Chapter 2

Characterization of planar graphs

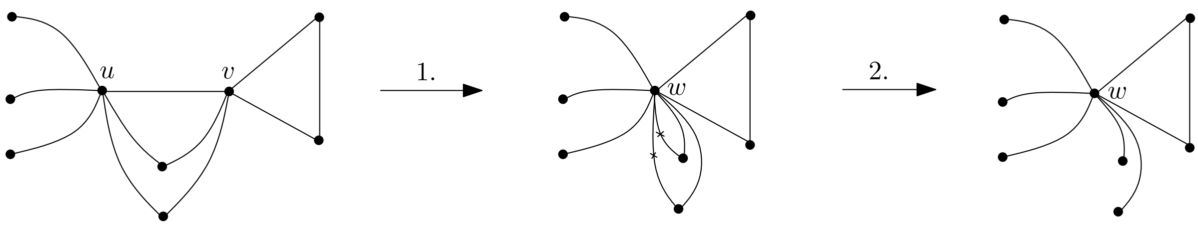
In this chapter we investigate various equivalent conditions for graphs to be planar. In Section 2.2 we present an algebraic algorithm for planarity testing. Then in the last section we briefly visit the third dimension.

Definition 2.1. Take a graph G and put additional vertices arbitrarily on the edges of G (but not on their crossings). This divides the original edges of G into smaller ones. Alternatively (and more precisely), we may say that we replace the edges of G by paths of length at least 1 whose internal vertices are disjoint. The resulting graph is called a **subdivision** of G.

Theorem 2.2 (Kuratowski, 1930)**.** *A graph G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$.*

Definition 2.3. A graph G contains H as a **minor** if H can be obtained from G by deleting edges and vertices and by *contracting* edges.
Contracting an edge uv consists of

1. removing the edge uv and identifying the vertices u and v, and then
2. removing all parallel edges.



Theorem 2.4 (Wagner, 1937)**.** *A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as a minor.*

In the literature, the following terminology is also used: G contains H as a **topological minor** if G contains a subdivision of H.

It is an easy exercise to show that if G contains a subdivision of H, then G contains H as a minor. Consequently, Kuratowski's theorem implies Wagner's theorem. The other implication, that Wagner's theorem implies Kuratowski's theorem, is also not hard to show, without knowing the proof of either of them. There is just a small catch: containing $K_{3,3}$ as a minor implies containing a subdivision of $K_{3,3}$, but containing K_5 as a minor implies containing a subdivision of K_5 or a subdivision of $K_{3,3}$.

2.1 Hanani–Tutte theorems

Hanani–Tutte theorems characterize planar graphs in terms of the parity of the numbers of crossings between their edges. We start with a very basic variant and then show two slightly stronger versions.

Theorem 2.5 (A "very weak" Hanani–Tutte theorem)**.** *A graph G is planar if and only if G can be drawn in the plane so that every two edges cross an even number of times.*

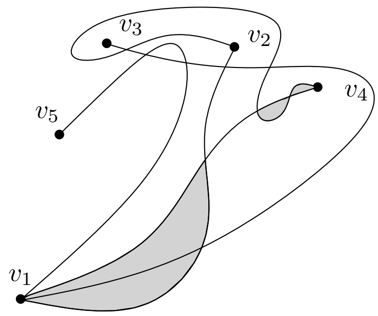
A drawing where every two edges cross an even number of times is also called an **even drawing**.

Sketch of the proof. "$\Rightarrow$" This direction is trivial: if G is planar then it can by definition be drawn without crossings, that is, each pair of edges cross zero times and zero is an even number.

"$\Leftarrow$" This direction can be proved in an easy way using Kuratowski's theorem. Namely, we only need to show that no subdivision of K_5 and $K_{3,3}$ has an even drawing.

As an example we show that K_5 has no even drawing. Suppose for contradiction that there exists an even drawing of K_5. Take a vertex v_1, and let the edges $v_1v_2, v_1v_3, v_1v_4, v_1v_5$ leave v_1 in this clockwise order in a small neighborhood of v_1. Of course, outside this neighborhood these edges may cross one another. Consider the image of the triangle $v_1v_2v_4$. It is a closed, possibly self-intersecting curve γ. It divides the plane into several regions. It is a simple exercise to show that these regions can be colored by two colors (say, black and white) so that no two regions whose boundaries share an arc get the same color. Notice that the initial portions of the edges v_1v_3 and

v_1v_5 around v_1 belong to regions of opposite colors. Assume that the initial portion of v_1v_3 in a small neighborhood of v_1 runs in a black region.



According to our assumptions, the edge v_1v_3 must cross each of the edges v_1v_2, v_2v_4, v_4v_1 an even number of times. Therefore, the curve v_1v_3 crosses γ an even number of times, and after each crossing it switches colors. This yields that v_3 must lie in a black region. Analogously, since the initial portion of the edge v_1v_5 runs in a white region, we can conclude that v_5 lies in a white region. Since v_3 and v_5 lie in regions of opposite colors, the edge v_3v_5 crosses gamma an odd number of times, contradicting our assumption that v_3v_5 crosses every edge an even number of times. □

Hanani [34] and Tutte [72] originally proved the following stronger version of the above theorem.

