A selection of questions from discrete geometry and graph drawing

1 Coloring simplicial 4-polytopes

Question 1.1 ([CKK+17]). *Is there a constant c such that for every simplicial* 4*-polytope* P *there is a coloring of its vertex set* $V(P)$ *with* c *colors such that no* 2*-face of* P *is monochromatic?*

Remark 1.1. *The simplex and the cyclic polytopes need only* 3 *colors. We have no example of a polytope that woud require more than* 3 *colors.*

Remark 1.2 ([HPPT14]). *There exist triangulations of* $\mathbb{R}^3$ *where arbitrarily many colors are needed.*

Remark 1.3. *The question seems to be open also for the special case of Delaunay triangulations of* $\mathbb{R}^3$*.*

References

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2 Empty triangles in monotone drawings

A *monotone* drawing of a graph is a drawing where every vertex has a different x-coordinate and every edge is represented by an x-monotone curve. A drawing is *simple* (or *good*) if every two edges intersect in at most one point, either an endpoint or a proper crossing. A drawing is *semisimple* if adjacent edges do not cross, but independent edges may cross in an arbitrary finite number of points. In particular, every triangle is drawn as a simple closed curve. An *interior-empty triangle* is a triangle with no vertex in the interior of the corresponding curve.

Question 2.1. *What is the minimum number of interior-empty triangles in a simple monotone drawing of a complete graph with n vertices?*

Question 2.2. *What is the minimum number of interior-empty triangles in a semisimple monotone drawing of a complete graph with n vertices?*

For both problems, the best known lower bound is $\Omega(n)$ and the best known upper bound $O(n^2)$. For straight-line or pseudolinear drawings, which are a special case of monotone simple drawings, the lower bound is $\Omega(n^2)$. On the other hand, there are non-monotone simple drawings with only $O(n)$ empty triangles. For monotone drawings, we consider only interior-empty triangles, since all exterior-empty triangles have to contain the leftmost and the rightmost vertex of the drawing, so there are at most $n-2$ of them.

3 Number of simple drawings and good abstract rotation systems

A drawing of a graph is *simple* if every pair of edges has at most one common point; either an endpoint or a crossing. Two simple drawings of a complete graph are *weakly isomorphic* if they have the same set of pairs of edges that cross; or equivalently, if they have the same rotation system.

Question 3.1. *Is it true that the number of weak isomorphism classes of simple drawings of K_n is at most $2^{O(n^2)}$?*

An *abstract rotation system* $\mathcal{R}$ on a set $V = \{v_1, \ldots, v_n\}$ is an n-tuple of cyclic $(n-1)$-permutations $\pi_{v_1}, \ldots, \pi_{v_n}$ where the set of elements of π_{v_i} is $V \setminus \{v_i\}$. A *subsystem of* $\mathcal{R}$ *induced* by a subset $W = \{w_1, \ldots, w_k\} \subset V$ is a $|W|$-tuple of cyclic permutations $\rho_{w_1}, \ldots, \rho_{w_k}$ where ρ_{w_i} is a restriction of π_{w_i} to the subset $W \setminus \{w_i\}$.

An abstract rotation system is *realizable* if it is a rotation system of a simple complete topological graph. An abstract rotation system $\mathcal{R}$ is *good* if every subsystem of $\mathcal{R}$ induced by a 4-element subset is realizable.

Question 3.2. *Is it true that the number of good abstract rotation systems on an n-element set is bounded by $2^{O(n^2)}$?*

The current best upper bound for both problems is $2^{n^2 \cdot \alpha(n)^{O(1)}}$ [K13] and the best lower bound is $2^{\Omega(n^2)}$ [PT04].

One possible approach to Problem 3.1 would be to use induction.

Question 3.3. *Let $\mathcal{C}$ be a weak isomorphism class of simple drawings of K_n. Is it true that there are at most $2^{O(n)}$ weak isomorphism classes of simple drawings of K_{n+1} that extend some drawing from $\mathcal{C}$? How many possible rotations can the $(n+1)$st vertex have?*

References

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[PT04] J. Pach and G. Tóth, How many ways can one draw a graph?, *Combinatorica* **26**(5) (2006), 559–576.

4 Cells in arrangements of pseudolines

A *pseudoline* in the plane is an image of a Euclidean line under a homeomorphism of the plane; in other words, a pseudoline is a homeomorphic image of the set $\mathbb{R}$, unbounded in both directions. An *arrangement of pseudolines* is a finite set of pseudolines such that every pair of them has exactly one crossing, and no other common intersection point. It is well known that every arrangement of pseudolines can be transformed by a homeomorphism to an arrangement of x-monotone pseudolines. It is also well known that every arrangement of n pseudolines in the plane has at least $n-2$ triangular cells. See the survey article by Felsner and Goodman [FG17] for more background.

Question 4.1. *Assume that $\mathcal{A}$ is an arrangement of pseudolines where each pseudoline is colored red or blue, and each color is used at least once. Is there always a triangular cell in $\mathcal{A}$ bounded by at least one red and at least one blue pseudoline?*

It is an easy exercise that the answer is positive in case $\mathcal{A}$ is an arrangement of lines.

