Chapter 4

Crossings: few and many

These crossovers are like rabbits... they have a tendency to multiply at a terrifying rate.

— *Yona Friedman [10]*

4.1 Turán's brick factory problem

In 1944, Turán posed the following problem [10]. Suppose that there are m kilns and n storage yards. How can we connect every kiln to every storage yard with paths so that the number of crossings of the paths is minimum? This problem can be modeled as follows. What is the minimum number of crossings of edges in a drawing of the graph $K_{m,n}$ in the plane?

Definition 4.1. Let G be an arbitrary graph. The **crossing number** of G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings of edges over all possible drawings of G in the plane. Here it is important to assume that no three edges cross at the same point.

The brick factory problem of Turán is then to find $\mathrm{cr}(K_{m,n})$.

Suppose that the vertices of $K_{m,n}$ are partitioned into two parts A and B with $|A| = m$ and $|B| = n$ and every vertex in A is connected to every vertex in B. The following simple straight-line drawing of $K_{n,m}$ gives the best known upper bound on $\mathrm{cr}(K_{m,n})$. Namely, place the vertices of A on the y-axis to the points

$$(0, -\lfloor m/2 \rfloor), (0, -\lfloor m/2 \rfloor + 1), \ldots, (0, -1), (0, 1), (0, 2), \ldots, (0, \lceil m/2 \rceil)$$

and the vertices of B on the x-axis to the points

$$(-\lfloor n/2 \rfloor, 0), (-\lfloor n/2 \rfloor + 1, 0), \ldots, (-1, 0), (1, 0), (2, 0), \ldots, (\lceil n/2 \rceil, 0),$$

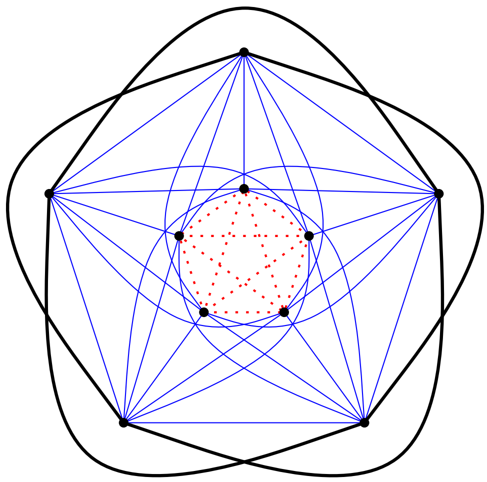

Figure 4.1: A cylindrical drawing of K_{10}.

and then join every vertex in A to every vertex in B by a straight-line segment. The number of crossings in this drawing is exactly $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.

Conjecture 4.2 (Zarankiewicz [71]). *We have*

$$\mathrm{cr}(K_{n,m}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Zarankiewicz actually published his conjecture as a theorem, but later his proof was found incomplete [10]. Zarankiewicz's conjecture has been verified for $m \leq 6$ [38].

The following conjecture about the crossing number of the complete graph K_n is usually known as Hill's conjecture.

Conjecture 4.3 (Harary–Hill [33], Guy [31]). *We have*

$$\mathrm{cr}(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

There are two families of drawings of K_n that attain the number of crossings stated in Hill's conjecture: cylindrical drawings and 2-page book drawings [9, 31, 33, 34]. In the *cylindrical drawing* of K_{2n}, n vertices are put on the boundary of each circular base of a cylinder in the vertices of a regular n-gon, and the vertices are connected by shortest arcs on the surface of the cylinder. Figure 4.1 shows a "deformed" cylindrical drawing of K_{10}.

In the "optimal" 2-*page book* drawing of K_n, there is a cycle of length n without crossings, forming the "spine" of the book, which can be drawn as

