

# Chapter 1

## Drawing planar graphs

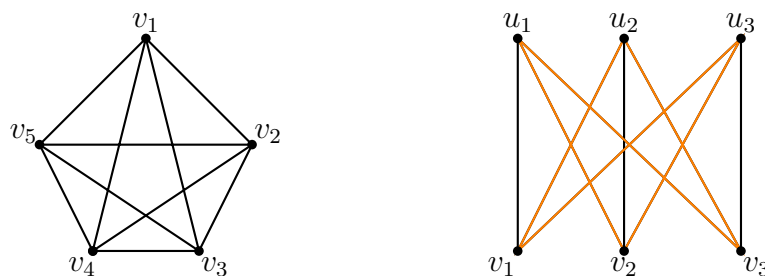
A **graph**  $G$  consists of a finite set  $V(G)$  of **vertices** (points) and a set  $E(G)$  of **edges**, where every edge is a 2-element subset  $\{u, v\} \subseteq V(G)$ ,  $u \neq v$ . For the sake of simplicity, an edge  $\{u, v\}$  is often denoted by  $uv$  (or  $vu$ ). Vertices  $u$  and  $v$  are called the **endpoints** of  $uv \in E(G)$ . If  $uv \in E(G)$ , then we say that  $u$  and  $v$  are **connected** (**joined** by an edge) in  $G$ , or that they are **adjacent**. A graph  $H$  is a **subgraph** of  $G$ , written  $H \subseteq G$ , if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . Given a  $k$ -element set  $V = \{v_1, v_2, \dots, v_k\}$ , the graphs  $P_k$  and  $C_k$  defined by

$$\begin{aligned}V(P_k) &= V, E(P_k) = \{v_1v_2, v_2v_3, \dots, v_{k-1}v_k\}; \\V(C_k) &= V, E(C_k) = \{v_1v_2, \dots, v_{k-1}v_k, v_kv_1\}\end{aligned}$$

are called a **path** of length  $k - 1$  and a **cycle** of length  $k$  respectively. Obviously  $P_k \subseteq C_k$ .

The most natural way of representing a graph in the plane is to assign distinct points to its vertices and connect two points by a simple curve if and only if the corresponding vertices are adjacent. A **simple curve** (also a **Jordan arc**) connecting two points  $u, v \in \mathbb{R}^2$  is a continuous non-self-intersecting curve  $\phi: [0, 1] \rightarrow \mathbb{R}^2$  with  $\phi(0) = u$  and  $\phi(1) = v$ . If no confusion is likely to occur, we often talk about the points and curves in the representation as vertices and edges, respectively. The actual positions of the points and the curves play no role in this representation. However, we usually require that a **drawing** of a graph satisfies the following conditions:

- (1) the edges pass through no vertices except their endpoints,
- (2) every two edges have only a finite number of intersection points,
- (3) every intersection point of two edges is either a common endpoint or a proper crossing (“touching” of the edges is not allowed), and

Figure 1.1: Non-planar graphs:  $K_5$  (left),  $K_{3,3}$ (right)

(4) no three edges pass through the same crossing.

Some drawings of a graph are much simpler than some others, and usually we also want to produce a visually pleasing diagram. For instance we may require our arcs to be straight-line segments, we may wish to avoid or minimize crossings, maximize angles between edges etc.

**Definition 1.1.** A graph  $G$  that can be represented in the plane so that no two arcs meet at a point different from their endpoints is said to be **embeddable in the plane** or **planar**. A particular representation of a planar graph satisfying this property is called a **plane graph**.

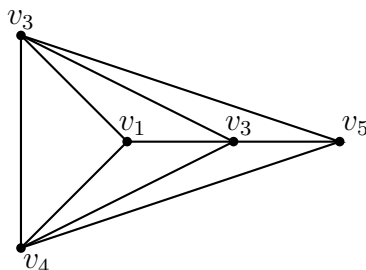
Intuitively, it is easy to “see” that the graphs  $K_5$  and  $K_{3,3}$  depicted in Figure 1.1 are not planar. However, to prove this precisely, one needs at least a polygonal version of the following “intuitively obvious” fact. A **simple closed curve** (also a **Jordan curve**) is a continuous map  $\varphi: [0, 1] \rightarrow \mathbb{R}^2$  that is injective on  $[0, 1)$  and satisfies  $\varphi(0) = \varphi(1)$ .