Definition 2.6. Two edges $\{a, b\}$ and $\{c, d\}$ are **independent** (also **non-adjacent**) if $\{a, b\} \cap \{c, d\} = \emptyset$; that is, they do not share any vertex.

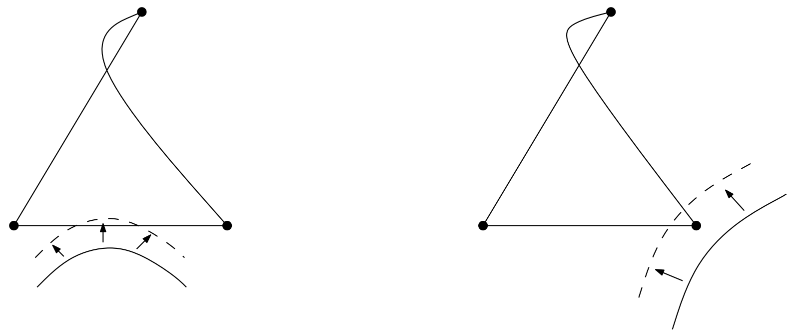
Theorem 2.7 (The strong Hanani–Tutte theorem, 1934 [34], 1970 [72])**.** *A graph G is planar if and only if G can be drawn in the plane so that any two independent edges cross an even number of times.*

A drawing where every two independent edges cross an even number of times is also called an **independently even drawing**.

Sketch of the proof. For the first direction, the same argument applies as before. For the second direction we again use Kuratowski's theorem and show as an example that K_5 has no independently even drawing. It is an easy exercise to show that this also implies that no subdivision of K_5 has an even independently even drawing.

Take an arbitrary drawing of K_5 in the plane; for example, the usual straight-line drawing with vertices on a circle, which has exactly five crossings of independent edges. We use as a fact that every drawing of K_5 in the plane can be obtained from any other by a continuous deformation of the plane and a sequence of continuous deformations of the individual edges. A proper proof of this fact would need the Jordan–Schönflies theorem [68].

We are going to prove that the parity of the total number N of crossings of all independent pairs of edges does not change during any continuous deformation of the edges. To see this, take an edge $e = v_4v_5$ of K_5 and slightly deform it. We only have to check how the intersection between this edge and the edges of the triangle $T = v_1v_2v_3$ changes. As we pull e through an edge or over a vertex vertex of T, the total number of crossings between e and T changes by two. The possible two cases are illustrated in the following figure:



Similar arguments apply for $K_{3,3}$ in place of K_5. □

An elementary proof of the strong Hanani–Tutte theorem, which does not use Kuratowski's theorem, was given by Pelsmajer, Schaefer and Štefankovič [60].

The weak Hanani–Tutte theorem was discovered later than the strong variant, by several different authors [12, 59, 60]. It does not directly from the strong variant, as the name would suggest, because it offers an additional conclusion.

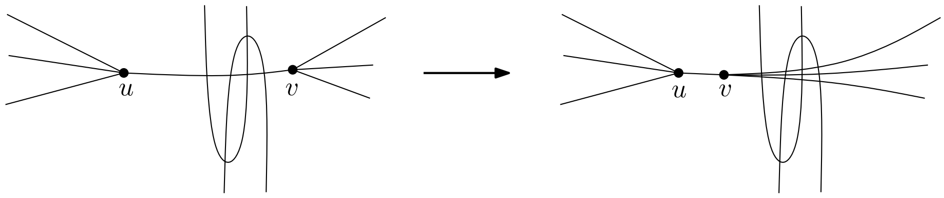
Definition 2.8. The **rotation** of a vertex v in a drawing of a graph is the clockwise cyclic order in which the edges incident to v leave the vertex v in the drawing in a small neighborhood of v. The collection of the rotations of all vertices in a drawing D is called the **rotation system** of D.

Theorem 2.9 (The weak Hanani–Tutte theorem, 2000+ [12, 59, 60])**.** *If D is a drawing of G where every two edges cross an even number of times, then G has a plane drawing with the same rotation system as D.*

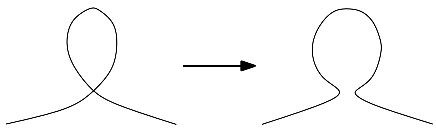
We show two different elementary proofs of Theorem 2.9, which do not need Kuratowski's theorem or any advanced topology.

Proof 1. (Pelsmajer, Schaefer and Štefankovič, 2007 [60]). We may assume that G is connected, since components may be redrawn arbitrarily far apart. Fix an even plane drawing D of G. We prove the result by induction on the number of edges in G. To make the induction possible, we prove the theorem for *multigraphs*, that is, a generalization of graphs where we allow *parallel edges* (more edges between the same pair of vertices) and *loops* (edges attached to the same vertex by both endpoints).