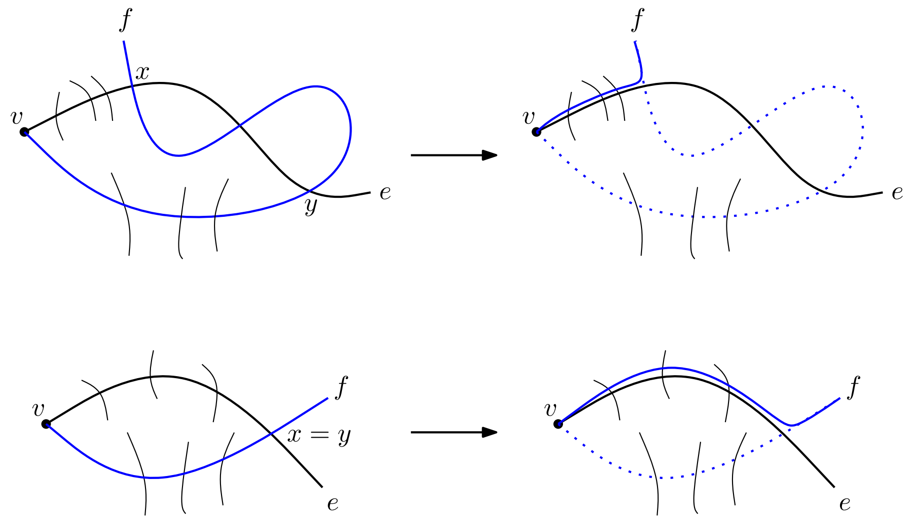

Figure 4.2: Reducing the number of crossings in the case when two adjacent edges cross.

a regular n-gon, for example. Then half of the other edges are drawn inside the cycle and the other half outside the cycle. Roughly speaking, the edges that are drawn inside are those whose slope is between $-45°$ and $45°$.

It is not hard to show that Zarankiewicz's conjecture implies an asymptotic version of Hill's conjecture [58]. We will prove this in Lemma 4.6.

Now we show that $\mathrm{cr}(K_{n,n})$ and $\mathrm{cr}(K_n)$ are of the order n^4. First we observe the following property of optimal drawings.

Lemma 4.4. *Let D be a drawing of a graph G with exactly $\mathrm{cr}(G)$ crossings. Then every two edges have at most one point in common (either an endpoint or a crossing).*

Proof. Suppose that e, f are two adjacent edges with a common vertex v that cross at least once. Let x be a crossing of e and f that is closest to v along e, and let y be a crossing of e and f that is closest to v along f. See Figure 4.2. The crossings x and y might be the same or different. Let e_{vx} be the portion of e between v and x, and let f_{vy} be the portion of f between v and y. Let a be the number of crossings of e_{vx} with other edges of G, and let b be the number of crossings of f_{vy} with other edges of G. Without loss of generality, assume that $b \geq a$. Replace a portion of f slightly longer than the part between v and x by a curve f' drawn along e_{vx}, from an appropriate side. In this way, we get rid of the crossing x (and perhaps some other crossings as well), and we exchange b old crossings on f for a new crossings on f'.

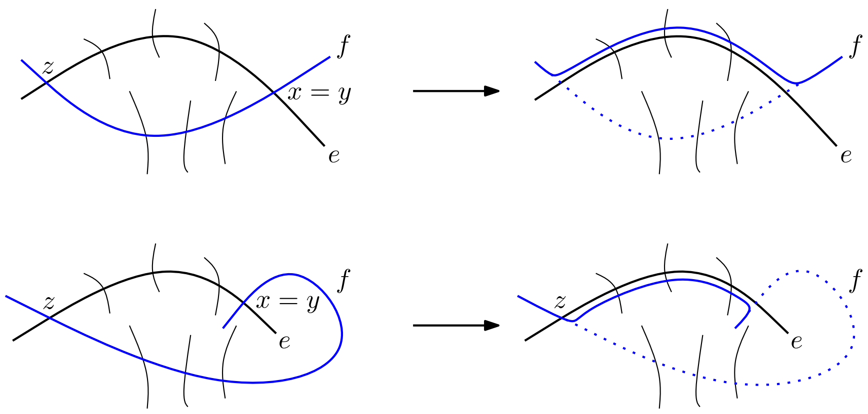

Figure 4.3: Reducing the number of crossings in the case when two edges cross more than once.

Now suppose that e, f are two independent edges with at least two crossings. Let z, z' be arbitrary two crossings of e with f. Let x be a crossing of e and f that is closest to z along e in the direction of z', and similarly, let y be a crossing of e and f that is closest to z along f in the direction of z'. The redrawing step is now analogous to the previous case where we substitute z for v. See Figure 4.3, where only the case $x = y$ is illustrated. Note that we cannot always get rid of both crossings x and z by this redrawing operation. □

There are alternative ways of proving Lemma 4.4. For example, we could first take any pair of crossings z, z' (or a vertex v and a crossing x) between the two edges, and redraw a portion of e or f between z and z' (between v and x). In this way, we could introduce self-crossings, but those may be removed rather easily.

