Improved enumeration of simple topological graphs

Jan Kynčl

Charles University, Prague

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Topological graph: drawing of an (abstract) graph in the plane

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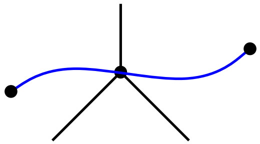

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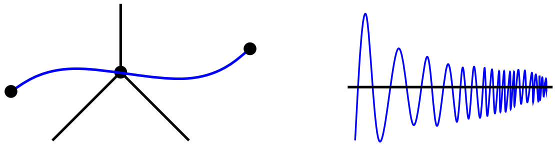

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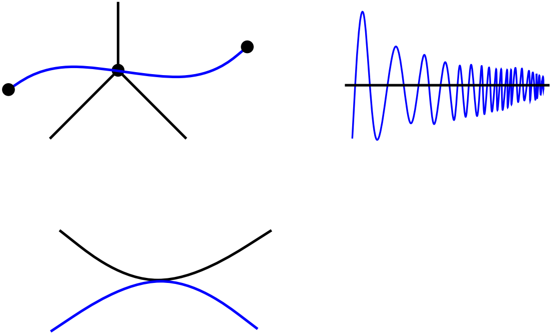

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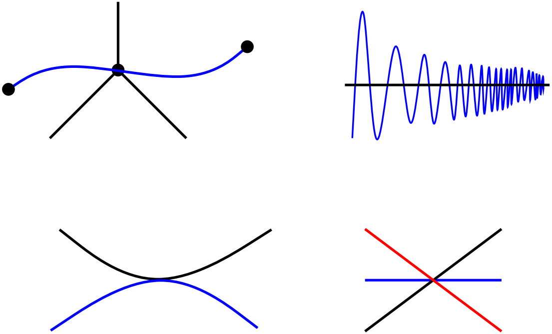
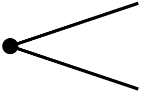


simple: any two edges have at most one common point



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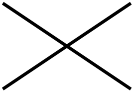

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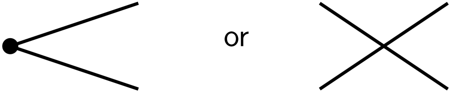

complete: $E = \binom{V}{2}$

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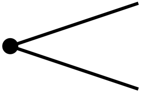

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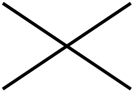

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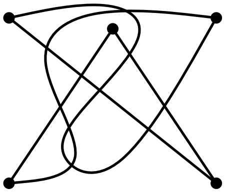

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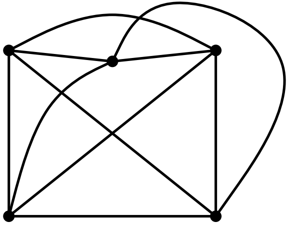

simple complete topological graph

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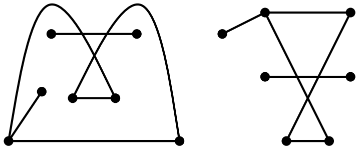

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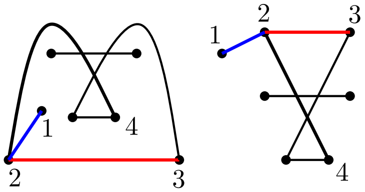

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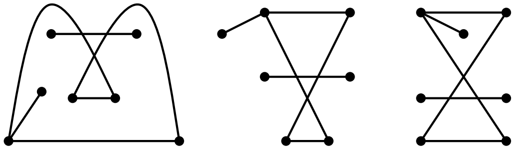

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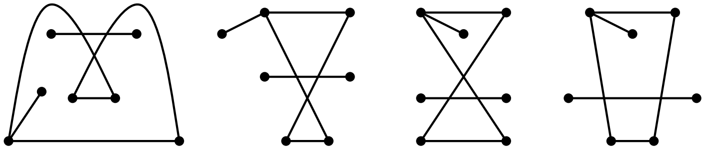

Weak isomorphism classes

$T_{\mathrm{w}}(G)$ = number of weak isomorphism classes of simple topological graphs that realize G

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Complete graphs

Theorem: (J. Pach and G. Tóth, 2006)

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Main Theorem 1:

$$T_{\mathrm{w}}(K_n) \le 2^{n^2 \cdot \alpha(n)^{O(1)}}$$

Weak isomorphism classes, complete graphs

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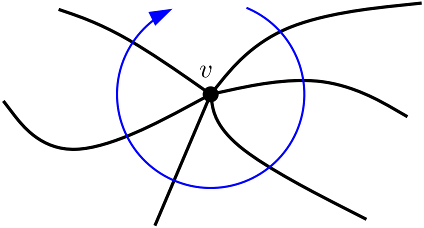

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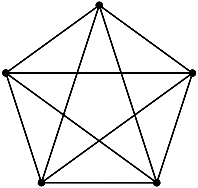
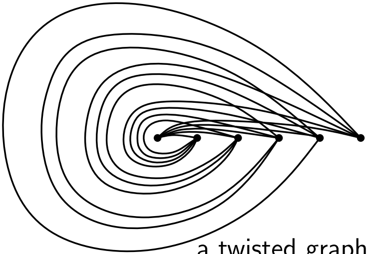
  Every simple complete topological graph with 4^{30^4} vertices contains one of the following subgraphs:



a convex graph C_5



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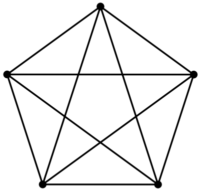

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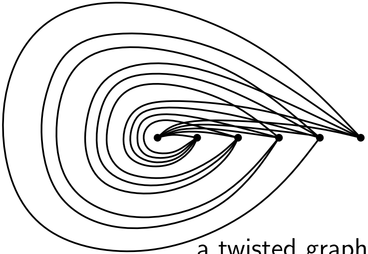

a twisted graph T_6

- an upper bound on the size of a set of permutations with bounded VC-dimension (J. Cibulka and JK, 2012)