**Theorem 1.2** (Jordan curve theorem). *The complement of a simple closed curve in the plane has exactly two connected components, one bounded and the other one unbounded.*

The closures of the two components of the complement of a simple closed curve  $\varphi$  are also called the **interior** and **exterior** of  $\varphi$ .

For the readers who are interested in the proof of the of the Jordan curve theorem, we recommend Thomassen’s proof [36]. Quite surprisingly, his proof is based on the fact that  $K_{3,3}$  is not planar.

Now suppose that  $K_{3,3}$  can be embedded in the plane. Then the arcs  $u_1v_2, v_2u_3, u_3v_1, v_1u_2, u_2v_3, v_3u_1$  would form a closed simple curve  $\varphi$ , and every arc  $u_iv_i$  ( $i = 1, 2, 3$ ) would be entirely in the interior of  $\varphi$  or in the exterior of  $\varphi$ . Assume without loss of generality that  $u_1v_1$  and  $u_2v_2$  lie in the interior of  $\varphi$ .

Figure 1.2: A straight-line embedding of  $K_5 - v_1v_5$ 

Then they should cross each other, contradicting our assumption. (Notice that we used the Jordan curve theorem in two places in this argument.) Similarly, one can check that  $K_5$  is not embeddable in the plane. In fact, a well-known theorem by Kuratowski (see Theorem 2.2) states that a graph is not planar if and only if it has a subgraph that can be obtained from  $K_5$  or  $K_{3,3}$  by replacing the edges with paths, all of whose interior vertices are distinct.

Deleting any edge (say  $v_1v_5$ ) from  $K_5$ , we obtain a planar graph. Moreover, this new graph can be embedded in the plane by using only straight-line segments (see Figure 1.2).

Does every planar graph have such a representation? As we shall see, the answer to this question is in the affirmative. Moreover, we will be able to impose some further restrictions on our drawings to ensure that the resulting diagrams are relatively balanced. But first we need some preparations.

## 1.1 Euler's formula

Let  $G$  be a graph. The **degree**  $d_G(v)$  (or simply  $d(v)$ ) of a vertex  $v \in V(G)$  is the number of vertices adjacent to  $v$  (recall that  $v$  is never adjacent to itself!). Denoting the number of vertices and edges of  $G$  by  $v(G)$  and  $e(G)$ , respectively, we clearly have

$$\sum_{v \in V(G)} d(v) = 2e(G). \quad (1.1)$$

Consequently,  $G$  must have a vertex whose degree is at most  $2e(G)/v(G)$ .

A graph  $G$  is said to be **connected** if for any two vertices  $v, v' \in V(G)$  there is a sequence of vertices  $v_1 = v, v_2, v_3, \dots, v_k = v'$  such that  $v_i v_{i+1} \in E(G)$  for all  $i$  ( $1 \leq i \leq k$ ). In other words,  $G$  is connected if for any

$v, v' \in V(G)$  there is a path  $P_k \subseteq G$  with  $v, v' \in V(P_k)$ . If  $G$  is connected and  $v(G) \geq 2$ , then  $d(v) \geq 1$  for any  $v \in V(G)$ , that is,  $G$  has no isolated vertices. It is easy to see that any connected graph has at least  $v(G) - 1$  edges, and equality holds if and only if  $G$  has no cycle as a subgraph.

The arcs of a plane graph partition the rest of the plane into a number of connected components, called **faces**. Exactly one of these faces is unbounded, which is called the **exterior face**. The number of faces of a plane graph  $G$  is denoted by  $f(G)$ .