Observe that if two edges e, f cross at least four times, it is not always possible to find two crossings, x and y, so that the portion of e between x and y contains no other crossings with f, and the portion of f between x and y contains no other crossings with e.

Theorem 4.5. *The limits* $\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2}$ *and* $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$ *exist and both are positive numbers.*

Proof. We will prove the theorem only for the graph K_n. The proof for the graph $K_{n,n}$ is similar and is left as an exercise. By Lemma 4.4, for every drawing of K_n with $\mathrm{cr}(K_n)$ crossings and for every four vertices in it, there are at most three possible crossings among the edges between these four vertices (in fact, there is at most one). This observation shows that $\frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$ never exceeds 3. Now, in order to show that the limit exists, it is sufficient to show that the sequence $\frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$, $n = 1, 2, 3, \ldots$, is an increasing sequence. The theorem will follow from the fact that every increasing upper bounded sequence whose first term is positive has a positive limit.

To complete the proof, we need to show that for every positive integer n we have $\frac{\mathrm{cr}(K_n)}{\binom{n}{4}} \leq \frac{\mathrm{cr}(K_{n+1})}{\binom{n+1}{4}}$. Expanding the binomial coefficients in both sides and ignoring the common factors in both sides, we observe that this inequality is equivalent to the inequality

$$\frac{\mathrm{cr}(K_n)}{n} \geq \frac{\mathrm{cr}(K_{n-1})}{n-4} \quad \text{or equivalently} \quad (n-4)\,\mathrm{cr}(K_n) \geq n\,\mathrm{cr}(K_{n-1}).$$

Fix a drawing D of K_n with exactly $\mathrm{cr}(K_n)$ crossings. Removing each vertex in D yields a copy of K_{n-1}, which has at least $\mathrm{cr}(K_{n-1})$ crossings in D. In total, this gives at least $n \cdot \mathrm{cr}(K_{n-1})$ crossings. But notice that every crossing in D is counted precisely $n-4$ times. Therefore, the number of the crossings in D is at least $\frac{n}{n-4} \cdot \mathrm{cr}(K_{n-1})$. This shows that $\mathrm{cr}(K_n) \geq \frac{n}{n-4} \cdot \mathrm{cr}(K_{n-1})$. □

Observe that the above proof tells us more than just the existence of a limit. It says that the sequence $\frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$ is an increasing sequence. Therefore, every term of this sequence is a lower bound for $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$.

Note that Zarankiewicz conjecture would imply that $\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2} = 1/4$ and Hill's conjecture would imply that $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}} = 3/8$.

Lemma 4.6 (Richter and Thomassen [58])**.** *If* $\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2} = 1/4$ *then* $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}} = 3/8$.

Proof. Let n be given and let D be a drawing of the graph K_{2n} with $\mathrm{cr}(K_{2n})$ crossings. If we color n vertices red and the remaining n vertices blue, the color classes induce a drawing of the bipartite graph $K_{n,n}$, which has at least $\mathrm{cr}(K_{n,n})$ crossings. There are exactly $\binom{2n}{n}$ such colorings. A crossing of edges uv and xy in D is counted if and only if u and v get different color and x and y get different color. The number of such colorings is exactly $4 \cdot \binom{2n-4}{n-2}$. Therefore, we get

$$4 \cdot \binom{2n-4}{n-2} \cdot \mathrm{cr}(K_{2n}) \geq \binom{2n}{n} \cdot \mathrm{cr}(K_{n,n}).$$

After simplifying, this gives

$$\frac{\mathrm{cr}(K_{2n})}{\binom{2n}{4}} \geq \frac{3}{2} \cdot \frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2}.$$

Since for every n, there are drawings of K_n attaining the number of crossings in Hill's conjecture, we have $\lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}} \leq 3/8$ and the lemma follows. □

4.2 Conway's thrackle conjecture

A **thrackle** is a graph drawn in the plane so that the edges are represented by simple curves, any pair of which either meet at a common vertex or cross precisely once. A graph is **thrackleable** if it can be drawn as a thrackle.

