Ramsey-type constructions for arrangements of segments

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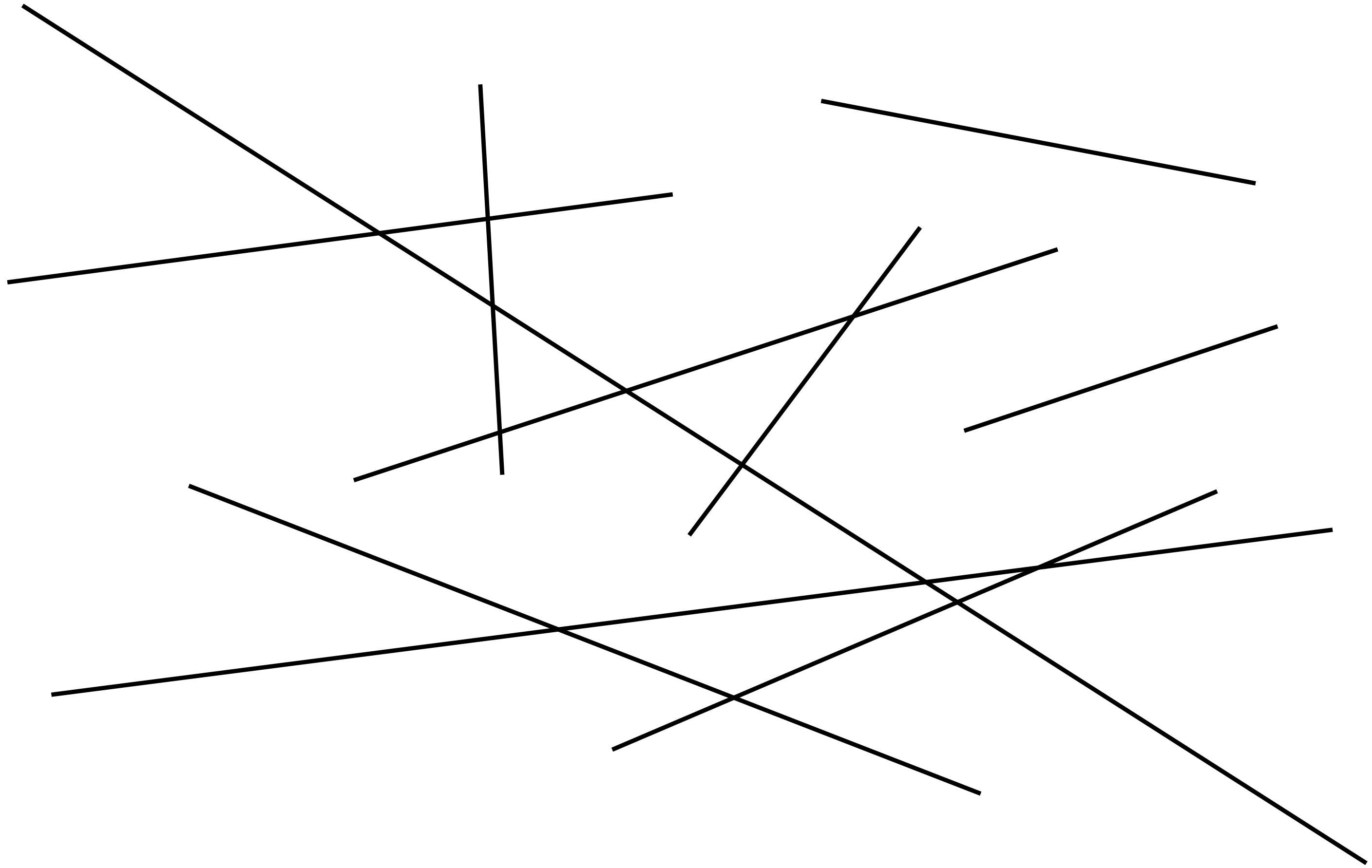
Arrangement of segments:



set of straight-line segments in general position in the plane

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Our improvement:

$r(k) \geq k^{\log 169/\log 8} > k^{2.4669}$

(for infinitely many values of k)

Upper bound k^5

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Dilworth's Theorem: A poset with $m \cdot n$ elements has a chain of size m or an anti-chain of size n

Previous constructions for the lower bound

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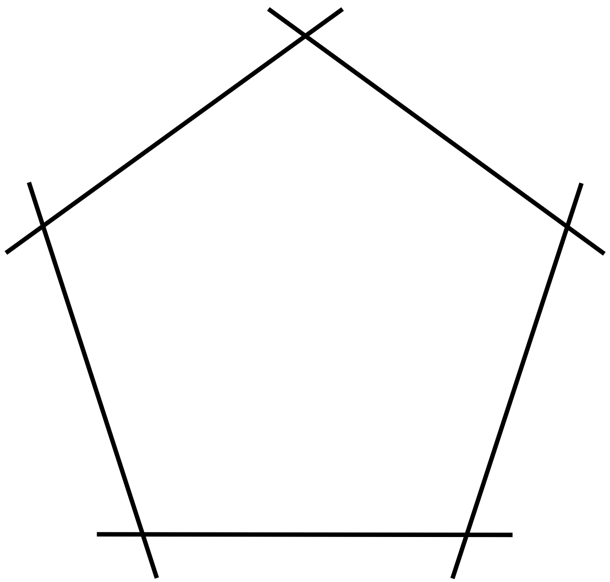
[Larman, Matoušek, Pach, Törőcsik, 1994]



