Monotone crossing number of complete graphs

Martin Balko, Radoslav Fulek and Jan Kynčl

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The story

In the beginning...

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Theorem: (Erdős–Szekeres)

- For every $k > 1$, there is a smallest $f(k)$ such that among every $f(k)$ points in general position in the plane some k points form a convex k-gon.
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- $f(6) = 17$ (Peters and Szekeres, 2006)

Combinatorial convexity

- $P = \{p_1, p_2, \ldots, p_n\}$ in general position, $x(p_1) < x(p_2) < \cdots < x(p_n)$

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- **signature function** $\sigma : T_n \subset [n]^3 \to \{-, +\}$

  $\sigma(i, j, k) =$'$+$' $\Leftrightarrow$ triangle $p_i p_j p_k$ is counter-clockwise

  $\Leftrightarrow$ point p_j is below segment $p_i p_k$

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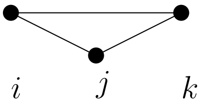

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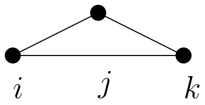

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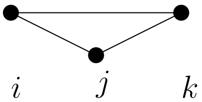

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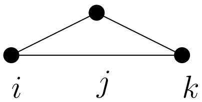
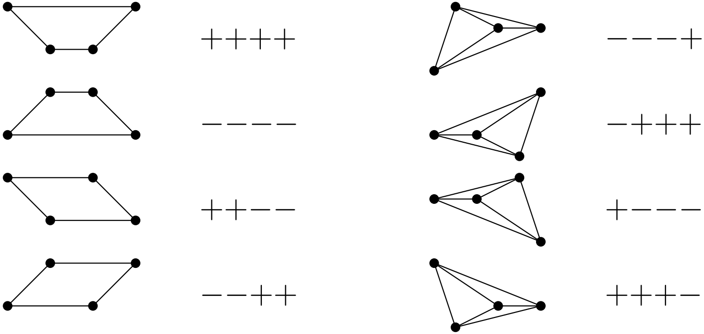


$\sigma(i, j, k) = -$

type of 4-tuple (i, j, k, l):

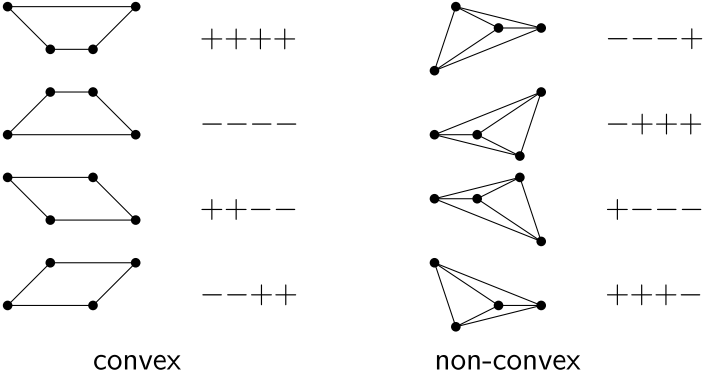
$$\sigma(i, j, k)\sigma(i, j, l)\sigma(i, k, l)\sigma(j, k, l)$$