We begin with the inductive step: if there are at least two vertices in G, then there is an edge $e = uv$ that has two different vertices. Pull v towards u until there remains no crossing between v and u.



Since e was an even edge, the edges incident to v remain even. The pull move will introduce self-crossings in curves that intersect e and are adjacent to v. To correct this, we remove each self-crossing by a local redrawing like in this figure.



Now that the edge uv no longer has any crossings, we contract it while keeping all resulting loops or parallel edges that might arise (we may call this operation a *multigraph edge contraction*). We obtain a new multigraph G' in which the rotations of u and v are combined appropriately. By the inductive assumption, there is a planar drawing of G' respecting the rotation system.

In such a drawing, we can simply split the vertex corresponding to u and v, reintroducing the edge e between them without any intersections. We obtain a plane drawing of G respecting the rotations of all its vertices from D. Notice that the condition on the rotation system was necessary here for the induction step.

If G contains only a single vertex v, then it might have several loops attached to it. Since all the loops in G are even, it cannot happen that we find edges leaving v in the order a, b, a, b since this would force an odd number of crossings between a and b. Hence, if we consider the regions enclosed within the two loops in a small enough neighborhood of v, either they are disjoint or one region contains the other. Then it is easy to show that there must be a loop e whose ends are consecutive in the rotation of v. Removing e we obtain a smaller multigraph G' which, by inductive assumption, can be drawn without crossings while respecting the rotation system. We can then reinsert the missing loop in the right location according to the rotation of v by making it small enough.

In the base case, we simply draw a single vertex with no edges. □

Proof 2. (Fulek, Pelsmajer, Schaefer and Štefankovič, 2012 [28]). Let $G = (V, E)$ where $E = \{e_1, e_2, \ldots, e_m\}$. For every $i \in [m]$, let $E_i = \{e_1, e_2, \ldots, e_i\}$. Let $E_0 = \emptyset$. Let D_0 be the original even drawing of G in the plane. In m successive steps, we construct drawings $D_1, D_2, \ldots, D_m$ such that for every $i \in [m]$, the edges of E_i have no crossings in D_i, and D_i has the same rotation system as D_0. In particular, D_m will satisfy the theorem.

Let $i \in [m]$ and assume that we have constructed D_{i-1}. For every edge f of G that crosses e_i in D_{i-1}, we do the following operations. Since f crosses e_i an even number of times, we can match the crossings together in consecutive pairs in the order as they are encountered along e_i. We cut the edge f at each of these crossings and reconnect the severed ends of f by drawing curves between the neighborhoods of the pairs of matched crossings close to e_i, from both sides of e_i, like in Figure 2.1.

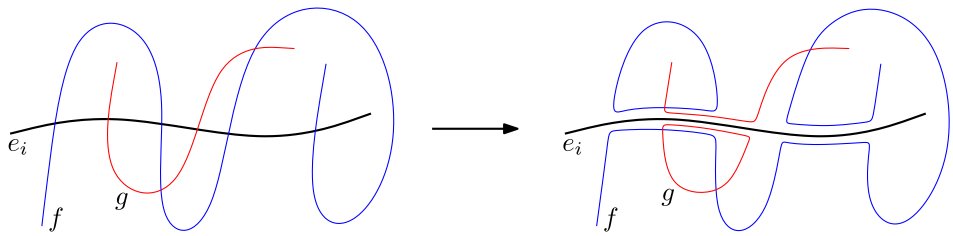


Figure 2.1: Cutting and reconnecting f along e_i.

By this operations, we removed all crossings of e_i with f. We might have created new crossings of f with other edges, but these always come in pairs as we draw the new portions of f from both sides of e_i. Moreover, the edges participating in these new crossings with f must cross e_i, so they do not belong to E_i. In general, the edge f is now represented by a “disconnected curve” consisting of one arc-component containing both endpoints of f, and several closed components. Therefore, we next try to connect some of these components together. As long as there are two components of f in the same face of the plane graph (V, E_i) in the current drawing, we connect them by a *tunnel* consisting of a pair of arcs running close to each other, see Figure 2.2.

Again, we might have created new crossings on f, but always in pairs, one on each side of the tunnel.

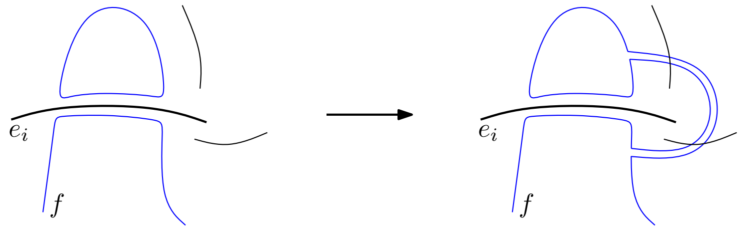


Figure 2.2: Connecting two components of f by a tunnel.