**Theorem 1.3** (Euler's formula). *If  $G$  is a connected plane graph, then*

$$v(G) - e(G) + f(G) = 2.$$

*Proof.* By induction on  $f$ . If  $f(G) = 1$ , then  $G$  has no cycle, thus by the above remark  $e(G) = v(G) - 1$  and the assertion is true. Assume that  $f(G) = f \geq 2$ , and we have already proved the theorem for all connected plane graphs having fewer than  $f$  faces. Obviously,  $G$  must contain a cycle. Delete any edge  $e$  that belongs to a cycle of  $G$ . For the resulting plane graph  $G - e$ ,  $f(G - e) = f(G) - 1$ , so we can apply the induction hypothesis to obtain

$$v(G) - (e(G) - 1) + (f(G) - 1) = 2.$$

□

Let  $G$  be a plane graph. If an edge (arc)  $e$  of  $G$  belongs to the boundary of only one face of  $G$ , then  $e$  is called a **bridge**. Let  $F(G)$  denote the set of faces of  $G$ . For any  $f \in F(G)$ , let  $s(f)$  be the number of **sides** of  $f$ , that is, the number of edges belonging to the boundary of  $f$ , where all bridges are counted twice. Obviously,

$$\sum_{f \in F(G)} s(f) = 2e(G). \quad (1.2)$$

**Corollary 1.4.** *Let  $G$  be any plane graph with at least three vertices. Then*

$$(i) \quad e(G) \leq 3v(G) - 6,$$

$$(ii) \quad f(G) \leq 2v(G) - 4.$$

*In both cases equality holds if and only if all faces of  $G$  have three sides.*

*Proof.* It is sufficient to prove the statement for connected plane graphs. Clearly,  $s(f) \geq 3$  for any face  $f \in F(G)$ . Then

$$3f(G) \leq \sum_{f \in F(G)} s(f) = 2e(G).$$

By Euler's formula, we obtain

$$v(G) - e(G) + \frac{2}{3}e(G) \geq 2,$$

$$v(G) - \frac{3}{2}f(G) + f(G) \geq 2,$$

as required.  $\square$

If  $s(f) = 3$  for some face  $f \in F(G)$ , then  $f$  is called a **triangle**. If all faces of  $G$  are triangles, then  $G$  is a **triangulation**. It is easy to show that any plane graph can be extended to a triangulation by the addition of edges (without introducing new vertices). Corollary 1.4 implies that, if  $G$  is a triangulation, then it is **maximal** in the sense that no further edges can be added to  $G$  without violating its planarity.

The **chromatic number**  $\chi(G)$  of a graph  $G$  is the minimum number of colors necessary to color the vertices of  $G$  so that no two vertices of the same color are adjacent. According to the **four-color theorem** of Appel and Haken, which settled a famous conjecture of Guthrie posed in the last century, the chromatic number of any planar graph is at most 4. (The graph in Figure 1.2 has chromatic number 4, showing that this bound cannot be improved.) The proof of Appel and Haken is quite complicated, and uses lengthy calculations by computers. However, a weaker statement can easily be deduced from Corollary 1.4.

**Corollary 1.5.** *If  $G$  is a planar graph, then  $\chi(G) \leq 5$ .*

*Proof.* By induction on  $v(G)$ . If  $v(G) \leq 5$ , then the statement is true, because one can assign a different color to each vertex of  $G$ . Assume that  $v(G) = v \geq 6$ , and that we have already established the result for all planar graphs with fewer than  $v$  vertices.

It follows from Equation (1.1) and Corollary 1.4(i) that  $G$  has a vertex  $u$  with  $d(u) \leq 5$ . If  $d(u) \leq 4$ , then we apply the induction hypothesis to the graph  $G - u$  obtained from  $G$  by the removal of  $u$  (and all edges incident to  $u$ ). We get that the vertices of  $G - u$  can be colored by 5 colors so that no two vertices of the same color are adjacent. Clearly, we can assign a color to  $u$ , different from the (at most 4) colors used for its neighbors.

Suppose next that  $u$  is adjacent to 5 vertices  $u_i$  ( $1 \leq i \leq 5$ ). Since  $G$  is planar, it cannot contain  $K_5$  as a subgraph. Thus, we can assume that, say  $u_1$  and  $u_2$  are not adjacent. Let  $G'$  denote the graph obtained from  $G - u$  by merging  $u_1$  and  $u_2$ . That is,  $V(G') = (V(G) \setminus \{u, u_1, u_2\}) \cup \{u'\}$  and  $E(G')$  consists of all edges of  $G$ , whose both endpoints belong to  $V(G) \setminus \{u, u_1, u_2\}$ , plus those pairs  $wu'$ , for which  $w \in V(G) \setminus \{u, u_1, u_2\}$  and either  $wu_1$  or  $wu_2$

belongs to  $E(G)$ . It is easy to see that  $G'$  is a planar graph, hence we can apply the induction hypothesis to obtain a proper coloring of the vertices of  $G'$  by 5 colors. If we assign the color of  $u'$  to both  $u_1$  and  $u_2$ , then we obtain a proper coloring of  $G - u$  such that the vertices  $u_i$  ( $1 \leq i \leq 5$ ) have at most 4 different colors. Therefore, we can again color  $u$  differently from its neighbors.  $\square$