Combinatorial convexity



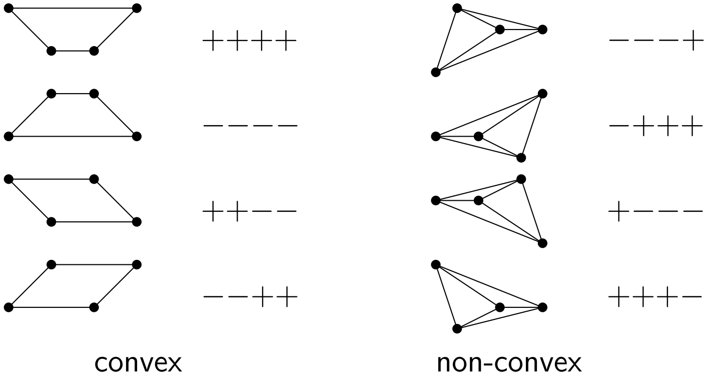
convex non-convex

Combinatorial convexity



generalized 4-cup: $\sigma(i,j,k) = \sigma(j,k,l) = \text{'+'}$

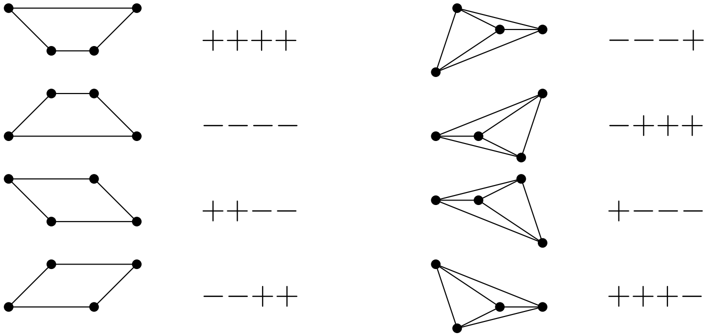
Combinatorial convexity



generalized 4-cup: $\sigma(i,j,k) = \sigma(j,k,l) = \text{'}+\text{'}$

generalized 4-cap: $\sigma(i,j,k) = \sigma(j,k,l) = \text{'}-\text{'}$

Combinatorial convexity



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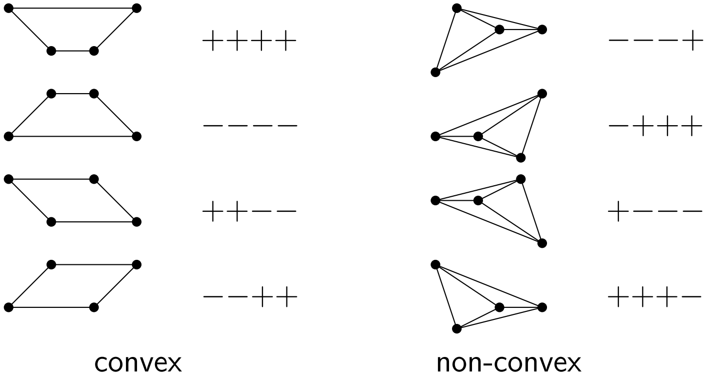
Conjecture: (Peters and Szekeres) For $n > 2^{k-2}$, any signature function on T_n induces a generalized convex k-gon.

Combinatorial convexity



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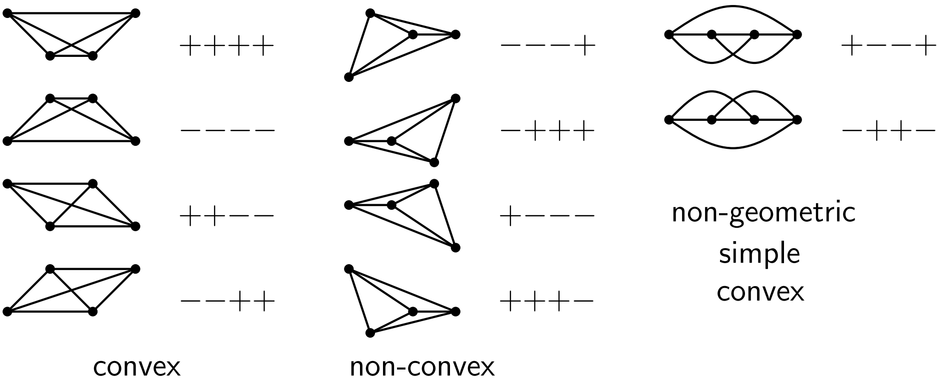
Conjecture: (Peters and Szekeres) For $n > 2^{k-2}$, any signature function on T_n induces a generalized convex k-gon.

- proved for $k = 5$ and $n = 9$

Question: What is the minimum number of generalized convex k-tuples? In particular, 4-tuples?

