Clustered planarity testing revisited

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and

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Clustered planarity

Graph: $G = (V, E)$, V finite, $E \subseteq \binom{V}{2}$

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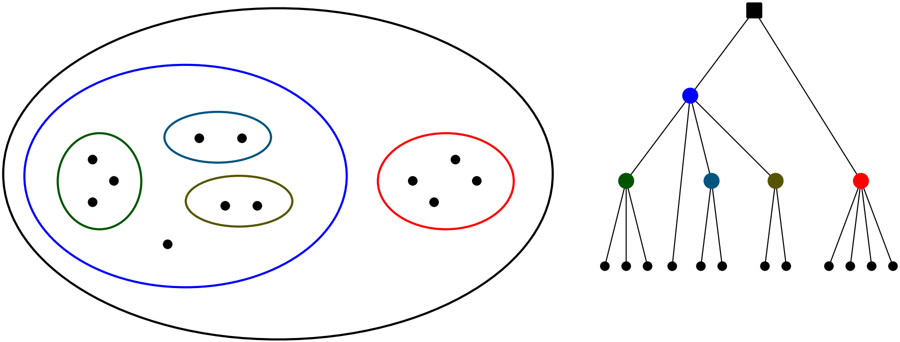
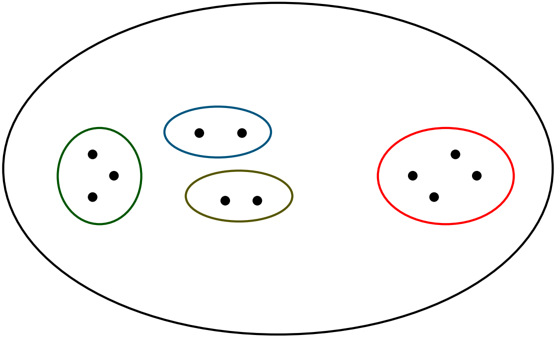
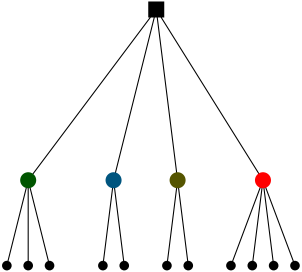
Clustered graph: (G, T) where T is a tree hierarchy of clusters



Clustered planarity

Flat clustered graph: nontrivial clusters form a partition of V





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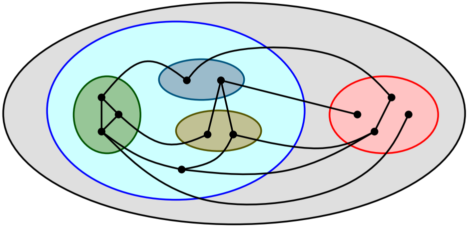

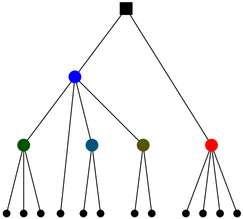

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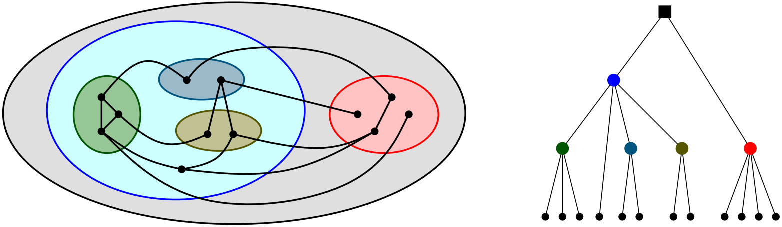

Such a representation is called a **clustered embedding** of (G, T).

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- embedded graphs with at most 2 vertices per face and cluster (Chimani et al., 2014)

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We do NOT aim for optimizing the running time.

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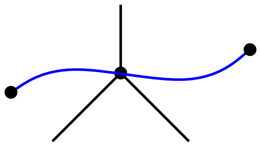

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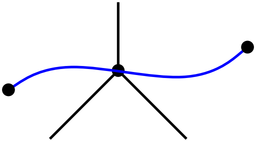

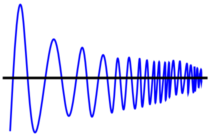

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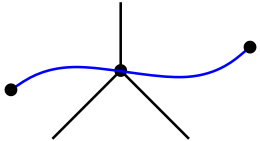

