one part, the vertices v_i such that the corresponding $d_i \times d_i$ grid has many points in B to the second part, and distribute the remaining vertices so that the sizes of the two parts are as equal as possible. We omit the details. $\square$

Using the inequality between the bisection width and the crossing number, we improve the upper bound from Theorem 5.19 as follows.

Theorem 5.23. *The maximum number of edges in a geometric graph with n vertices and no three pairwise crossing edges is $O(n \log^2 n)$.*

Proof. Let G be a geometric graph with n vertices and degree sequence $d_1, d_2, \ldots, d_n$. By Theorem 5.21, we have $b(G) \le 1.58\sqrt{16\mathrm{cr}(G) + \sum_{i=1}^{n} d_i^2}$. For every edge e, the edges crossing e in G form a planar subgraph. Hence, $\mathrm{cr}(G) \le |E(G)| \cdot 3n$. By the estimate $d_i \le n$, we have $\sum_{i=1}^{n} d_i^2 \le n \cdot \sum_{i=1}^{n} d_i = 2n|E(G)|$. Therefore, $b(G) \le 1.58\sqrt{50n|E(G)|} \le 12\sqrt{n|E(G)|}$.

By this inequality, there is a set of at most $12\sqrt{n|E(G)|}$ edges that separates the graph G into two parts G_1, G_2 with n_1 and n_2 vertices, respectively, so that $n/3 \le n_1, n_2 \le 2n/3$.

Let $f_3(n)$ be the maximum number of edges in a geometric graph with n vertices and no three pairwise crossing edges. By induction on n we prove that $f_3(n) \le cn \log^2 n$ for some constant c and $n \ge 2$. For $n = 2$ this is true with $c \ge 1/(4 \log 2)$. Let $n \ge 3$. Let G be a geometric graph with n vertices, no three pairwise crossing edges and with $f_3(n)$ edges. Consider the partition from the previous paragraph. By induction hypothesis, we have

$$f_3(n) \le f_3(n_1) + f_3(n_2) + b(G) \le cn_1 \log^2 n_1 + cn_2 \log^2 n_2 + 12\sqrt{nf_3(n)}.$$

Since $x \to x \log^2 x$ is a convex function on $(0, \infty)$, we have

$$f_3(n) \le c \cdot \lfloor 2n/3 \rfloor \cdot \log^2(\lfloor 2n/3 \rfloor) + c \cdot \lceil n/3 \rceil \cdot \log^2(\lceil n/3 \rceil) + 12\sqrt{n f_3(n)}.$$

The rest of the computation is left as an exercise. □

5.4 Crossing lemma revisited

We start with a corollary of Theorem 5.21.

Corollary 5.24. *Let G be a graph with degree sequence $d_1, d_2, \ldots, d_n$, and let $G_1, G_2, \ldots, G_j$ be edge-disjoint subgraphs of G. Then the sum of their bisection widths satisfies*

$$\sum_{i=1}^{j} b(G_i) \le 2\sqrt{j} \cdot \sqrt{16\mathrm{cr}(G) + \sum_{k=1}^{n} d_k^2}.$$

Proof. By the inequality between the arithmetic mean and the quadratic mean (or the Cauchy–Schwarz inequality), we have

$$\sum_{i=1}^{j} b(G_i) \le \sqrt{j} \cdot \sqrt{\sum_{i=1}^{j} (b(G_i))^2}.$$

For $i = 1, 2, \ldots, j$, let $d_{1,i}, d_{2,i}, \ldots, d_{n,i}$ be the degree sequence of G_i. By Theorem 5.21 applied to G_i, we have

$$(b(G_i))^2 \le 4\left(16\mathrm{cr}(G_i) + \sum_{k=1}^{n} d_{k,i}^2\right).$$

This implies that

$$\sum_{i=1}^{j} (b(G_i))^2 \le 4\left(16 \sum_{i=1}^{j} \mathrm{cr}(G_i) + \sum_{k=1}^{n} \sum_{i=1}^{j} d_{k,i}^2\right).$$

Since the graphs G_i are edge-disjoint, we have

$$\sum_{i=1}^{j} \mathrm{cr}(G_i) \le \mathrm{cr}(G) \quad \text{and}$$

$$\sum_{i=1}^{j} d_{k,i}^2 \le \left(\sum_{i=1}^{j} d_{k,i}\right)^2 = d_k^2.$$

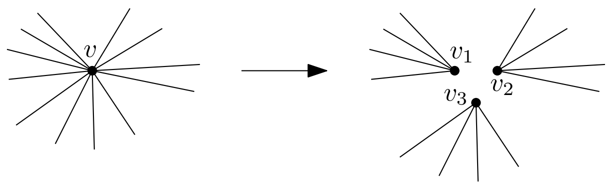

Figure 5.7: Splitting a vertex of large degree.