More types of 4-tuples



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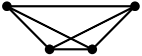

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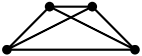

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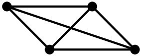

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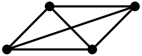
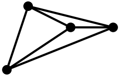
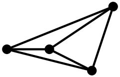


$--++$

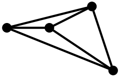
convex



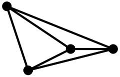
$---+$



$-+++$

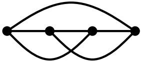


$+---$

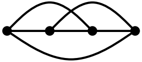


$+++-$

non-convex

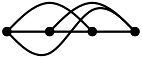


$+--+$



$-++-$

non-geometric
simple
convex

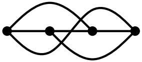


$+-++$

$\mathrm{cr} = 3, \mathrm{iocr} = 2$

“double-convex”?

ugly



$+-+-$

$\mathrm{cr} = 3, \mathrm{iocr} = 1$

“convex”

Monotone drawings of complete graphs

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Monotone drawing: edges are x-monotone curves

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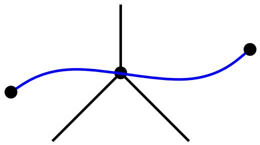
Monotone drawing: edges are x-monotone curves

forbidden:



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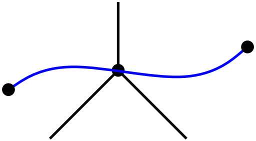

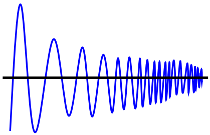

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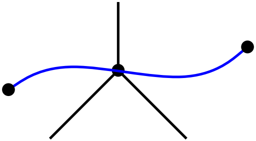

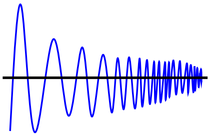

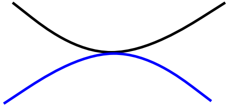

Monotone drawings of complete graphs

Monotone drawing: edges are *x*-monotone curves

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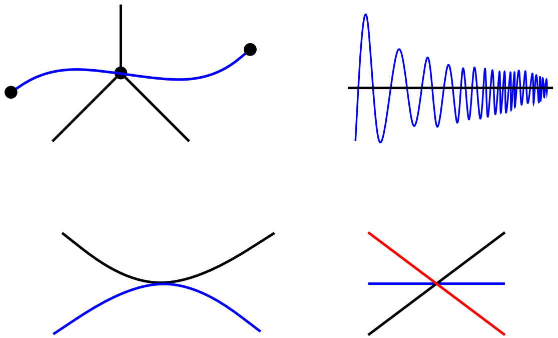
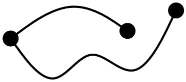
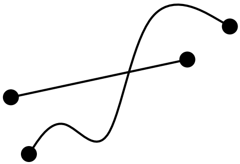


simple: any two edges have at most one common point



or



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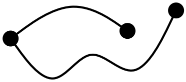

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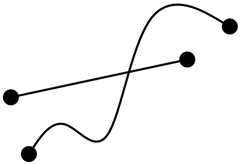
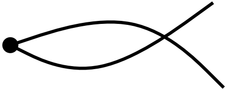


semisimple: adjacent edges do not cross

NO



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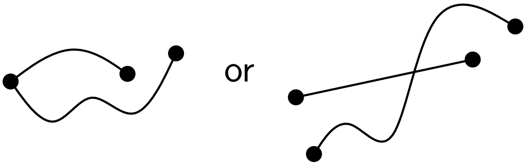