After performing these operations for all edges that crossed e_i in D_{i-1}, we have removed all crossings from e_i, and did not introduce any new crossings on the edges of E_{i-1}. It may still be the case that some edges are represented by disconnected curves, however. In this situation we just remove all the closed components. We need to verify that the resulting drawing D_i is still even. Suppose for contradiction that some two edges, f and g, cross an odd number of times in D_i. Then they are both in the same face of (V, E_i) in D_i. All closed components of f and g that we removed are thus in a different face, and cannot cross the arc-component of g and f, respectively. Since every two closed curves cross an even number of times, by removing the closed components, we changed the number of crossings between f and g by an even number. This implies that the number of crossings between f and g in D_{i-1} was odd, a contradiction. $\square$

For the reader interested in more information about the Hanani–Tutte theorems, their history and future, other variants, and applications, we highly recommend the surveys by Schaefer [63, 64].

2.2 Algebraic algorithm for planarity testing

Planarity testing is the following decision problem: given a graph G, is G planar? Many algorithms for planarity testing exist; the first linear-time algorithm was published by Hopcroft and Tarjan [39]. However, most of the algorithms are rather complicated.

By the strong Hanani–Tutte theorem (Theorem 2.7), planarity testing can be reduced to solving a system of linear equations over $\mathbb{Z}_2$ [63, Section 1.4.1]. Pach and Tóth described a slightly more general method for proving the NP-completeness of the odd crossing number [59], where they also took the rotation system into account. We now describe the method in detail.

Let G be a given graph and let D be an arbitrary drawing of G. For example, we may place the vertices of G on a circle in an arbitrary order and draw every edge as a straight-line segment. We use the "obvious" fact that every drawing of G can be obtained from any other drawing of G by a homeomorphism of the plane, which aligns the vertices of the two drawings, followed by a sequence of finitely many continuous deformations (isotopies) of the edges that keep the endpoints fixed, and maintain the property that every pair of edges have only finitely many points in common at each moment of the deformation (we will call such deformations *generic*). This fact can be proved using the Jordan–Schönflies theorem.

We will assume that the positions of the vertices are fixed. The algorithm will test whether the edges of the initial drawing D can be continuously deformed to form an independently even drawing of G. By the strong Hanani–Tutte theorem (Theorem 2.7), the existence of such a drawing is equivalent to G being planar. During a generic continuous deformation from D to some other drawing D', three types of combinatorially interesting events can happen:

1) two edges exchanging their order around their common vertex and creating a new crossing,

2) an edge passing over an another edge, forming a pair of new crossings and a lens between them,

3) an edge e passing over a vertex v not incident to e, creating a crossing with every edge incident to v.

Each of the three events has also a corresponding inverse event, where crossings are eliminated. However, the effects on the parity of the number of crossings between edges are the same. The parity of the number of crossings between a pair of independent edges is affected only during the event 3) or its inverse, in which case we change the parity of the number of crossings of e with all the edges incident to v; see Figure 2.3. We call such an event an *edge-vertex switch* and we will denote it by the ordered pair (e, v). We will consider drawings up to the equivalence generated by events of type 1) and 2) and their inverses. Every edge-vertex switch (e, v) can be performed independently of others, for any initial drawing, by deforming the edge e along a curve connecting an interior point of e with v. We can thus represent the deformation from D to D' by the set of edge-vertex switches that were performed an odd number of times during the deformation.

A drawing D of G can then be represented by a vector $\mathbf{v} \in \mathbb{Z}_2^M$ where M is the number of unordered pairs of independent edges in G. The component

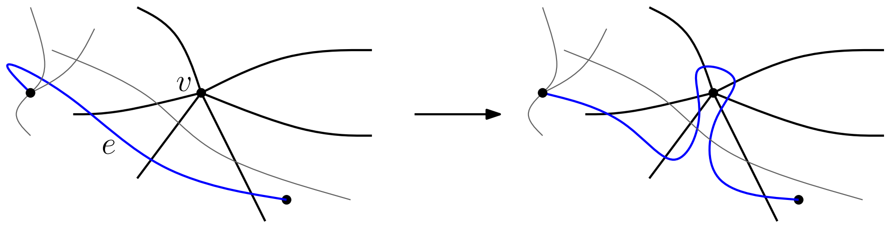


Figure 2.3: A continuous deformation of e resulting in an edge-vertex switch (e, v).

of $\mathbf{v}$ corresponding to a pair $\{e, f\}$ is 1 if e and f cross an odd number of times and 0 otherwise. We remark that the space $\mathbb{Z}_2^M$ can also be considered as the space of subgraphs of the complement of the line graph $L(G)$.

Let e be an edge of G and v a vertex of G such that $v \notin e$. Performing an edge-vertex switch (e, v) corresponds to adding the vector $\mathbf{w}^{(e,v)} \in \mathbb{Z}_2^M$ whose only components equal to 1 are those indexed by pairs $\{e, f\}$ where f is incident to v. The set of all drawings of G that can be obtained from D by edge-vertex switches then corresponds to an affine subspace $\mathbf{v} + W$ where W is the subspace generated by the set $\{\mathbf{w}^{(e,v)}; v \in V(G), e \in E(G), v \notin e\}$.