## 1.2 Straight-line drawing

In this section we are going to show that every planar graph  $G$  can be embedded in the plane so that the arcs representing the edges of  $G$  are straight-line segments that can meet only at their endpoints. An embedding with this property is called a **straight-line embedding** of  $G$ . The existence of such an embedding was discovered independently by Fáry, Tutte and Wagner, but it also follows from an ancient theorem of Steinitz.

The proof presented here is based on a simple canonical way of constructing a plane graph, which will allow us to use an inductive argument for finding a proper position of the vertices one by one.

We need the following observation.

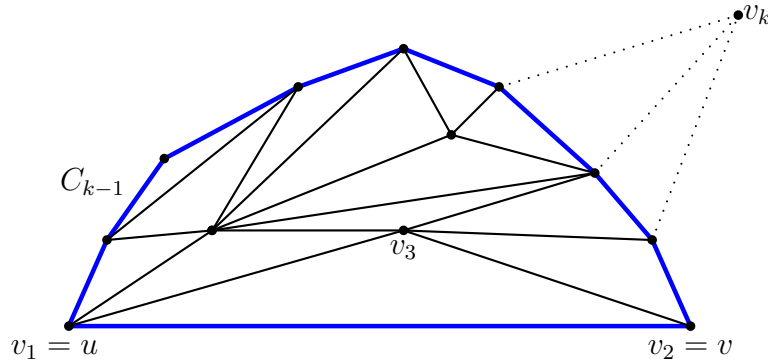
**Lemma 1.6.** *Let  $G$  be a plane graph, whose exterior face is bounded by a cycle  $u_1, u_2, \dots, u_k$ . Then there is a vertex  $u_i$  ( $i \neq 1, k$ ) not adjacent to any  $u_j$  other than  $u_{i-1}$  and  $u_{i+1}$ .*

*Proof.* If there are no two non-consecutive vertices along the boundary of the exterior face that are adjacent, then there is nothing to prove. Otherwise, pick an edge  $u_i u_j \in E(G)$ , for which  $j > i+1$  and  $j-i$  is minimal. Then  $u_{i+1}$  cannot be adjacent to any element of  $\{u_1, \dots, u_{i-1}, u_{j+1}, \dots, u_k\}$  by planarity, nor can it be adjacent to any other vertex of the exterior face different from  $u_i$  and  $u_{i+2}$ , by minimality of  $j-i$ .  $\square$

Let  $G$  be a graph (or a plane graph), and let  $U \subseteq V(G)$ . The subgraph of  $G$  **induced** by  $U$  is a graph (a plane graph) whose vertex set is  $U$  and whose edge set consists of all edges of  $E(G)$  such that both of their endpoints belong to  $U$ .

Now we are in the position to establish the following useful theorem.

**Theorem 1.7** (Canonical Construction of Triangulations). *Let  $G$  be a triangulation on  $n$  vertices, with exterior face  $uvw$ . Then there is a labeling of the vertices  $v_1 = u, v_2 = v, v_3, \dots, v_n = w$  satisfying the following conditions for every  $k \in \{4, \dots, n\}$ :*

Figure 1.3:  $G_{k-1}$  and  $v_k$  in the exterior

- (i) the boundary of the exterior face of the subgraph  $G_{k-1}$  of  $G$  induced by  $\{v_1, v_2, \dots, v_{k-1}\}$  is a cycle  $C_{k-1}$  containing the edge  $uv$ ;
- (ii)  $v_k$  is in the exterior face of  $G_{k-1}$ , and its neighbors in  $V(G_{k-1})$  are some (at least two) consecutive elements along the path obtained from  $C_{k-1}$  by removal of the edge  $uv$ . (See Figure 1.3)

*Proof.* The vertices  $v_n, v_{n-1}, \dots, v_3$  will be defined by reverse induction. Set  $v_n = w$ , and let  $G_{n-1}$  be the graph obtained from  $G$  by the deletion of  $v_n$ . Since  $G$  is a triangulation, the neighbors of  $w$  form a cycle  $C_{n-1}$  containing  $uv$ , and this cycle is the boundary of the exterior face of  $G_{n-1}$ .