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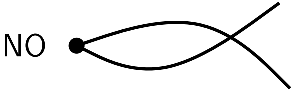
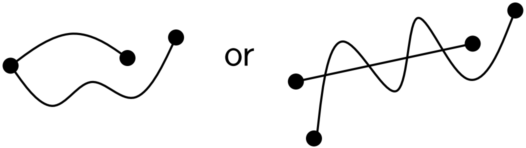
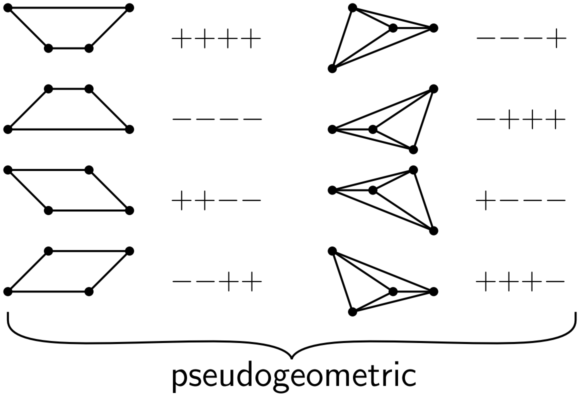




Hierarchy of signature functions:



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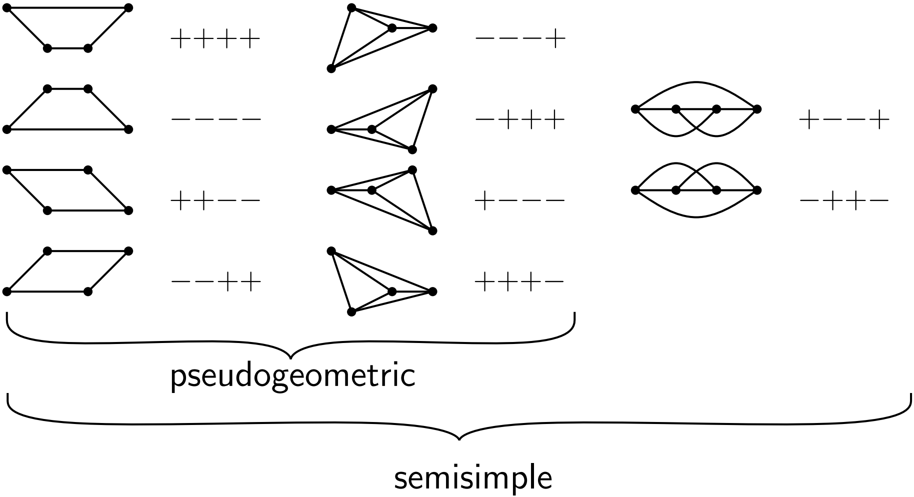

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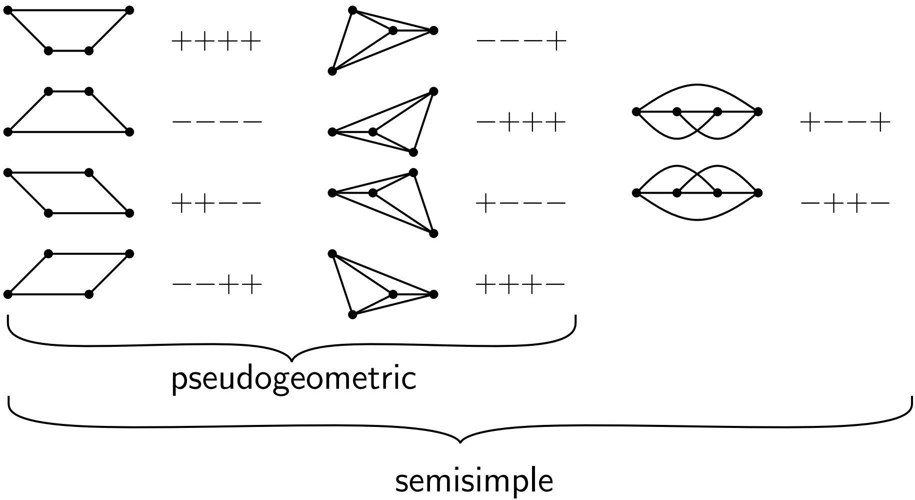

simple = semisimple + NO 

Crossing numbers

cr(G) = crossing number of G = minimum number of crossings in a drawing of G

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monotone semisimple odd crossing number of G = minimum number of <u>pairs</u> of edges with <u>odd</u> number of crossings in a monotone <u>semisimple</u> drawing of G