Since independently even embeddings of G are represented by the zero vector, G is planar if and only if $\mathbf{0} \in \mathbf{v} + W$, equivalently, $\mathbf{v} \in W$. This is equivalent to the solvability of a system of M linear equations over $\mathbb{Z}_2$, where each variable corresponds to one edge-vertex switch. In general, the vectors $\mathbf{w}^{(e,v)}$ are not linearly independent, so the number of variables in the system could be slightly reduced.

Before running the algorithm, we check whether $|E(G)| < 3|V(G)|$; if not, G is not planar by a consequence of Euler's formula (Corollary 1.4). After a straightforward preprocessing (selecting the initial drawing and finding which pairs of edges cross oddly), the algorithm solves a system of $O(|E(G)||V(G)|) = O(|V(G)|^2)$ linear equations in $O(|E(G)|^2) = O(|V(G)|^2)$ variables. This can be performed in $O(|V(G)|^6$ time by Gaussian elimination, or in $O(|V(G)|^{2\omega}) \leq O(|V(G)|^{4.746})$ time using the algorithm by Ibarra, Moran and Hui [40]. Here $O(n^\omega)$ is the complexity of multiplication of two square $n \times n$ matrices; the best current algorithms for matrix multiplication give $\omega < 2.3729$ [30, 75]. Since our linear system is sparse, it is also possible to use Wiedemann's randomized algorithm [74], with expected running time $O(n^4 \log^2 n)$ in our case.

2.3 Intersection representations of planar graphs

One of the most important theorems about representation of planar graphs is the Koebe–Andreev–Thurston theorem, also known as the circle packing theorem.

Theorem 2.10 (The Koebe–Andreev–Thurston theorem, 1936–1970–1985)**.** *The vertices $v \in V(G)$ of any planar graph G can be represented by closed disks D_v in the plane such that D_u and D_v are tangent to each other if and only if $uv \in E(G)$, otherwise D_u and D_v are disjoint.*

Theorem 2.11 (de Fraysseix, de Mendez, Rosenstiehl, 1994 [27])**.** *The vertices $v \in V(G)$ of any planar graph G can be represented by non-overlapping triangles T_v in the plane so that T_u and T_v have a point of contact if and only if $uv \in E(G)$.*

These two theorems give rise to the following question:

Question. *Is it true that the vertices of every planar graph can be represented by (pseudo-)segments so that two of them intersect if and only if the corresponding vertices are adjacent? A collection of* ***pseudosegments*** *is a collection of simple curves such that every two of them cross at most once and do not touch.*

The following two theorems answer part of this problem.

Theorem 2.12 (Hartman, Newman, Ziv, 1991 [38]; de Fraysseix, de Mendez, Pach, 1994 [26])**.** *True for bipartite planar graphs.*

Theorem 2.13 (Castro, Cobos, Dana, Márquez, Noy, 2002 [14])**.** *True for triangle-free planar graphs.*

The problem was finally solved by Chalopin, Gonçalves and Ochem for pseudosegments.

Theorem 2.14 (Chalopin, Gonçalves, Ochem, 2010 [16])**.** *Every planar graph has an intersection representation by pseudosegments in the plane.*

Chalopin and Gonçalves then strengthened the proof to representations by segments.

Theorem 2.15 (Chalopin, Gonçalves, 2009 [15])**.** *Every planar graph has an intersection representation by segments in the plane.*

2.4 Embeddings of graphs in three dimensions

Our next subject are graphs in higher dimensions. A graph can be drawn in $\mathbb{R}^3$ in the following way: the vertices are points in $\mathbb{R}^3$ and the edges are simple curves such that they do not pass through any vertex and do not cross any other edge.

Definition 2.16. (i) Let γ_1, γ_2 be two simple closed curves in $\mathbb{R}^3$. Notice that we cannot always transform γ_1 into γ_2 just by deforming the space (such a deformation is called *ambient isotopy*), since the curves may be knotted in different ways. If we allow deformations of the curve during which the curve may cross itself, then it is possible to deform γ_1 into a circle γ which bounds a disc D. By reversing this deformation while dragging the disc D with the curve, we obtain a disc-like surface D_1, which may intersect itself, and whose boundary is γ_1. Then γ_1 and γ_2 are called **linked** if the number of times γ_2 intersects D_1 from "above" is different from the number of times it intersects D_1 from "below".

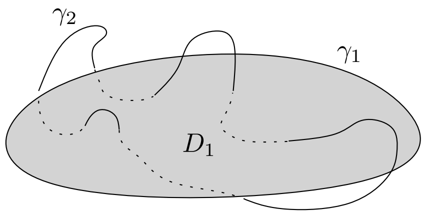


Figure 2.4: Two unlinked curves in $\mathbb{R}^3$.

(ii) Two cycles C_1, C_2 in an embedding of a graph in $\mathbb{R}^3$ are **linked** if the corresponding closed curves γ_1, γ_2 are linked.

(iii) G is a **linkless graph** if it can be drawn in $\mathbb{R}^3$ so that no two disjoint cycles are linked.