Therefore,

$$\sum_{i=1}^{j}(b(G_i))^2 \leq 4\left(16\mathrm{cr}(G)+\sum_{k=1}^{n} d_k^2\right)$$

and the corollary follows. □

The following theorem strengthens the crossing lemma for C_4-free graphs.

Theorem 5.25 (Pach, Spencer and Tóth, 2000 [52]). *Let G be a graph with n vertices, e edges, and with no C_4 as a subgraph. If $e \geq 1000n$, then*

$$\mathrm{cr}(G) \geq c \cdot \frac{e^4}{n^3}$$

where c is a positive constant.

In the original paper [52], the theorem is proved with $c = 1/10^8$. We prove it with $c = 1/10^7$.

Proof. The idea of the proof is the following. We recursively cut G into smaller parts by removing few edges. When a part has fewer than s vertices (where the s will be chosen later), we stop the recursion. The number of edges in all resulting parts will be small, at most $3e/4$. If the crossing numbers of the parts are small, we would delete less than $e/4$ edges, which would be a contradiction. This will imply that the crossing number of G is large. The same idea can be also used to prove the crossing lemma for general graphs.

First we modify the graph G so that all degrees are at most $\Delta = \lfloor 4e/n \rfloor$. Let D be a drawing of G with $\mathrm{cr}(G)$ crossings. For every vertex v with degree more than Δ, do the following. Split the neighbors of v into $k = \lceil d(v)/\Delta \rceil$ sets $A_1, A_2, \ldots, A_k$ of size at most Δ, so that each set forms an interval of consecutive vertices in the rotation at v in D. Then remove the vertex v from G and replace it by k vertices $v_1, v_2, \ldots, v_k$ placed close to the original location of v in D, on a small circle, and connect v_i to all the vertices in A_i by an edge so that these new edges do not cross. See Figure 5.7. Let G' be the resulting graph. Clearly, $e(G') = e$ and G' has no C_4 as a subgraph.

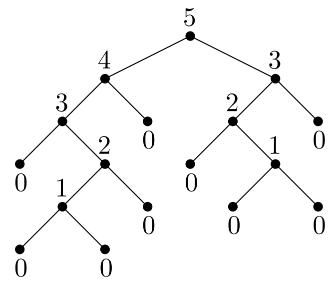

Figure 5.8: The heights of the nodes in the tree T.

Since we did not create new crossings, we have $\mathrm{cr}(G') \le \mathrm{cr}(G)$. A vertex v of degree $d(v)$ was replaced by $\lceil d(v)/\Delta \rceil \le d(v)/\Delta + 1$ new vertices. This implies that

$$v(G') \le \sum_{v \in V(G)} \left(\frac{d(v)}{\Delta} + 1 \right) = n + \frac{2e}{\Delta} = n + \frac{2e}{\lfloor 4e/n \rfloor} \le n + \frac{2e}{4e/n - 1}$$

$$= n + \frac{2en}{4e - n} = n + \frac{2en - 1/2}{4e - n} + \frac{1}{8e - 2n} \le \frac{3n}{2} + \frac{1}{2} \le 2n.$$

Thus, if we prove that $\mathrm{cr}(G') \ge ce^4/v(G')^3$, this will imply that $\mathrm{cr}(G) \ge (c/8) \cdot e^4/n^3$. Hence, for the rest of the proof we assume that all degrees in G are at most $4e/n$.

Now we describe the recursive decomposition of G in detail. Let V be the vertex set of G. In step i, we will have a decomposition of V into several subsets, and these subsets will be arranged in a rooted tree T_i whose root is V, each node has either two children or is a leaf, and each node that is not a leaf is the union of its two children. The leaves of T_i are exactly the sets of the decomposition. In the beginning, we have a single set V in the decomposition, and the corresponding tree T_0 has just one vertex—the root.