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mon-iocr(G) **= mon-ocr**$_{-}(G)$ **=**
monotone independent odd crossing number of G = minimum number of <u>pairs</u> of <u>independent</u> edges with <u>odd</u> number of crossings in a monotone drawing of G

Crossing numbers

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$$\text{cr}(G) \leq \text{mon-cr}(G) \leq \overline{\text{cr}}(G)$$

$$\text{mon-iocr}(G) \leq \text{mon-ocr}_+(G) \leq \text{mon-cr}(G)$$

Crossing numbers of complete graphs

n	5	6	7	8	9	10	11	12
$\overline{\text{cr}}(K_n)$	1	3	9	19	36	62	102	153
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Conjecture: (Hill; Guy)

$$\mathrm{cr}(K_n) = Z(n) := \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

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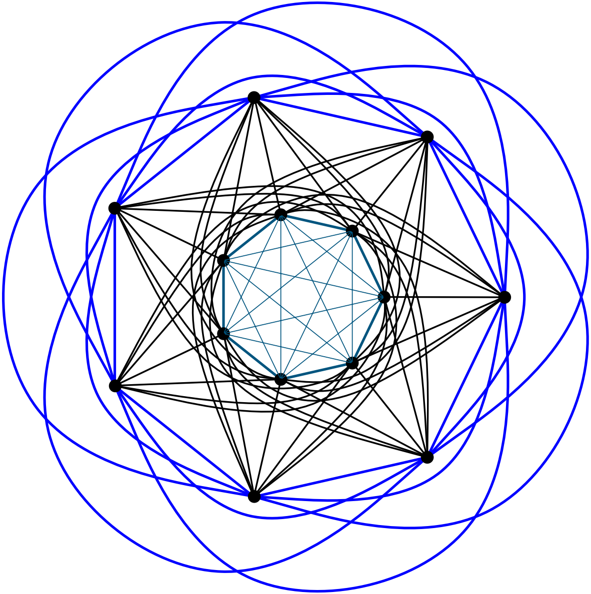
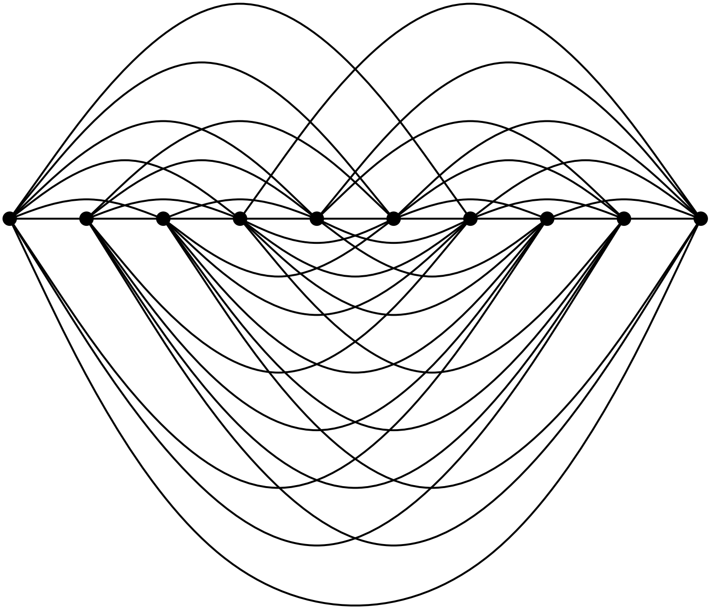
known:

$$\mathrm{cr}(K_n) \leq Z(n)$$

cylindrical drawings:



2-page book drawings:



Meanwhile...

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Theorem: (B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, and G. Salazar, The 2-page crossing number of K_n, arXiv:1206.5669 (2012))

The 2-page book crossing number of K_n is $Z(n)$.