Let  $k \in \{4, \dots, n\}$  be fixed and assume that  $v_n, v_{n-1}, \dots, v_k$  have already been determined so that the subgraph  $G_{k-1}$  induced by  $V(G) \setminus \{v_k, v_{k+1}, \dots, v_n\}$  satisfies conditions (i) and (ii). Let  $C_{k-1}$  denote the boundary of the exterior face of  $G_{k-1}$ . Applying Lemma 1.6 to  $G_{k-1}$ , we obtain that there is a vertex  $u'$  on  $C_{k-1}$ , different from  $u$  and  $v$ , which is adjacent only to two other vertices of  $C_{k-1}$  (that is, to its immediate neighbors). Letting  $v_{k-1} = u'$ , the subgraph  $G_{k-2} \subseteq G$  induced by  $V(G) \setminus \{v_{k-1}, v_k, \dots, v_n\}$  obviously meets the requirements.  $\square$

Using this theorem, we can easily prove the main result of this section.

**Corollary 1.8.** *Every planar graph has a straight-line embedding in the plane.*

*Proof.* It is sufficient to show that the statement is true for any **maximal** planar graph, that is, for any graph that can be represented by a triangulation (see Exercise 1.4 and the remark after Corollary 1.4).

Let  $G$  be any triangulation with the canonical labeling  $v_1 = u, v_2 = v, v_3, \dots, v_n = w$ , described above. We will determine the positions  $f(v_k) = (x(v_k), y(v_k))$  of the vertices by induction on  $k$ .

Set  $f(v_1) = (0, 0)$ ,  $f(v_2) = (2, 0)$ ,  $f(v_3) = (1, 1)$ . Assume that  $f(v_1), f(v_2), \dots, f(v_{k-1})$  have already been defined for some  $k \geq 4$  so that by connecting the images of the adjacent vertex pairs by segments, we obtain a straight-line embedding of  $G_{k-1}$  whose exterior face is bounded by the segments corresponding to the edges of  $C_{k-1}$ . Suppose further that

$$\begin{aligned} x(u_1) &< x(u_2) < \dots < x(u_m), \\ y(u_i) &> 0 \quad \text{for } 1 < i < m, \end{aligned} \tag{1.3}$$

where  $u_1 = u, u_2, u_3, \dots, u_m = v$  denote the vertices of  $C_{k-1}$  listed in cyclic order. By condition (ii) of Theorem 1.7,  $v_k$  is connected to  $u_p, u_{p+1}, \dots, u_q$  for some  $1 \leq p \leq q \leq m$ . Let  $x(v_k)$  be any number strictly between  $x(u_p)$  and  $x(u_q)$ . If we choose  $y(v_k) > 0$  to be sufficiently large and connect  $f(v_k) = (x(v_k), y(v_k))$  to  $f(u_p), f(u_{p+1}), \dots, f(u_q)$  by segments, then we obtain a straight-line embedding of  $G_k$  meeting all the requirements (including the auxiliary conditions (1.3) for the vertices of  $C_k$ ).  $\square$

Note that by the same method we can also establish the existence of straight-line embeddings with some special geometric properties. For example, we can require that the segments corresponding to the edges of  $C_{k-1}$  form a convex polygon for every  $k \geq 4$ .

**Corollary 1.9.** *Let  $G$  be a planar graph with  $n$  vertices and  $3n - 6$  edges. Then there are a labeling of the vertices  $v_1, v_2, \dots, v_n$  and a straight-line embedding of  $G$  such that for every  $4 \leq k \leq n$ ,*

- (i) *the image of the subgraph of  $G$  induced by  $\{v_1, v_2, \dots, v_{k-1}\}$  is a triangulated convex polygon  $C_{k-1}$ , and*
- (ii) *the image of  $v_k$  lies in the exterior of  $C_{k-1}$ .*

The same technique can be used to obtain a different kind of representation of planar graphs, found by Rosenstiehl and Tarjan.