Conjecture 4.7. *In every thrackle, the number of edges is at most equal to the number of vertices.*

Conway's thrackle conjecture is analogous to the following combinatorial theorem known as nonuniform Fisher's inequality [8], which generalizes Fisher's inequality [24], and was originally proved by de Bruijn and Erdős [11].

Theorem 4.8 (a nonuniform Fisher's inequality, 1948 [11])**.** *If $A_1, A_2, \ldots, A_m$ are distinct subsets of a finite set X such that every two of the subsets have precisely one element in common, then $m \leq |X|$.*

Proof. Let $n = |X| \geq 1$. If some of the sets A_i is empty then $m \leq 1$. If some of the elements $x \in X$ is contained in all the sets A_i, then the sets $A_i \setminus \{x\}$ are pairwise disjoint, and thus we can select a unique point from each of the sets A_i, which implies that $m \leq |X|$. If some of the sets A_i is equal to X, then every other set A_j has only one element, and again, $m \leq |X|$. For the rest of the proof assume that $1 \leq |A_i| \leq n-1$ for every i and that $\bigcap_{i=1}^m A_i = \emptyset$.

For every $x \in X$, let $\deg(x)$ be the number of sets A_i containing x. Observe that if $x \notin A_i$, then $|A_i| \geq \deg(x)$: indeed, every two sets containing x must intersect A_i and these intersections must be disjoint.

Draw a rectangular table (a matrix) with rows indexed by the elements of X and the columns indexed by the sets A_i (or by the numbers $1, 2, \ldots, m$). Write a '1' at the position (x, A_i) if $x \in A_i$ and '0' otherwise. By our assumption, every column and every row has at least one 0-entry. Obviously, the total number of entries in the table is mn. We will now count the number of entries in the table in two other ways, while "stepping" only on the 0-entries. First, we will count according to the columns. We have $n - |A_i|$ 0-entries in the ith column, thus

$$mn = \sum_{i=1}^{m} \sum_{x \in X; x \notin A_i} \frac{n}{n - |A_i|} = \sum_{x \notin A_i} \frac{n}{n - |A_i|}. \tag{4.1}$$

Now we count according to the rows. We have $m - \deg(x)$ 0-entries in row x, thus

$$mn = \sum_{x \in X} \sum_{i \in \{1,\ldots,m\}; x \notin A_i} \frac{m}{m - \deg(x)} = \sum_{x \notin A_i} \frac{m}{m - \deg(x)}. \tag{4.2}$$

Suppose that $m > n$. We observed that if $x \notin A_i$, then $|A_i| \geq \deg(x)$. Since $|A_i| \geq 1$, this further implies the following inequalities:

$$\begin{aligned} m|A_i| &> n \cdot \deg(x), \\ mn - m|A_i| &< mn - n \cdot \deg(x), \\ \frac{n - |A_i|}{n} &< \frac{m - \deg(x)}{m}, \\ \frac{n}{n - |A_i|} &> \frac{m}{m - \deg(x)}. \end{aligned}$$

Summing the last inequality over all $x \notin A_i$, we get

$$\sum_{x \notin A_i} \frac{n}{n - |A_i|} > \sum_{x \notin A_i} \frac{m}{m - \deg(x)},$$

which contradicts equations (4.1) and (4.2). □

An example of a thrackleable graph is the cycle C_5. This can be easily seen from the star-like drawing of C_5 (Figure 4.4). We now show that C_4 cannot be drawn as a thrackle. If the vertices of C_4 are a, b, c, d and each vertex is joined to the next vertex in this order, then in every thrackle drawing of C_4, there is only one possible configuration for the path $abcd$ shown in Figure 4.5. These three edges create a triangle whose one side is the edge bc. The edge da must cross the edge bc, so it has to get inside the triangle and when it goes out of the triangle it either crosses the edge bc for the second time or it must cross one of the other edges. None of these situations is allowed in a thrackle.

Clearly, every subgraph of a thrackle is also a thrackle. This together with the previous observation shows that if G is thrackleable then G has no C_4 as a subgraph.

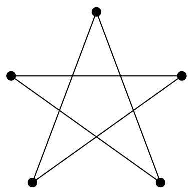

Figure 4.4: A thrackle drawing of C_5.