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Main Theorem:

$$\text{mon-cr}(K_n) = Z(n)$$

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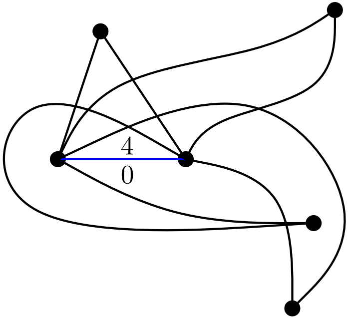
- k**-edges**



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- ***k*-edges**
- **$\leq k$-edges**
- **$\leq\leq k$-edges**

Lemma 1: For every simple drawing D of K_n we have

$$\operatorname{cr}(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k) E_k(D),$$

equivalently,

$$\operatorname{cr}(D) = 2\sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2}(1+(-1)^n) E_{\leq\leq \lfloor n/2 \rfloor - 2}(D).$$

Outline of the proof

(common with Ábrego et al., 2012)

Lemma 2: For every 2-page book drawing D of K_n and $0 \le k < n/2 - 1$, we have

$$E_{\le\le k}(D) \ge 3\binom{k+3}{3}.$$

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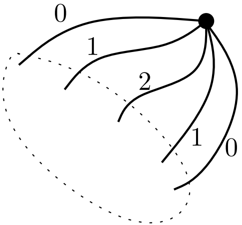

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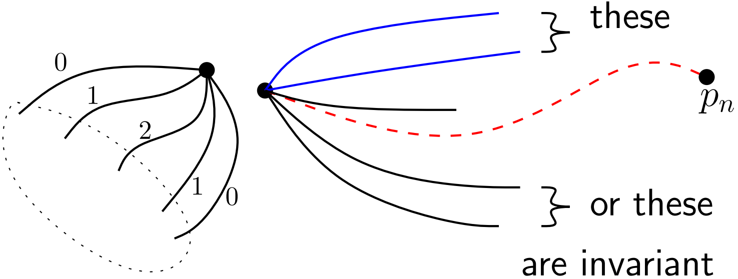

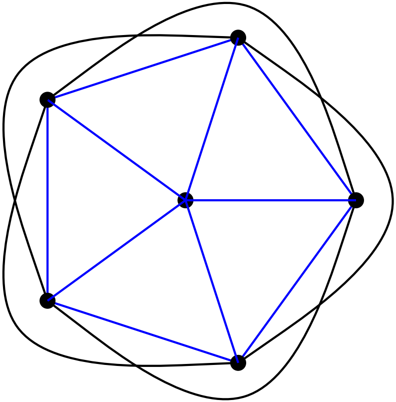

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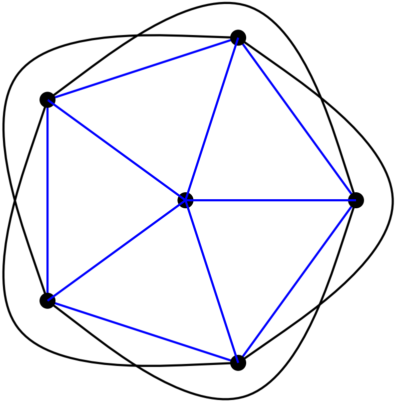

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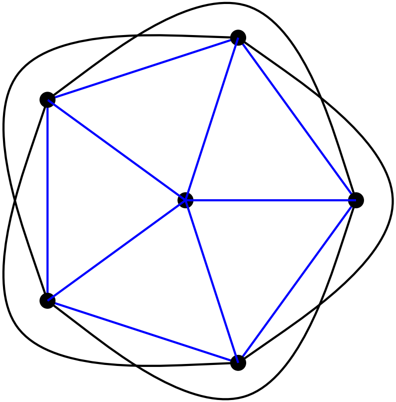

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Open questions

- Is $\text{mon-iocr}(K_n)) \geq Z(n)$?

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- Is $\text{mon-iocr}(K_n)) \geq Z(n)$?
- Let $n \geq 3$ and let D be a simple drawing of K_n. Suppose that $0 \leq k < n/2 - 1$. Is

$$E_{\leq\leq\leq k}(D) \geq 3\binom{k+4}{4}?$$