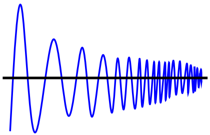

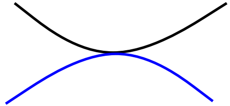

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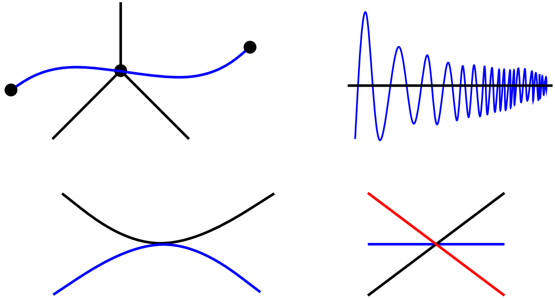

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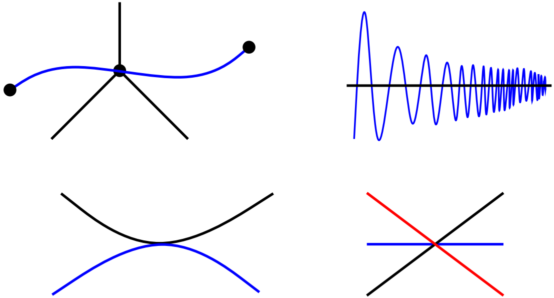

embedding = drawing with no crossings

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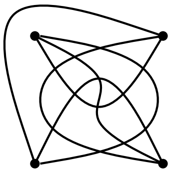

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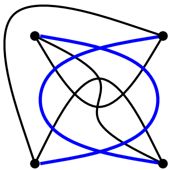

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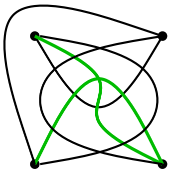

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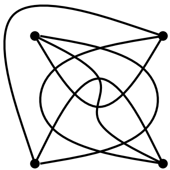

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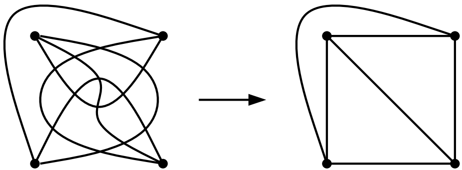

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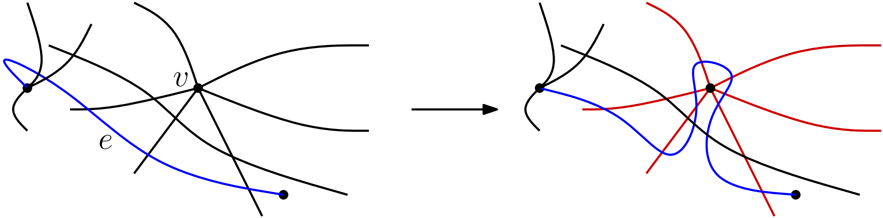

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- solve the linear system!

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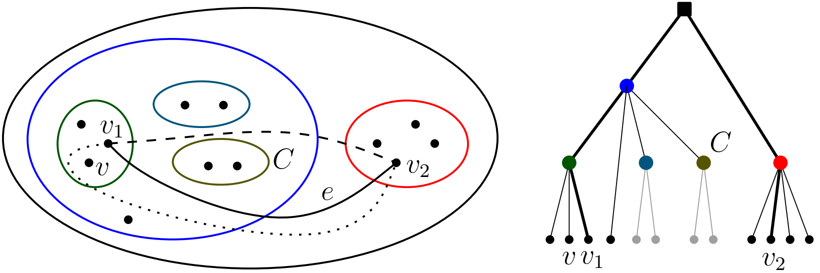

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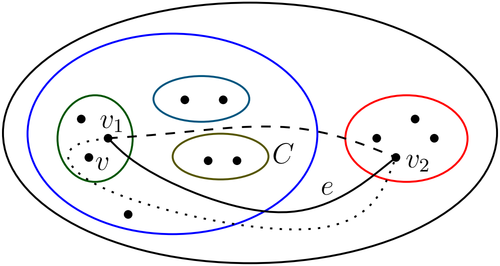

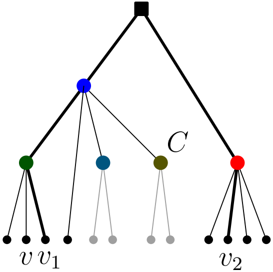

a different algorithm: Gutwenger, Mutzel and Schaefer (2014)

Main result

Theorem: (Hanani–Tutte for two clusters)