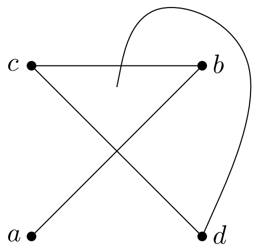

Figure 4.5: An unsuccessful attempt of drawing C_4 as a thrackle.

Theorem 4.9 (Erdős, Kővári–Sós–Turán, 1954 [39]). *Any graph G with n vertices with no C_4 as a subgraph has at most $n^{3/2}$ edges.*

Proof. Suppose that G is a graph with n vertices with no C_4 as its subgraph. We count the number of paths of length 2 in G in two ways. Since G has no C_4, every pair of its vertices have at most one common neighbor and therefore the number of 2-paths in G is at most $\binom{n}{2}$. Now we count the number of 2-paths as follows. Let v be a vertex of G of degree d. Every pair of the neighbors of v form a path of length 2 and conversely every such path is obtained in this way precisely once (just consider the middle point of the 2-path). So, the number of 2-paths in G is equal to

$$\sum_{i=1}^{n} \binom{d_i}{2}$$

where d_i's are the degrees of the vertices of G. So we have $\sum_{i=1}^{n} \binom{d_i}{2} \leq \binom{n}{2}$. Since the function $f(x) = \binom{x}{2} = x(x-1)/2$ is a convex function, we can use Jensen's inequality to conclude that

$$\binom{n}{2} \geq \sum_{i=1}^{n} \binom{d_i}{2} \geq n \cdot \binom{\frac{\sum_{i=1}^{n} d_i}{n}}{2} = n \cdot \binom{\frac{2|E(G)|}{n}}{2}.$$

Thus,

$$n \geq n - 1 \geq \frac{2|E(G)|}{n} \cdot \left(\frac{2|E(G)|}{n} - 1\right) \geq \left(\frac{2|E(G)|}{n} - 1\right)^2$$

and therefore $|E(G)| \leq \frac{1}{2}(n^{3/2} + n) \leq n^{3/2}$. □

Corollary 4.10. *If G is a thrackle with n vertices then $|E(G)| \leq n^{3/2}$.*

Proof. Since no thrackle has C_4 as a subgraph, the assertion is true by Theorem 4.9. □

Notice that the previous corollary is still very far from Conway's thrackle conjecture. Now we try to obtain a better upper bound on the number of the edges of a thrackle. We need the following useful lemmas to obtain an $O(n)$ upper bound on the number of edges of a thrackle.

Lemma 4.11. *Let C_1 and C_2 be two closed curves (possibly self-intersecting) that may cross but do not touch each other. The number of crossings of C_1 and C_2 is even.*

Proof. The closed curve C_1 divides the plane into regions and each of these regions can be colored black or white so that every two adjacent regions have different colors. Now, a point traveling along C_2 observes a change of color every time it crosses C_1. Therefore, after returning to its initial position, the color must have changed an even number of times. □

Lemma 4.12. *Every graph G has a bipartite subgraph H such that $|E(H)| \geq |E(G)|/2$.*

Proof. Let H be a bipartite subgraph of G with the maximum number of edges. Without loss of generality we can assume that H has all the vertices of G. Let A and B be a bipartition of the vertices of H. Let v be an arbitrary vertex of G. Assume that $v \in A$. The bipartite subgraph of G induced by the bipartition $A \setminus \{v\}, B \cup \{v\}$ cannot have more edges than the graph H because of the way we have chosen H. This means that in the graph G, v has at least as many neighbors in B as in A. So, the degree of v in H is at least half the degree of v in G. This argument is valid for every vertex v. Therefore H has the property that each of its vertices has degree at least half the degree in the graph G. Therefore $|E(H)| \geq |E(G)|/2$. □

Theorem 4.13. *Every bipartite thrackleable graph is planar.*

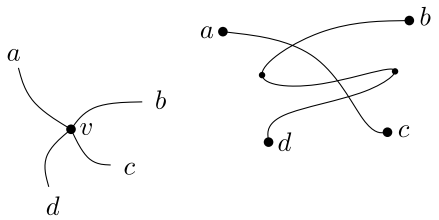

Figure 4.6: A neighborhood of v and the paths $a \dots c$ and $b \dots d$ in $D(H)$.