**Corollary 1.10.** *The vertices and the edges of any planar graph can be represented by horizontal and vertical segments, respectively, such that*

- (i) *no two segments have an interior point in common,*
- (ii) *two horizontal segments are connected by a vertical segment if and only if the corresponding vertices are adjacent.*



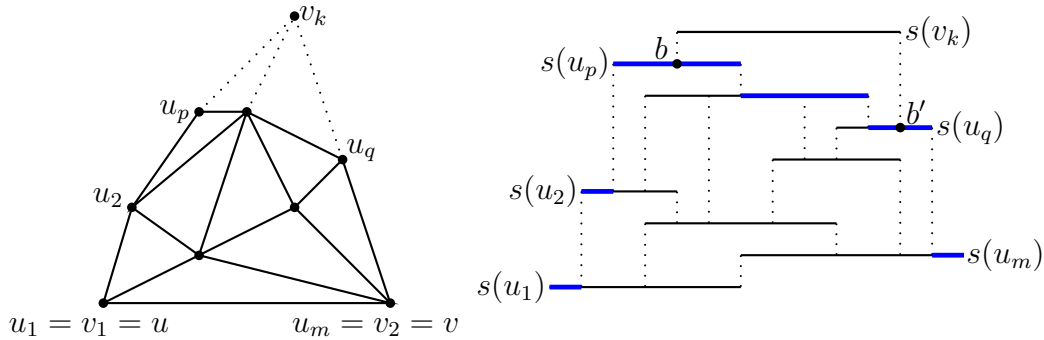


Figure 1.4: Illustration to Corollary 1.10. The thick segments are the pieces of the upper envelope of  $s(v_1), s(v_2), \dots, s(v_{k-1})$ .

*Proof.* As in the proof of Corollary 1.8, it is sufficient to establish the statement for triangulations.

Let  $G$  be any triangulation with canonical labeling  $v_1 = u, v_2 = v, v_3, \dots, v_n$ . To every  $v_k$  we shall assign a horizontal segment  $s(v_k)$  whose endpoints are  $(x_k, k)$  and  $(x'_k, k)$ . Set  $x_1 = 0, x'_1 = 2, x_2 = 2, x'_2 = 4, x_3 = 1, x'_3 = 3$ . Assume that  $s(v_1), s(v_2), \dots, s(v_{k-1})$  have already been determined for some  $k \geq 4$  so that the segments corresponding to adjacent vertex pairs can be connected by vertical segments, that is, the subgraph  $G_{k-1} \subseteq G$  induced by  $\{v_1, v_2, \dots, v_{k-1}\}$  has a representation satisfying conditions (i) and (ii). Let  $u_1 = u, u_2, \dots, u_m = v$  denote the vertices of the exterior face of  $G_{k-1}$ , listed in cyclic order. Suppose further that the upper envelope of the segments  $s(v_1), s(v_2), \dots, s(v_{k-1})$  consists of some portion of  $s(u_1), s(u_2), \dots, s(u_m)$ , in this order. (A point  $b \in s(v_i)$  belongs to the **upper envelope** of  $s(v_1), s(v_2), \dots, s(v_{k-1})$ , if the vertical ray starting from  $b$  and pointing upwards does not intersect any other segment  $s(v_j), 1 \leq j \leq k-1$ ).

By condition (ii) of Theorem 1.7,  $v_k$  is connected to  $u_p, u_{p+1}, \dots, u_q$  for some  $1 \leq p \leq q \leq m$ . Let  $b$  and  $b'$  be any interior points of those portions of  $s(u_p)$  and  $s(u_q)$ , respectively, that belong to the upper envelope of  $s(v_1), s(v_2), \dots, s(v_{k-1})$ . Letting  $x_k$  and  $x'_k$  be equal to the  $x$ -coordinates of  $b$  and  $b'$ , respectively, and drawing the vertical segments connecting  $s(v_k)$  with  $s(v_p), \dots, s(v_q)$ , we obtain a representation of  $G_k$  with the required properties (see Figure 1.4).  $\square$