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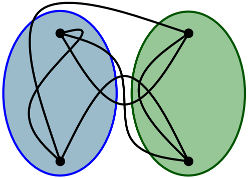

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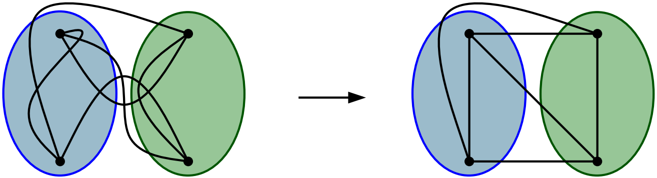

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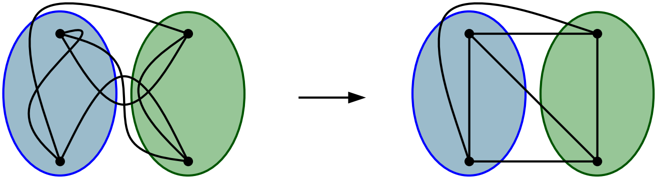

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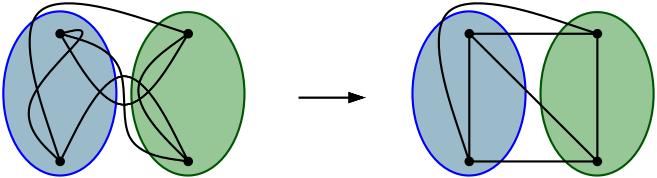

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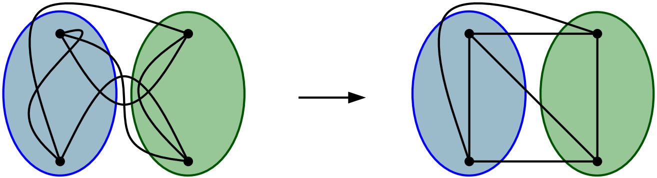

- Hanani–Tutte for c-connected clustered graphs
- weak Hanani–Tutte for two clusters
- generalization: weak Hanani–Tutte for strip planarity (Fulek, 2014)

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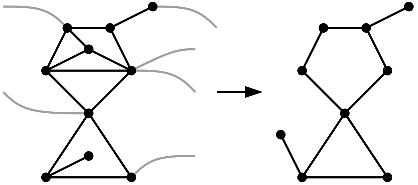

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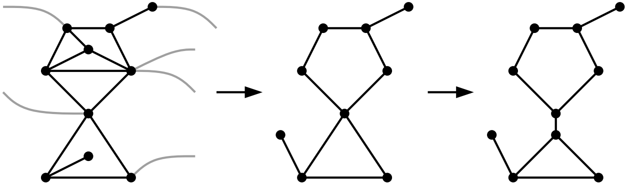

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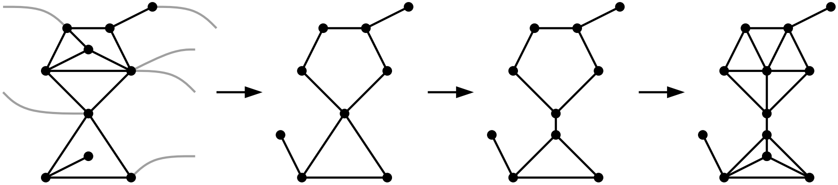

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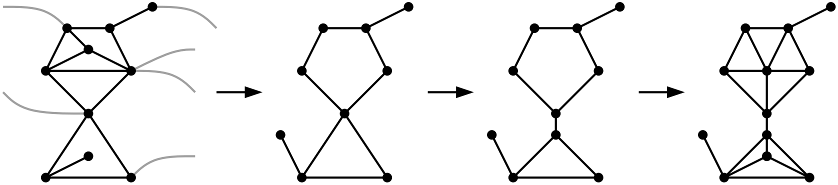

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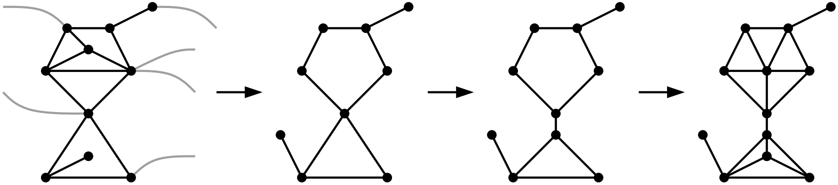