Proof (sketch). First we prove that if G is bipartite and thrackleable then G contains no subdivision of K_5. Let $D(G)$ be a thrackle drawing of G. Suppose for contradiction that there is a subdivision H of K_5 in G. Clearly, H is also bipartite and the drawing $D(H)$ of H in $D(G)$ is a thrackle. Let v, a, b, c, d be the vertices of $D(H)$ of degree 4 (notice that $D(H)$ has five vertices of degree 4 and the other vertices are of degree 2). Suppose that the neighborhood of v looks like in Figure 4.6. Let C_1 be the closed curve formed by the paths $v \dots a$, $a \dots c$ and $c \dots v$. Let C_2 be the closed curve formed by the paths $v \dots b$, $b \dots d$ and $d \dots v$. Since H is bipartite, each of the two closed curves is formed by an even number of edges. Since $D(H)$ is a thrackle, every edge of C_1 must cross every edge of C_2. On the other hand, Lemma 4.11 ensures that C_1 and C_2 will cross an even number of times. This is a contradiction since C_1 and C_2 intersect an even number of times at interior points of their edges and one more time at the point v. Similarly, it can also be shown that no subdivision of $K_{3,3}$ can be both bipartite and thrackleable and therefore G has no subdivision of K_5 or $K_{3,3}$. Thus G is a planar graph. □

A graph drawn in the plane is called an **odd thrackle** (also a **generalized thrackle**) if every two independent edges cross an odd number of times and every two adjacent edges cross an even number of times (that is, they have an odd number of intersecting points, including the vertex they share). Theorem 4.13 is still true if we replace "thrackle" with "odd thrackle". In fact, we have the reverse implication as well.

Theorem 4.14. *A bipartite graph G is planar if and only if it can be drawn as an odd thrackle.*

Proof. Let D be a plane drawing of G. Deform the plane so that the two vertex classes of G are separated by the x-axis. It is an interesting exercise to show that the deformation can be chosen in such a way that every edge crosses the x-axis exactly once. However, we do not need this stronger observation as we use only the fact that every edge crosses the x-axis an odd number of times. Now cut the plane along the x-axis, move the lower part of the drawing one unit down, and reflect this lower part over the y-axis. Then in the empty strip between the lines $y = 0$ and $y = -1$, reconnect the severed edges by straight-line segments (the ith end from the left on the x-axis with the ith end from the right on the line $y = -1$), and remove all self-crossings that were created. By this, we introduced an odd number of crossings between every pair of edges, since every two segments in the strip cross. Finally, in a small neighborhood of every vertex v, do a similar trick: deform the neighborhood of v so that all the edges are directed in the lower halfplane, cut the edges by a horizontal line, move the part containing v and reflect it over a vertical line passing through v, and reconnect the edges. This operation introduces one crossing on every pair of edges incident with v. The resulting drawing is an odd thrackle.

If D is a drawing of G that is an odd thrackle, we perform the same procedure, and in the end we obtain a drawing of G where every two edges cross an even number of times. By the weak (or strong) Hanani–Tutte theorem (Theorem 2.9 or 2.7), G is planar. □

Now, we are able to prove the following upper bound for the number of edges of a thrackle.

Corollary 4.15. *If G is a thrackle or an odd thrackle then $|E(G)| \leq 4|V(G)|$.*

Proof. Suppose that G is a thrackle with n vertices. By Lemma 4.12, there is a bipartite subgraph H of G with at least $\frac{|E(G)|}{2}$ edges. Since H is a bipartite thrackle, it is a drawing of a planar graph by Theorem 4.14. Thus, by Euler's formula, we have $\frac{|E(G)|}{2} \leq |E(H)| \leq 2n - 4$. □

For thrackles, one can obtain a better upper bound, $|E(G)| \leq 3|V(G)|$, using the fact that there are no cycles of length four. This is left as an exercise.

The upper bound has been further improved several times during recent years.

