Simple realizability of complete abstract topological graphs simplified

Jan Kynčl

Charles University, Prague

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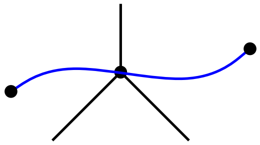

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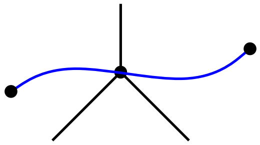

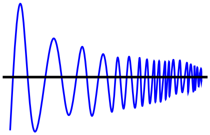

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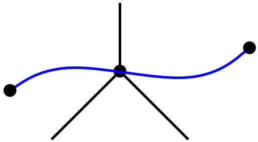

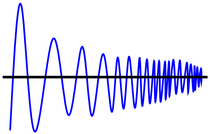

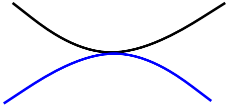

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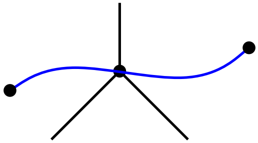
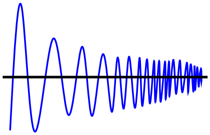
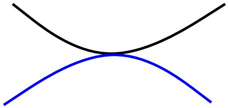
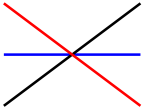
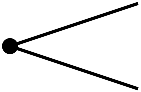








simple: any two edges have at most one common point



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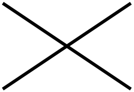

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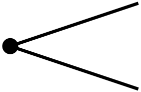

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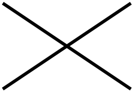

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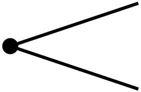

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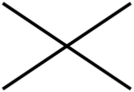

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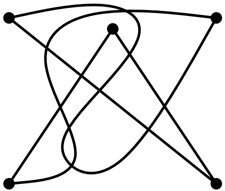
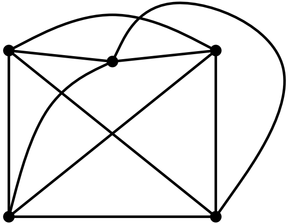


topological graph



simple complete topological graph

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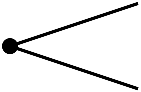

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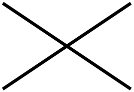

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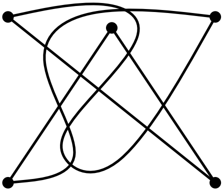

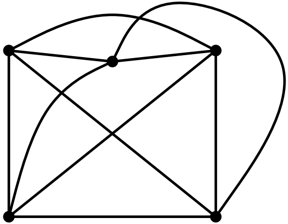

topological graph

drawing

simple complete topological graph

simple drawing of K_5

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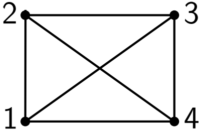

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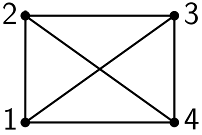

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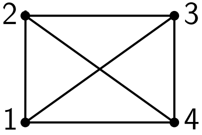

$A = (K_5, \emptyset)$ is not simply realizable

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instance: AT-graph A
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“The proof in [..] only gives a highly complex testing procedure, but no description in terms of forbidden minors or crossing configurations.”

— M. Chimani, 2011

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- Ábrego, Aichholzer, Fernández-Merchant, Hackl, Pammer, Pilz, Ramos, Salazar and Vogtenhuber (2015) generated a list of simple drawings of K_n for $n \leq 9$

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3) computing the minimum crossing numbers of pairs of edges

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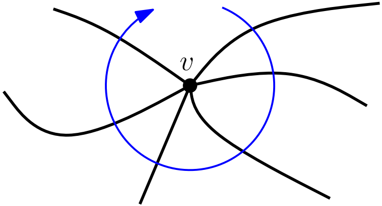

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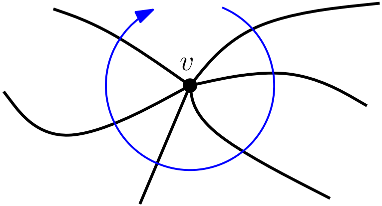