Theorem 2.17 (Robertson–Seymour–Thomas)**.** *A graph G has a linkless embedding in $\mathbb{R}^3$ if and only if G has no minor belonging to the Petersen family (shown in Figure 2.5).*

Example 2.1. *K_6 is not a linkless graph, that is, it cannot be drawn without two linked cycles.*

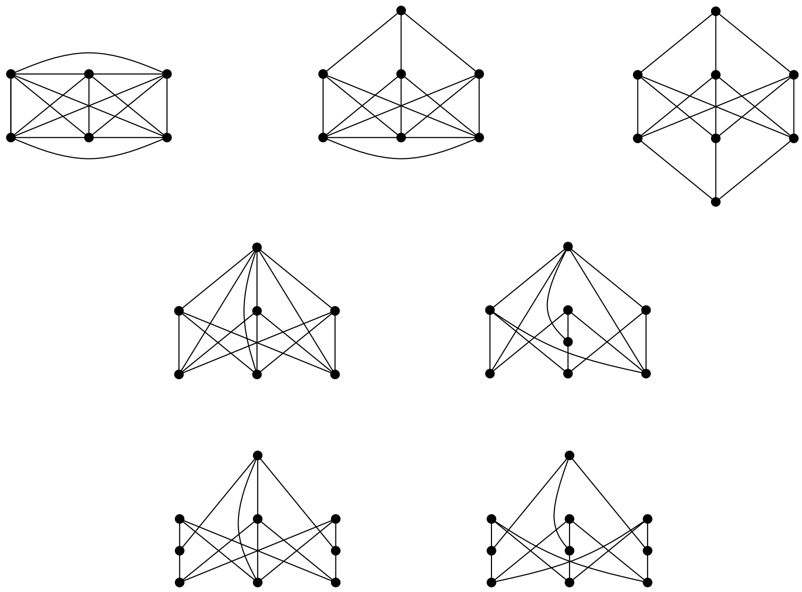


Figure 2.5: The Petersen family.

Bibliography

[1] E. Ackerman, On the maximum number of edges in topological graphs with no four pairwise crossing edges, *Discrete Comput. Geom.* **41**(3) (2009), 365–375.

[2] E. Ackerman, On topological graphs with at most four crossings per edge, unpublished manuscript, `http://sci.haifa.ac.il/~ackerman/publications/4crossings.old`, 2013.

[3] E. Ackerman and G. Tardos, On the maximum number of edges in quasi-planar graphs, *J. Combin. Theory Ser. A* **114**(3) (2007), 563–571.

[4] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack and M. Sharir, Quasi-planar graphs have a linear number of edges, *Combinatorica* **17**(1) (1997), 1–9.

[5] M. Ajtai, V. Chvátal, M. M. Newborn and E. Szemerédi, Crossing-free subgraphs, *Theory and practice of combinatorics*, 9–12, *North-Holland Math. Stud.*, 60, North-Holland, Amsterdam, 1982.

[6] N. Alon, P. Seymour and R. Thomas, Planar separators, *SIAM J. Discrete Math.* **7**(2) (1994), 184–193.

[7] A. Andrzejak, B. Aronov, S. Har-Peled, R. Seidel and E. Welzl, Results on k-sets and j-facets via continuous motion, *Proceedings of the Fourteenth Annual Symposium on Computational Geometry*, 192–199, ACM, New York, NY, 1998.

[8] L. Babai, On the nonuniform Fisher inequality, *Discrete Math.* **66**(3) (1987), 303–307.

[9] J. Blažek and M. Koman, A minimal problem concerning complete plane graphs, in: *Theory of Graphs and its Applications, Proc. Sympos. Smolenice, 1963*, 113–117, Publ. House Czechoslovak Acad. Sci., Prague, 1964.

[10] L. Beineke and R. Wilson, The early history of the brick factory problem, *The Mathematical Intelligencer* **32**(2) (2010), 41–48.

[11] N. G. de Bruijn and P. Erdős, On a combinatorial problem, *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen Indagationes mathematicae* **51** (1948), 1277–1279; *Indagationes Math.* 10, 421–423 (1948).

[12] G. Cairns and Y. Nikolayevsky, Bounds for generalized thrackles, *Discrete Comput. Geom.* **23**(2) (2000), 191–206.

[13] V. Capoyleas and J. Pach, A Turán-type theorem on chords of a convex polygon, *J. Combin. Theory Ser. B* **56**(1) (1992), 9–15.

[14] N. de Castro, F. J. Cobos, J. C. Dana, A. Márquez and M. Noy, Triangle-free planar graphs as segment intersection graphs, *Graph drawing and representations (Prague, 1999), J. Graph Algorithms Appl.* **6**(1) (2002), 7–26.

[15] J. Chalopin and D. Gonçalves, Every planar graph is the intersection graph of segments in the plane: extended abstract, *Proceedings of the forty-first annual ACM symposium on Theory of computing (STOC 09),* 631–638, ACM New York, NY, USA, 2009; full version: `http://pageperso.lif.univ-mrs.fr/~jeremie.chalopin/publis/CG09.long.pdf`

[16] J. Chalopin, D. Gonçalves and P. Ochem, Planar graphs have 1-string representations, *Discrete Comput. Geom.* **43**(3) (2010), 626–647.