General graphs

Main Theorem 2: Let G be a graph with n vertices and m edges. Then

$$T_{\mathrm{w}}(G) \le 2^{O(n^2 \log(m/n))}.$$

If $m < n^{3/2}$, then

$$T_{\mathrm{w}}(G) \le 2^{O(mn^{1/2} \log n)}.$$

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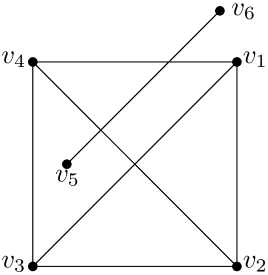
Corollary: There are at most $2^{O(n^{3/2} \log n)}$ intersection graphs of n pseudosegments.

Weak isomorphism classes, general graphs

tools for upper bounds:

- topological spanning tree $\mathcal{T}$



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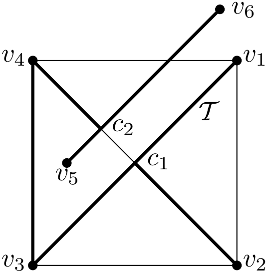

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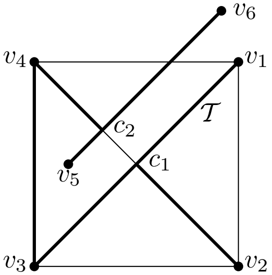

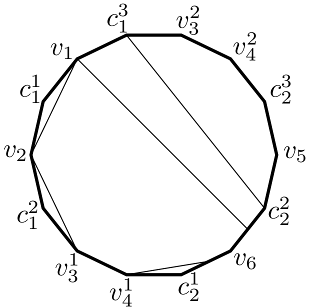

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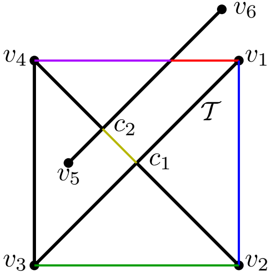

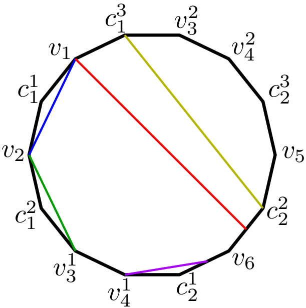

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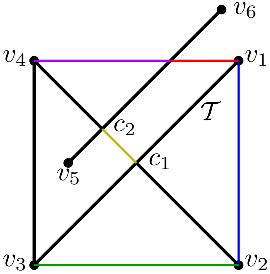

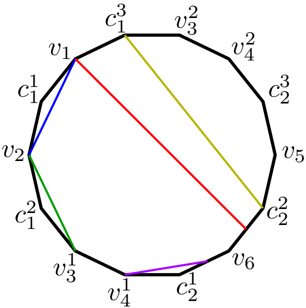

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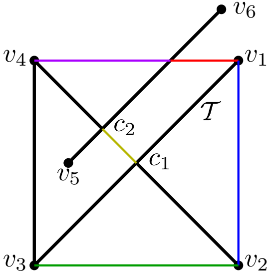

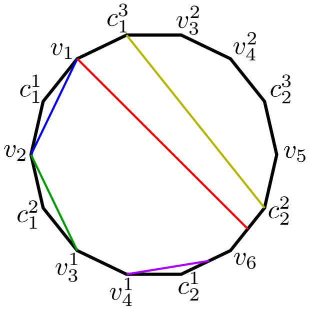

- types of (pseudo)chords $\Rightarrow$ pairs of crossing edges
- bounded number of crossings of the pseudochords

Isomorphism classes

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Complete graphs

Theorem: (JK, 2009)

$$2^{\Omega(n^4)} \leq T(K_n) \leq 2^{(1/12+o(1))(n^4)}$$

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Theorem: Let G be a graph with n vertices, m edges and no isolated vertices. Then

$$T(G) \leq 2^{1 \cdot m^2 + O(mn)}.$$

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Theorem: Let G be a graph with n vertices, m edges and no isolated vertices. Then

$$T(G) \leq 2^{m^2+11.51mn+O(n\log n)} \leq 2^{23.118m^2}+c, \text{ and}$$

$$T(G) \leq 2^{m^2+2mn(\log(1+\frac{m}{4n})+3.443)+O(n\log n)} \leq 2^{11.265m^2}+c.$$

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"very sparse" graphs $\rightarrow$ rooted connected planar loopless maps (T.R.S. Walsh and A. B. Lehman, 1975)

$$T(G) \leq 2^{(\log_2(256/27)+o(1))m^2} \leq 2^{3.246m^2} + c$$

Isomorphism classes, general graphs

tools for upper bounds:

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- simple quadrangulations of a disc
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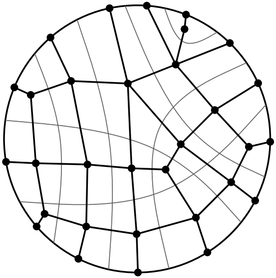

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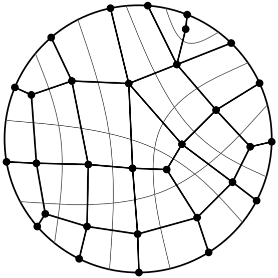

- chord diagrams (R. C. Read, 1979) and arrangements of pseudochords with fixed boundary

Open questions

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- Or, more generally, is $T_w(H) \leq T_w(G)$ for $H \subseteq G$?
- What is the number of weak isomorphism classes of drawings of G where every two edges have at most two common points?