Theorem 4.16 (Lovász, Pach and Szegedy, 1997 [44])**.** *If G is a thrackle then $|E(G)| \leq 2|V(G)|$.*

Theorem 4.17 (Cairns and Nikolayevsky, 2000 [12])**.** *If G is a thrackle then $|E(G)| \leq 1.5|V(G)|$.*

Theorem 4.18 (Fulek and Pach, 2011 [28])**.** *If G is a thrackle then $|E(G)| \leq \frac{167}{117}|V(G)| < 1.428|V(G)|$.*

Theorem 4.19 (Xu, 2014 [70])**.** *If G is a thrackle then $|E(G)| \leq 1.4|V(G)|$.*

4.3 The crossing lemma

We recall that the **crossing number** of a graph G, denoted by $\mathrm{cr}(G)$, is the smallest possible number of crossing in a drawing of G in the plane. Here we consider drawings with not necessarily straight-line edges, and such that no three edges cross at the same point.

Lemma 4.20. *If G is a graph with $n \geq 3$ vertices, then*

$$\mathrm{cr}(G) \geq e(G) - (3n - 6).$$

Proof. Let D be a drawing of G with $k = \mathrm{cr}(G)$ crossings. By removing one edge from a pair of edges that cross, we decrease the number of crossings. Therefore, by removing at most k edges from D, we obtain a plane drawing of a graph with at least $e(G) - k$ edges. By Corollary 1.4, we have $e(G) - k \leq 3n - 6$, which proves the lemma. □

The following lower bound on the crossing number of a graph is known under different names, including the crossing lemma, the crossing number theorem, or the crossing number inequality.

Theorem 4.21 (The crossing lemma)**.** *If G is a graph with n vertices and $e \geq 4n$ edges, then*

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{e^3}{n^2}.$$

The crossing lemma was proved independently by Leighton [41, Theorem 7.6] with constant 1/375 and by Ajtai, Chvátal, Newborn and Szemerédi [5] with constant 1/100. The constant was later improved by Pach and Tóth [54] to 1/33.75 (if $e \geq 7.5n$), by Pach, Radoičić, Tardos and Tóth [49] to $1024/31827 \approx 1/31.08$ (if $e \geq 6.44n$), and by Ackerman [2] to 1/29 (if $e \geq 6.95n$).

Proof. The idea of the proof is to use the weak bound from Lemma 4.20 and amplify it using a probabilistic trick. We do not apply the weaker bound directly to G, but to sufficiently sparse induced subgraphs, containing, in average, cn^2/e vertices and $c'n^2/e$ edges.

Let D be a drawing of G with $\mathrm{cr}(G)$ crossings. We choose a random subset $V' \subset V(G)$ by including each vertex independently with probability p (which we choose later). Let G' be the subgraph induced by V' and D' the corresponding subdrawing of D. Let x be the number of crossings in D'. We have

$$\mathbf{E}[|V'|] = np, \qquad \mathbf{E}[e(G')] = ep^2, \qquad \mathbf{E}[x] = \mathrm{cr}(G)p^4.$$

By Lemma 4.20, we have $x \geq e(G') - 3|V'|$, hence $\mathrm{cr}(G)p^4 \geq ep^2 - 3np$. Setting $p = 4n/e$ (which is at most 1) we get

$$\mathrm{cr}(G) \geq \frac{e^3}{64n^2}.$$

□

The order of magnitude of the lower bound in Theorem 4.21 cannot be improved. To see this, take a graph G consisting of $n^2/(2e)$ disjoint complete graphs as equal in size as possible. Then in each component of G there are $\Theta(e/n)$ vertices, $\Theta(e^2/n^2)$ edges, and it can be drawn with $O(e^4/n^4)$ crossings. Therefore, G can be drawn with $O(e^3/n^2)$ crossings, which matches asymptotically the lower bound from the crossing lemma.

The following construction by Pach and Tóth [54] shows that the constant from the crossing lemma is not far from optimal. Suppose that $n \ll e \ll n^2$. Take for the vertex set the vertices of the $\sqrt{n} \times \sqrt{n}$ grid and connect two vertices by an edge if and only if their distance is at most $\sqrt{2e/\pi n}$. Then

$$\mathrm{cr}(G) \leq \left(\frac{16}{27\pi}\right)\frac{e^3}{n^2} \approx \frac{1}{16.65}\frac{e^3}{n^2}.$$

The following lemma can be used to improve the lower bound from Theorem 4.21.