[17] J. Černý, Geometric graphs with no three disjoint edges, *Discrete Comput. Geom.* **34**(4) (2005), 679–695.

[18] D. Conlon, A new upper bound for diagonal Ramsey numbers, *Ann. of Math.* **170** (2009), 941–960.

[19] T. K. Dey, Improved bounds for planar k-sets and related problems, *Discrete Comput. Geom.* **19**(3) (1998), 373–382.

[20] R. P. Dilworth, A decomposition theorem for partially ordered sets, *Ann. of Math. (2)* **51** (1950), 161–166.

[21] A. Dumitrescu and G. Tóth, Ramsey-type results for unions of comparability graphs, *Graphs Combin.* **18**(2) (2002), 245–251.

[22] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53**, (1947) 292–294.

[23] P. Erdős, L. Lovász, A. Simmons and E. G. Straus, Dissection graphs of planar point sets, *A survey of combinatorial theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971)*, 139–149, North-Holland, Amsterdam, 1973.

[24] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.

[25] R. A. Fisher, An examination of the different possible solutions of a problem in incomplete blocks, *Ann. Eugenics* **10** (1940), 52–75.

[26] H. de Fraysseix, P. O. de Mendez and J. Pach, Representation of planar graphs by segments, *Intuitive geometry (Szeged, 1991), Colloq. Math. Soc. János Bolyai* **63**, 109–117, North-Holland, Amsterdam, 1994.

[27] H. de Fraysseix, P. O. de Mendez and P. Rosenstiehl, On triangle contact graphs, *Combin. Probab. Comput.* **3**(2) (1994), 233–246.

[28] R. Fulek, M. J. Pelsmajer, M. Schaefer and D. Štefankovič, Adjacent crossings do matter, *J. Graph Algorithms Appl.* **16**(3) (2012), 759–782.

[29] R. Fulek and J. Pach, A computational approach to Conway's thrackle conjecture, *Comput. Geom.* **44**(6-7) (2011), 345–355.

[30] F. L. Gall, Powers of tensors and fast matrix multiplication, arXiv:1401.7714 (2014).

[31] H. Gazit and G. Miller, Planar separators and the Euclidean norm, *Algorithms (Tokyo, 1990)*, 338–347, *Lecture Notes in Comput. Sci.*, 450, Springer, Berlin, 1990.

[32] W. Goddard, M. Katchalski and D. J. Kleitman, Forcing disjoint segments in the plane, *European J. Combin.* **17**(4) (1996), 391–395.

[33] R. K. Guy, A combinatorial problem, *Nabla (Bull. Malayan Math. Soc)* **7** (1960), 68–72.

[34] H. Hanani, Über wesentlich unplättbare Kurven im drei-dimensionalen Raume, *Fundamenta Mathematicae* **23** (1934), 135–142.

[35] F. Harary and A. Hill, On the number of crossings in a complete graph, *Proc. Edinburgh Math. Soc. (2)* **13** (1963), 333–338.

[36] H. Harborth, Special numbers of crossings for complete graphs, *Discrete Math.* **244** (2002), 95–102.

[37] S. Har-Peled, A simple proof of the existence of a planar separator, arXiv:1105.0103v5, 2013.

[38] I. B.-A. Hartman, I. Newman and R. Ziv, On grid intersection graphs, *Discrete Math.* **87**(1) (1991), 41–52.

[39] J. Hopcroft and R. E. Tarjan, Efficient planarity testing, *J. Assoc. Comput. Mach.*, **21**(4) (1974), 549–568.

[40] O. H. Ibarra, S. Moran and R. Hui, A generalization of the fast LUP matrix decomposition algorithm and applications, *J. Algorithms* **3**(1) (1982), 45–56.

[41] G. Károlyi, J. Pach and G. Tóth, Ramsey-type results for geometric graphs, I, *Discrete Comput. Geom.* **18**(3) (1997), 247–255.

[42] D. J. Kleitman, The crossing number of $K_{5,n}$, *J. Combinatorial Theory* **9** (1970), 315–323.

[43] T. Kővári, V. Sós and P. Turán, On a problem of K. Zarankiewicz, *Colloquium Math.* **3** (1954) 50–57.

[44] D. Larman, J. Matoušek, J. Pach and J. Törőcsik, A Ramsey-type result for convex sets, *Bull. London Math. Soc.* **26**(2) (1994), 132–136.

[45] F. T. Leighton, *Layouts for the shuffle-exchange graph and lower bound techniques for VLSI*, PhD thesis, Laboratory for Computer Science, Massachusetts Institute of Technology, 1982.

[46] R. J. Lipton and R. E. Tarjan, A separator theorem for planar graphs, *SIAM J. Appl. Math.* **36**(2) (1979), 177–189.

[47] L. Lovász, On the number of halving lines, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **14** (1971), 107–108.