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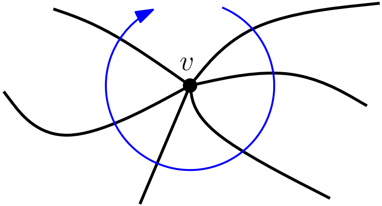

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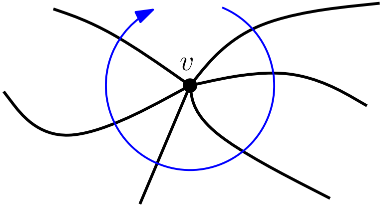

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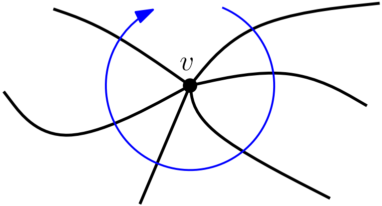

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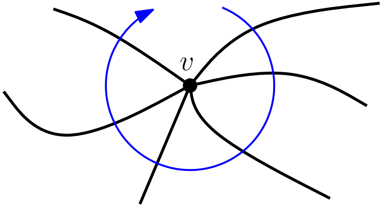

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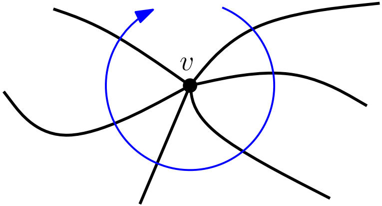

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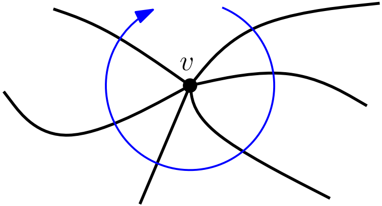

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Ábrego et al. (pers. com.) verified that an abstract rotation system (ARS) of K_9 is realizable if and only if the ARS of every 5-tuple is realizable, and conjectured that this is true for any K_n.

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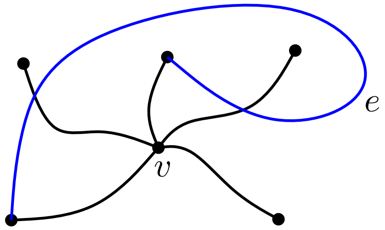

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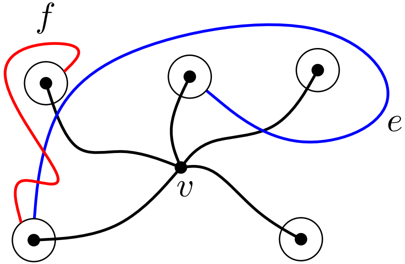
- drill small holes around the vertices, fix the endpoints of the edges on the boundaries of the holes



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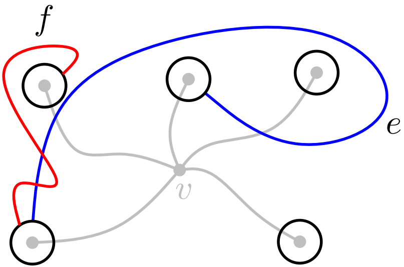

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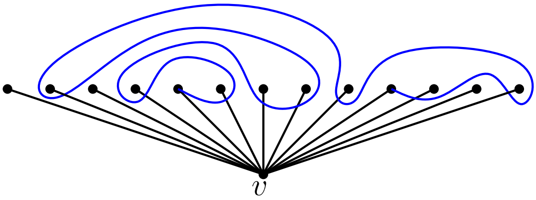
We need to verify that

- $\mathrm{cr}(e) = 0$,
- $\mathrm{cr}(e, f) \leq 1$, and
- $\mathrm{cr}(e, f) = 1 \Leftrightarrow \{e, f\} \in \mathcal{X}$.