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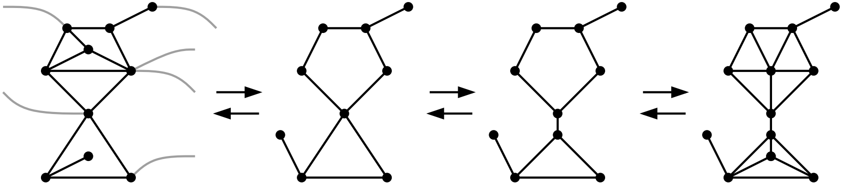
- remove the interiors of the wheels, contract the new edges, and draw the rest of G



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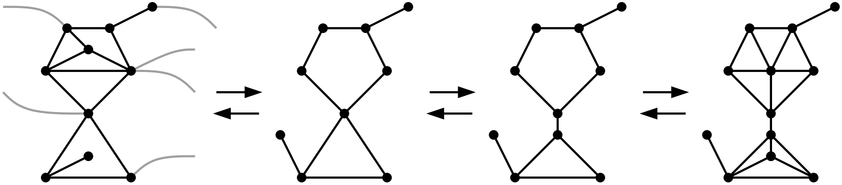
- draw two disjoint discs around A and B



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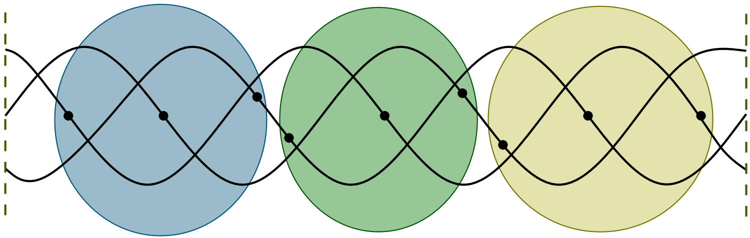

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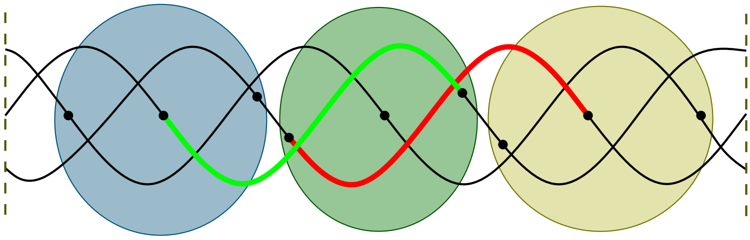

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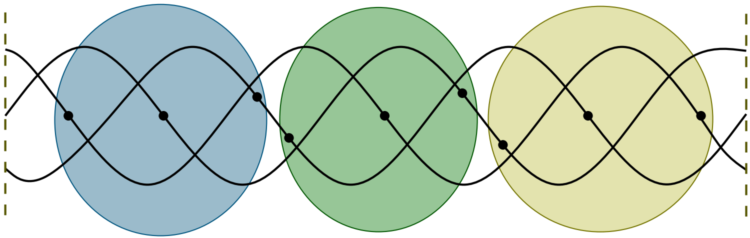

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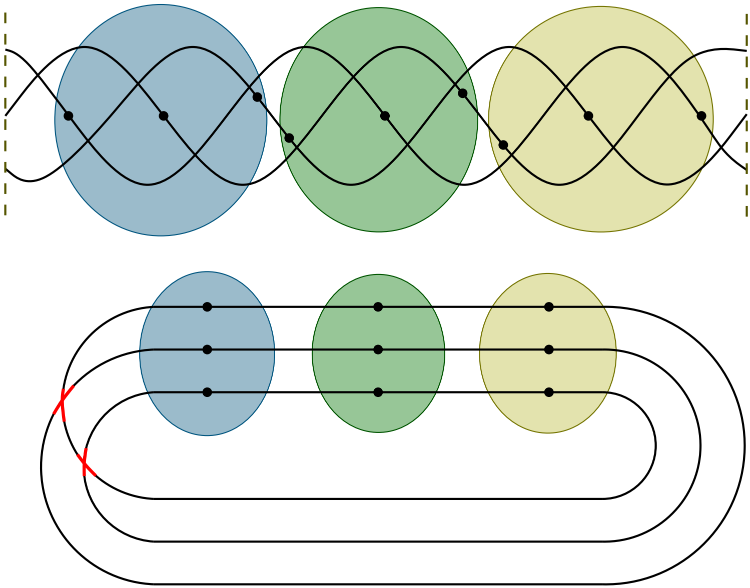

Are there other counterexamples???