[48] L. Lovász, J. Pach and M. Szegedy, On Conway's thrackle conjecture, *Discrete Comput. Geom.* **18**(4) (1997), 369–376.

[49] G. L. Miller and W. Thurston, Separators in two and three dimensions, *STOC '90 Proceedings of the twenty-second annual ACM symposium on Theory of computing,* 300–309, ACM New York, NY, 1990.

[50] L. Mirsky, A dual of Dilworth's decomposition theorem, *Amer. Math. Monthly* **78** (1971), 876–877.

[51] G. Nivasch, An improved, simple construction of many halving edges, *Surveys on discrete and computational geometry*, 299–305, *Contemp. Math.*, 453, Amer. Math. Soc., Providence, RI, 2008.

[52] J. Pach and P. Agarwal, *Combinatorial geometry,* Wiley-Interscience Series in Discrete Mathematics and Optimization, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1995, ISBN: 0-471-58890-3.

[53] J. Pach, R. Radoičić, G. Tardos and G. Tóth, Improving the crossing lemma by finding more crossings in sparse graphs, *Discrete Comput. Geom.* **36**(4) (2006), 527–552.

[54] J. Pach, R. Radoičić and G. Tóth, Relaxing planarity for topological graphs, *More sets, graphs and numbers*, 285–300, *Bolyai Soc. Math. Stud.* 15, Springer, Berlin, 2006.

[55] J. Pach, F. Shahrokhi and M. Szegedy, Applications of the crossing number, *Algorithmica* **16**(1) (1996), 111–117.

[56] J. Pach, J. Spencer and G. Tóth, New bounds on crossing numbers, *Discrete Comput. Geom.* **24**(4) (2000), 623–644.

[57] J. Pach, W. Steiger and E. Szemerédi, An upper bound on the number of planar k-sets, *Discrete Comput. Geom.* **7**(2) (1992), 109–123.

[58] J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica* **17**(3) (1997), 427–439.

[59] J. Pach and G. Tóth, Which crossing number is it anyway?, *J. Combin. Theory Ser. B* **80**(2) (2000), 225–246.

[60] M. J. Pelsmajer, M. Schaefer and D. Štefankovič, Removing even crossings, *J. Combin. Theory Ser. B* **97**(4) (2007), 489–500.

[61] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **s2-30**(1) (1930), 264–286.

[62] R. B. Richter and C. Thomassen, Relations between crossing numbers of complete and complete bipartite graphs, *Amer. Math. Monthly* **104**(2) (1997), 131–137.

[63] M. Schaefer, Hanani-Tutte and related results, *Geometry—intuitive, discrete, and convex*, vol. 24 of *Bolyai Soc. Math. Stud.*, 259–299, János Bolyai Math. Soc., Budapest (2013).

[64] M. Schaefer, Toward a theory of planarity: Hanani-Tutte and planarity variants, *J. Graph Algorithms Appl.* **17**(4) (2013), 367–440.

[65] J. Spencer, E. Szemerédi and W. Trotter, Jr., Unit distances in the Euclidean plane, *Graph theory and combinatorics (Cambridge, 1983)*, 293–303, Academic Press, London, 1984.

[66] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combin. Probab. Comput.* **6**(3) (1997), 353–358.

[67] E. Szemerédi and W. T. Trotter, Jr., A combinatorial distinction between the Euclidean and projective planes, *European J. Combin.* **4**(4) (1983), 385–394.

[68] C. Thomassen, The Jordan-Schönflies theorem and the classification of surfaces, *Amer. Math. Monthly* **99**(2) (1992), 116–130.

[69] G. Tóth, Note on geometric graphs, *J. Combin. Theory Ser. A* **89**(1) (2000), 126–132.

[70] G. Tóth, Point sets with many k-sets, *Discrete Comput. Geom.* **26**(2) (2001), 187–194.

[71] G. Tóth and P. Valtr, The Erdős–Szekeres theorem: upper bounds and related results, in: J.E. Goodman et al. (eds.), *Combinatorial and Computational Geometry*, *MSRI Publications* **52** (2005), 557–568.

[72] W. T. Tutte, Toward a theory of crossing numbers, *J. Combinatorial Theory* **8** (1970), 45–53.

[73] P. Valtr, On geometric graphs with no k pairwise parallel edges, *Discrete Comput. Geom.* **19**(3) (1998), 461–469.

[74] D. H. Wiedemann, Solving sparse linear equations over finite fields, *IEEE Trans. Inform. Theory* **32**(1) (1986), 54–62.

[75] V. V. Williams, Multiplying matrices faster than Coppersmith-Winograd, *Proceedings of the Forty-fourth Annual ACM Symposium on Theory of Computing*, STOC '12, 887–898, ACM, New York, NY, USA (2012).

[76] Y. Xu, *Generalized thrackles and graph embeddings*, M.Sc. thesis, Simon Fraser University, 2014. http://summit.sfu.ca/item/14748

[77] K. Zarankiewicz, On a problem of P. Turan concerning graphs, *Fund. Math.* **41** (1954) 137–145.