We set $s = e^2/(16n^2)$. Let $i \ge 0$. If $i \ge 1$ and all the leaves in T_i have at most s vertices, we stop. Otherwise, if $i \ge 1$, let W be a leaf of T_i with more than s vertices. If $i = 0$, let $W = V$. We cut $G[W]$ into two parts W_1, W_2 with at most $2|W|/3$ vertices each, by removing $b(G[W])$ edges from $G[W]$. Note that here we used just the definition of the bisection width. We attach W_1 and W_2 as children of W to T_i and obtain a tree T_{i+1}.

Let T be the tree obtained by the decomposition algorithm. For a node W of T, the *height* of W is the length of the longest path from W to a leaf in its subtree. That is, the leaves of T are exactly the nodes of height 0, the nodes whose both children are leaves are the nodes of height 1 and so on. See Figure 5.8. For $i \ge 0$, let $\mathcal{A}_i$ be the set of nodes of T of height i. Observe that each $\mathcal{A}_i$ is a decomposition of V. Let h be the height of the root of T.

Since G has no C_4 as a subgraph, Theorem 4.9 implies that $e \le n^{3/2} \Rightarrow e^2 \le n^3 \Rightarrow n \ge e^2/n^2 = 16s$. It follows that T has at least 16 leaves, in particular, $h \ge 2$. It also follows that every set of $\mathcal{A}_0$ has at least $s/3$ (and less than s) vertices, and every set of $\mathcal{A}_1$ has at least s vertices. By induction, for every $i = 1, 2, \dots, h$, every set of $\mathcal{A}_i$ has at least $(3/2)^{i-1} \cdot s$ vertices. This implies that $|\mathcal{A}_0| \le 3n/s$ and $|\mathcal{A}_i| \le (n/s) \cdot (2/3)^{i-1}$ for every $i = 1, 2, \dots, h$.

By Theorem 4.9, the total number of edges in $\bigcup_{W \in \mathcal{A}_0} G[W]$ is at most $|\mathcal{A}_0| \cdot s^{3/2} \le (3n/s) \cdot s^{3/2} = 3ns^{1/2} = 3e/4$. Therefore, we have deleted at least $e/4$ edges during the decomposition.

During the decomposition algorithm, we deleted $b(G[W])$ edges from every $G[W]$ such that $W \in \mathcal{A}_h \cup \mathcal{A}_{h-1} \cup \dots \cup \mathcal{A}_1$. By Corollary 5.24 applied to each decomposition $\mathcal{A}_i$ with $i \in \{1, 2, \dots, h\}$, we have

$$\frac{e}{4} \le \sum_{i=1}^{h} \sum_{W \in \mathcal{A}_i} b(G[W]) \le \sum_{i=1}^{h} 2\sqrt{|\mathcal{A}_i|} \cdot \sqrt{16\mathrm{cr}(G) + \sum_{k=1}^{n} d_k^2}$$

where $d_1, d_2, \dots, d_k$ is the degree sequence of G. Since $d_i \le \Delta \le 4e/n$, we have $\sum_{k=1}^{n} d_k^2 \le 16e^2/n$. Further we have

$$\sum_{i=1}^{h} \sqrt{|\mathcal{A}_i|} \le \sqrt{\frac{n}{s}} \cdot \sum_{i=0}^{h-1} \left(\sqrt{\frac{2}{3}}\right)^i \le \frac{4n^{3/2}}{e} \cdot \frac{1}{1 - \sqrt{2/3}} \le \frac{22n^{3/2}}{e}.$$

Putting this together, we get

$$\frac{e}{4} \le \frac{44n^{3/2}}{e} \cdot \sqrt{16\mathrm{cr}(G) + 16e^2/n},$$

which implies that

$$\mathrm{cr}(G) \ge \frac{e^4}{16 \cdot (4 \cdot 44)^2 \cdot n^3} - \frac{e^2}{n} \ge \frac{2e^4}{10^6 \cdot n^3} - \frac{e^2}{n}.$$

By our assumption, $e \ge 1000n$, so $e^2/n \le e^4/(10^6 n^3)$. Therefore, we have

$$\mathrm{cr}(G) \ge \frac{e^4}{10^6 \cdot n^3}$$

and we are finished. □

The proof of Theorem 5.25 can be also adapted to give an alternative proof of the crossing lemma: we only choose a different threshold $s = e/(2n)$. Similarly, Theorem 5.25 can be generalized to give an improved lower bound on the crossing number for graphs with no $K_{s,t}$ as a subgraph.

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