### 1.3 Drawing a planar graph on a grid

In the previous section, we have seen that any planar graph has a straight-line embedding (Corollary 1.8). However, the solution has a serious drawback: as we embed the vertices recursively in the plane, we may be forced to map a new vertex far away from all previous points, so that the size of the picture may increase exponentially with the number of vertices. To put it differently, if we want to view the resulting drawing on a computer screen, then many points will bunch together and become indistinguishable. To handle this problem, in this section we shall restrict our attention to straight-line drawings where each point is mapped into a **grid point**, that is, a point with integer coordinates. Our goal is to minimize the size of the grid needed for the embedding of any planar graph of  $n$  vertices. The set of all grid points  $(x, y)$  with  $0 \leq x \leq m$ ,  $0 \leq y \leq n$  is said to be an  **$m$  by  $n$  grid**.

**Theorem 1.11.** *Any planar graph with  $n$  vertices has a straight-line embedding in the  $2n - 4$  by  $n - 2$  grid.*

*Proof.* It suffices to prove the theorem for triangulations. Let  $G$  be a triangulation with exterior face  $uvw$ , and let  $v_1 = u, v_2 = v, v_3, \dots, v_n = w$  be a canonical labeling of the vertices (see Theorem 1.7).

We are going to show by induction on  $k$  that  $G_k$ , the subgraph of  $G$  induced by  $\{v_1, v_2, \dots, v_k\}$ , can be straight-line embedded on the  $2k - 4$  by  $k - 2$  grid, for every  $k \geq 3$ . Let  $f_3$  be the following embedding of  $G_3$ :

$$f_3(v_1) = (0, 0), f_3(v_2) = (2, 0), f_3(v_3) = (1, 1).$$

Suppose now that for some  $k \geq 4$  we have already found an embedding  $f_{k-1}(v_i) = (x_{k-1}(v_i), y_{k-1}(v_i))$ ,  $1 \leq i \leq k - 1$ , with the following properties:

- (a)  $f_{k-1}(v_1) = (0, 0)$ ,  $f_{k-1}(v_2) = (2k - 6, 0)$ ;
- (b) If  $u_1 = u, u_2, \dots, u_m = v$  denote the vertices of the exterior face of  $G_{k-1}$  in cyclic order, then

$$x_{k-1}(u_1) < x_{k-1}(u_2) < \dots < x_{k-1}(u_m);$$

- (c) The segments  $f_{k-1}(u_i)f_{k-1}(u_{i+1})$ ,  $1 \leq i < m$ , all have slope  $+1$  or  $-1$ .

Note that (c) implies that the Manhattan (or Iowa) distance  $|x_{k-1}(u_j) - x_{k-1}(u_i)| + |y_{k-1}(u_j) - y_{k-1}(u_i)|$  between the images of any two vertices  $u_i$  and  $u_j$  on the exterior face of  $G_{k-1}$  is even. Consequently, if we take a line with slope  $+1$  through  $u_i$  and a line with slope  $-1$  through  $u_j$ , then they always intersect at a grid point  $P(u_i, u_j)$ .

Let  $u_p, u_{p+1}, \dots, u_q$  be the neighbors of  $v_k$  in  $G_k$  ( $1 \leq p < q \leq m$ ). Clearly,  $P(u_p, u_q)$  is a good candidate for  $f_k(v_k)$ , except that we may not be able to connect it to e.g.  $f_{k-1}(u_p)$  by a segment avoiding  $f_{k-1}(u_{p+1})$ . To resolve this problem, we have to modify  $f_{k-1}$  before embedding  $v_k$ . We shall move the images of  $u_{p+1}, u_{p+2}, \dots, u_m$  one unit to the right, and then move the images of  $u_q, u_{q+1}, \dots, u_m$  to the right by an additional unit. That is, let

$$x_k(u_i) = \begin{cases} x_{k-1}(u_i), & \text{for } 1 \leq i \leq p, \\ x_{k-1}(u_i) + 1, & \text{for } p < i < q, \\ x_{k-1}(u_i) + 2, & \text{for } q \leq i \leq m, \end{cases}$$