3a) characterization of the homotopy classes



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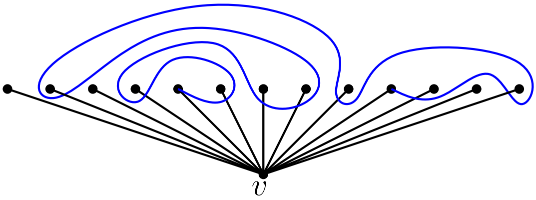

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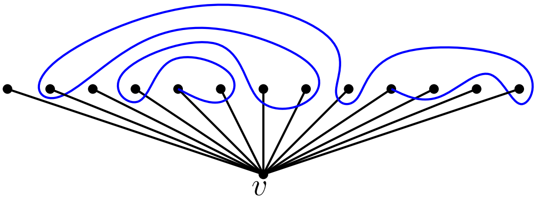
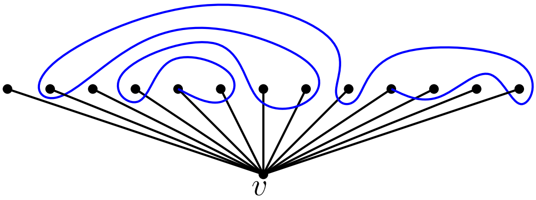


3b) parity of the crossing numbers (4- and 5-tuples)

3c) crossings of adjacent edges (5-tuples)

3a) characterization of the homotopy classes



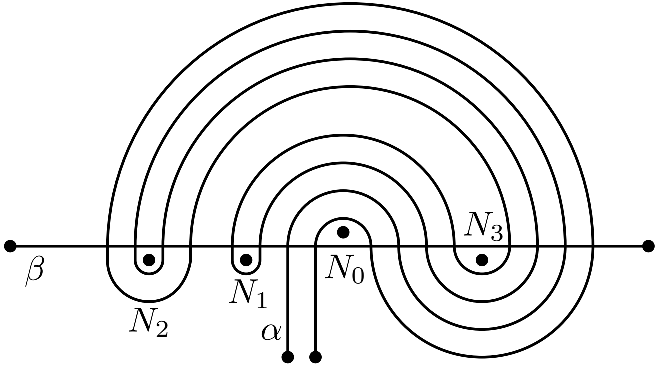
3b) parity of the crossing numbers (4- and 5-tuples)

3c) crossings of adjacent edges (5-tuples)

3d) multiple crossings of independent edges (5-tuples)

Picture hanging without crossings

remove one nail:



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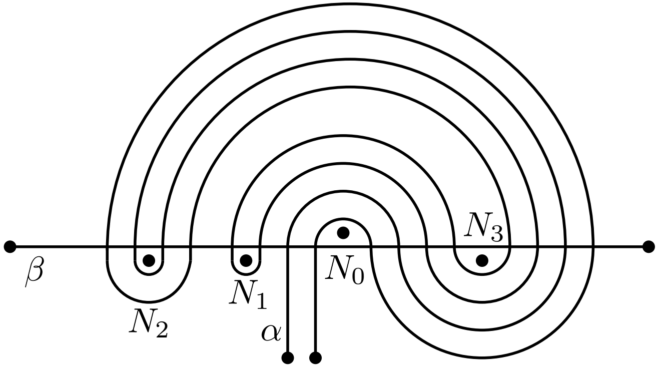

similar concept with crossings:
Demaine et al., Picture-hanging puzzles, 2014.

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Example:

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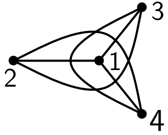
independent $\mathbb{Z}_2$-realization of A:



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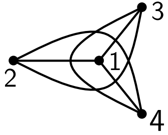

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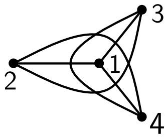

$A = (K_5, \emptyset)$ is not independently $\mathbb{Z}_2$-realizable (Hanani–Tutte)

def.: Call an AT-graph $(G, \mathcal{X})$ even (or an even G) if $|\mathcal{X}|$ is even, and odd (or an odd G) if $|\mathcal{X}|$ is odd.

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Theorem 3: Every complete AT-graph that is not independently $\mathbb{Z}_2$-realizable has an AT-subgraph on at most six vertices that is not independently $\mathbb{Z}_2$-realizable.

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More precisely, a complete AT-graph is independently $\mathbb{Z}_2$-realizable if and only if it contains no even K_5 and no odd $2K_3$ as an AT-subgraph.