References

[FG17] S. Felsner and J. E. Goodman, Pseudoline arrangements, *Handbook of Discrete and Computational Geometry*, Third edition, Edited by Jacob E. Goodman, Joseph O'Rourke and Csaba D. Tóth, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2018. ISBN: 978-1-4987-1139. Electronic version: `http://www.csun.edu/~ctoth/Handbook/HDCG3.html`.

5 Monochromatic configurations in 2-colored plane

Let $T_{a,b,c} = \{A, B, C\}$ be a triple of points on a line or in the plane with distances $|AB| = c, |BC| = a, |AC| = b$. We say that two triples T', T'' are *congruent* if T'' can be obtained from T' by a sequence of translations, rotations and reflections.

Question 5.1. *Assume that the plane is colored with black and white color. Is there always a monochromatic triple congruent to $T_{1,1,2}$?*

In general, the question of the existence of monochromatic triples is open for most triples $T_{a,b,c}$, and the answer is known only in several special cases. See the survey article by Graham [G17] for more background (and more open problems).

References

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6 Colorful islands

We say that a finite set in $\mathbb{R}^d$ is in *general position* if each of its subsets of size at most $d+1$ is affinely independent. A partition of a finite set X into m parts is called an *m-coloring* of X, the parts are called *color classes*, and we also say that X is *m-colored*. We allow the color classes to be empty. A subset $Y \subseteq X$ is called *j-colorful* if Y contains points from at least j distinct color classes. Let X be a subset of $\mathbb{R}^d$, and $Y \subseteq X$. The convex hull of Y, denoted by $\operatorname{conv} Y$, is called an *island* (*spanned by* X) if $X \cap \operatorname{conv} Y = Y$. Equivalently, we say that the set X *spans* Y. If $\operatorname{conv} Y$ is an island spanned by X and $|Y| = k$, then we also say that $\operatorname{conv} Y$ is a *k-island*. If $Y \subseteq X$, we say that the island $\operatorname{conv} Y$ is *j-colorful* if Y is j-colorful. Notice that when X is in general position and $k \leq d+1$, then a k-island spanned by X is a $(k-1)$-dimensional simplex with vertices in X.

The following question generalizes an earlier conjecture by Kano and Suzuki [KK17, Conjecture 3], generalizing a classical partition theorem by Akyiama and Alon [AA89].

Question 6.1 ([HKV17])**.** *Let k, m, d be integers satisfying $k, m \geq d \geq 2$. Let X be an m-colored set of kn points in general position in $\mathbb{R}^d$. Suppose that X admits a partition into n pairwise disjoint d-colorful k-tuples. Is it true that then X spans n pairwise disjoint d-colorful k-islands?*

The result of Akyiama and Alon [AA89] is a special case of Question 6.1 for $k = m = d$. Kano and Suzuki [KK17, Conjecture 3] conjectured the case $k = d$ with arbitrary $m \geq d$. Question 6.1 has been solved in other special cases by several groups of researchers [ACF+10, BKS00, HKV17, IUY00, KK17, S02]. Very recently, Blagojević, Rote, Steinmeyer and Ziegler [BRSZ17] proved Conjecture 6.1 for $m = d$ and all $k \geq d$.

References

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7 Variants of the Hanani–Tutte theorem on surfaces

A drawing of a graph G on a surface S is an *embedding* if no two edges cross. We say that two edges in a graph are *independent* if they do not share a vertex. A drawing of a graph is *independently even* if every pair of independent edges in the drawing crosses an even number of times.

The (strong) Hanani–Tutte theorem states that a graph is planar if it has an independently even drawing in the plane.

The following question is often expressed as "do adjacent crossings matter?".

Question 7.1 ([FK19]). *Let S be a surface other than the plane or the projective plane. Assume that a graph G has a drawing on S where only adjacent edges are allowed to cross. Does G have an embedding on S?*

Question 7.1 can also be formulated in terms of the *independent crossing number* of G on S, which may be denoted by $\mathrm{cr}_{-S}(G)$ using the notation in Schaefer's survey on crossing numbers [Sch17]. For a given surface S, Question 7.1 then asks whether $\mathrm{cr}_{-S}(G) = 0$ implies $\mathrm{cr}_S(G) = 0$.

The strong Hanani–Tutte theorem and its possible generalizations can be weakened in several other ways. For example, instead of an independently even drawing of G we may consider the edges of G oriented and require a drawing of G where for every pair of independent edges e and f, the number of crossings in which e crosses f from the left is equal to the number of crossings in which e crosses f from the right. This can be formulated in terms of the *independent algebraic crossing number* of G on S, denoted by $\mathrm{iacr}_S(G)$ [Sch17], which is well-defined on orientable surfaces. We denote the orientable surface of genus g by M_g.

Question 7.2 ([FK19])**.** *Let $g \geq 1$. Does $\mathrm{iacr}_{M_g}(G) = 0$ imply $\mathrm{cr}_{M_g}(G) = 0$?*

The following question was suggested to us by Jeff Erickson.

Question 7.3 ([FK19])**.** *Let S be a surface other than the plane or the projective plane. Assume that a graph G has an independently even drawing on S where the union of every pair of adjacent edges can be covered by a topological disc. Does G have an embedding on S?*

References

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