$$\begin{aligned} y_k(u_i) &= y_{k-1}(u_i), & \text{for } 1 \leq i \leq m, \\ f_k(u_i) &= (x_k(u_i), y_k(u_i)), & \text{for } 1 \leq i \leq m, \end{aligned}$$

and let  $f_k(v_k)$  be the point of intersection of the lines of slope  $+1$  and  $-1$  through  $f_k(u_p)$  and  $f_k(u_q)$ , respectively. Of course,  $f_k(v_k)$  is a grid point that can be connected by disjoint segments to the points  $f_k(u_i)$ ,  $p \leq i \leq q$ , without intersecting the polygon  $f_k(u_1)f_k(u_2)\dots f_k(u_m)$ . However, as we move the image of some  $u_i$ , it may be necessary to move some other points (not on the exterior face) as well, otherwise we may create crossing edges.

In order to tell exactly which set of points has to move together with the image of a given exterior vertex  $u_i$ , we define recursively a total order ' $\prec$ ' on  $\{v_1, v_2, \dots, v_n\}$ . Originally, let  $v_1 \prec v_3 \prec v_2$ . If the order has already been defined on  $\{v_1, v_2, \dots, v_{k-1}\}$ , then insert  $v_k$  just before  $u_{p+1}$ . According to this rule, obviously

$$u_1 \prec u_2 \prec \dots \prec u_m.$$

Now we can extend the definition of  $f_k$  to the interior vertices of  $G_{k-1}$ , as follows. For any  $1 \leq i \leq k-1$ , let

$$x_k(v_i) = \begin{cases} x_{k-1}(v_i), & \text{if } v_i \prec u_{p+1}, \\ x_{k-1}(v_i) + 1, & \text{if } u_{p+1} \preceq v_i \prec u_q, \\ x_{k-1}(v_i) + 2, & \text{if } u_q \preceq v_i, \end{cases}$$

$$\begin{aligned} y_k(v_i) &= y_{k-1}(v_i), \\ f_k(v_i) &= (x_k(v_i), y_k(v_i)). \end{aligned}$$

Evidently,  $f_k$  satisfies conditions (a), (b) and (c).

To complete the proof, it remains to verify that  $f_k$  is a straight-line embedding, that is, no two segments cross each other. A slightly stronger statement follows by straightforward induction.

**Claim 1.12.** *Let  $f_{k-1} = (x_{k-1}, y_{k-1})$  be the straight-line embedding of  $G_{k-1}$ , defined above, and let  $\alpha_1, \alpha_2, \dots, \alpha_m \geq 0$ . For any  $1 \leq i \leq k-1$ ,  $1 \leq j \leq m$ , let*

$$\begin{aligned} x(v_i) &= x_{k-1}(v_i) + \alpha_1 + \alpha_2 + \dots + \alpha_j \text{ if } u_j \preceq v_i \prec u_{j+1}, \\ y(v_i) &= y_{k-1}(v_i). \end{aligned}$$

*Then  $f'_{k-1} = (x, y)$  is also a straight-line embedding of  $G_{k-1}$ .*

The claim is trivial for  $k = 4$ . Assume that it has already been confirmed for some  $k \geq 4$ , and we want to prove the same statement for  $G_k$ . The vertices of the exterior face of  $G_k$  are  $u_1, \dots, u_p, v_k, u_q, \dots, u_m$ . Fix now any nonnegative numbers  $\alpha(u_1), \dots, \alpha(u_p), \alpha(v_k), \alpha(u_q), \dots, \alpha(u_m)$ . Applying the induction hypothesis to  $G_{k-1}$  with  $\alpha_1 = \alpha(u_1), \dots, \alpha_p = \alpha(u_p), \alpha_{p+1} = \alpha(v_k), \alpha_{p+2} = \dots = \alpha_{q-1} = 0, \alpha_q = \alpha(u_q), \alpha_{q+1} = \alpha(u_{q+1}), \dots, \alpha_m = \alpha(u_m)$ , we obtain that the restriction of  $f'_k$  to  $G_{k-1}$  is a straight-line embedding. To see that the edges of  $G_k$  incident to  $v_k$  do not create any crossing, it is enough to notice that  $f_k$  and  $f'_k$  map  $\{v_k, u_{p+1}, \dots, u_{q-1}\}$  to two translations of the same set.  $\square$

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