Lemma 4.22. *The maximum number of edges in a graph with $n \geq 3$ vertices that can be drawn in the plane so that every edge crosses at most one other edge is $4n - 8$.*

Proof (sketch). Let G' be a maximal plane subgraph of G. The edges in $E(G) - E(G')$ are all split into two by some edge of G'. We call these pars *half-edges*. It is easy to prove by induction that in each face f of G' with $s(f)$ sides we can have at most $s(f) - 2$ half-edges. Now we only use Euler's formula and its corollaries:

$$\begin{aligned} e(G) &= e(G') + (e(G) - e(G')) \\ &\leq e(G') + \frac{1}{2}\sum_f (s(f) - 2) = 2e(G') - f(G') \\ &= e(G') + (e(G') - f(G')) \leq 3n - 6 + (n - 2) = 4n - 8. \end{aligned}$$

For an optimal construction take a planar graph whose all faces are quadrilaterals and then add two diagonals of each quadrilateral. □

Denote by $e_k(n)$ the maximum number of edges in a graph on n vertices that can be drawn in the plane so that every edge crosses at most k other edges. Lemma 4.22 says that $e_1(n) = 4n - 8$. It can be also proved that $e_2(n) = 5n - 10$, $e_3(n) = 6n - 12$, $e_4(n) = 7n - 14$. It can be conjectured that $e_k(n) = (k + 3)(n - 2)$. As a consequence we have:

$$\operatorname{cr}(G) > (e - 3n) + (e - 4n) + (e - 5n) + (e - 6n) + (e - 7n) = 5e - 25n.$$

4.4 Incidences and unit distances

Let P be a set of n points and L a set of m lines in the plane. An **incidence** between P and L is a pair (p, ℓ) such that $p \in P, \ell \in L$ and $p \in \ell$.

Theorem 4.23 (Szemerédi–Trotter, 1983 [63])**.** *The maximum number of incidences between n points and m lines in the plane is $O(n^{2/3}m^{2/3} + m + n)$.*

Proof. (Székely [62]) Let P be the given set of points and L the given set of lines. We may assume without loss of generality that every line is incident to at least one point and that every point is incident to at least one line. Define a graph G drawn in the plane as follows. The vertex set of G is P, and two vertices are joined by an edge drawn as a straight line segment if the two vertices are consecutive points of P on one of the lines from L. This drawing shows that $\operatorname{cr}(G) \leq \binom{m}{2}$. The number of points on any of the lines of L is one greater than the number of edges drawn along that line. Therefore, the number of incidences among the points and the lines is at most $e(G) + m$. Theorem 4.21 finishes the proof: either $e(G) \leq 4n$, in which case the number of incidences is at most $4n+m$, or $\binom{m}{2} \geq \operatorname{cr}(G) \geq e(G)^3/n^2$, in which case $e(G) \leq O(n^{2/3}m^{2/3})$ and the number of incidences is thus at most $O(n^{2/3}m^{2/3}) + m$. □

Theorem 4.24 (Spencer, Szemerédi and Trotter, 1984 [61])**.** *The maximum number of unit distances determined by n points in the plane is $O(n^{4/3})$.*

Proof. (Székely [62]) Draw a multigraph G in the plane in the following way. The vertex set of G is the set of n given points. Draw a unit circle around each point; in this way, consecutive points on the unit circles are connected by circular arcs. These arcs form the edges of the multigraph G. The number of edges of G is equal to the number of point-circle incidences, and this is equal to twice the number of unit distances. Discard the circles that contain at most two points. By this, we delete at most $2n$ edges from G and obtain a multigraph G'. In G', there are no loops, and every two vertices are connected by at most two edges, each of them coming from a different circle, since at most two unit circles can pass through two given points. Then, for every two vertices joined by two edges in G', delete one of the edges. The resulting drawing is a drawing of a graph G''. For the number of edges of G'', we have $e(G'') \geq e(G')/2 \geq (e(G) - 2n)/2 = e(G)/2 - n$. The number of crossings of G'' in this drawing is at most n^2, since any pair of circles intersect in at most two points. By Theorem 4.21, $e(G'')^3/n^2 = O(n^2)$ and so $e(G'') = O(n^{4/3})$. This implies that the number of unit distances is at most $e(G)/2 \leq e(G'') + n \leq O(n^{4/3})$. □

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