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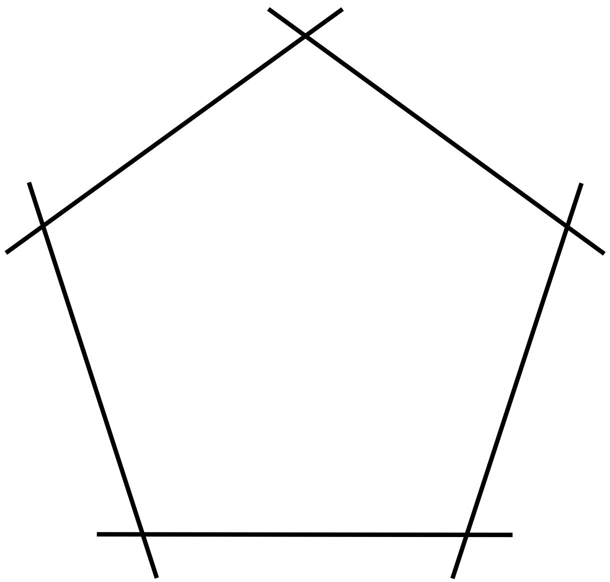

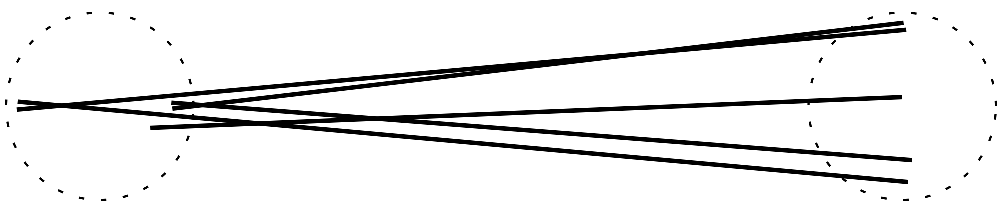

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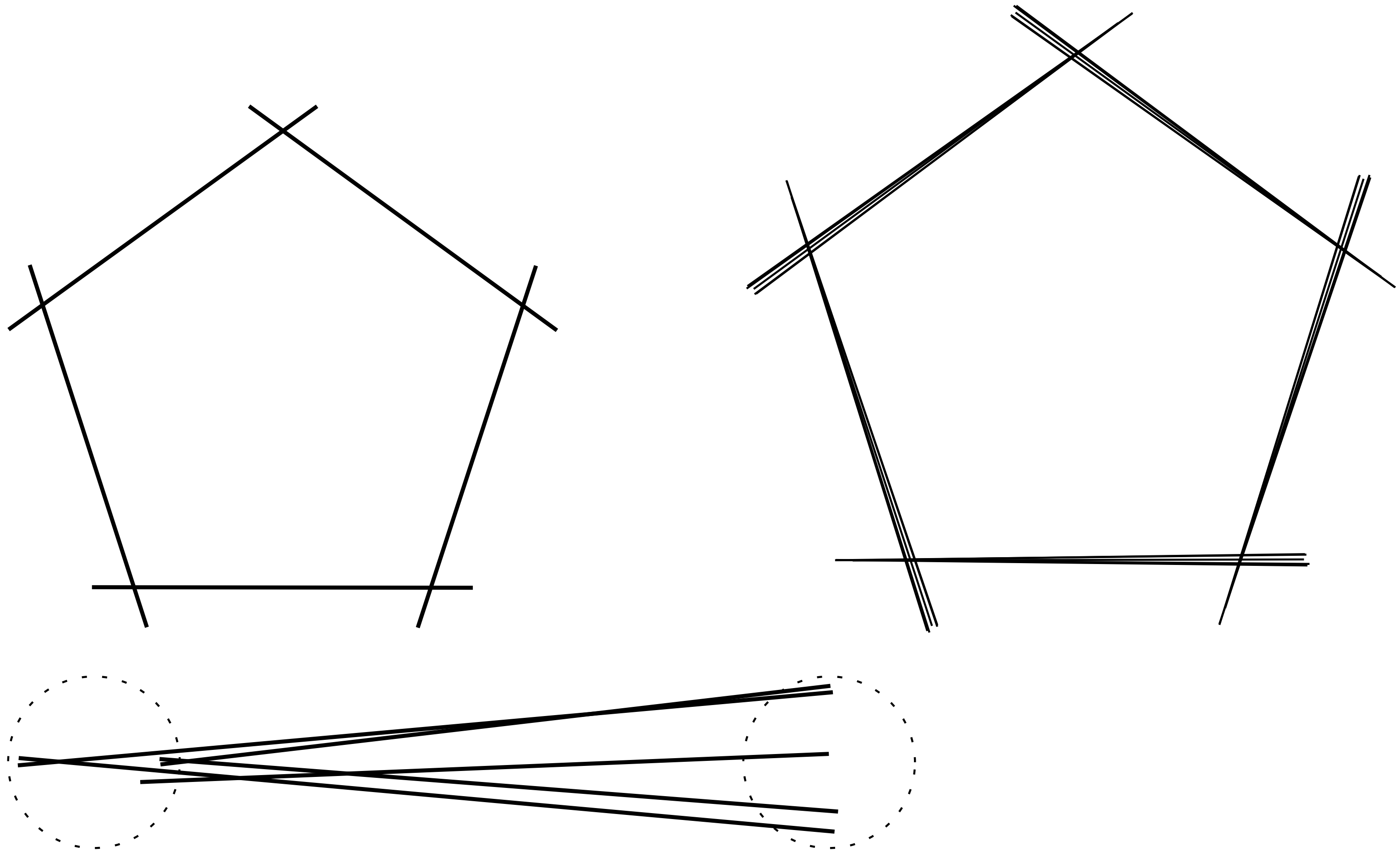
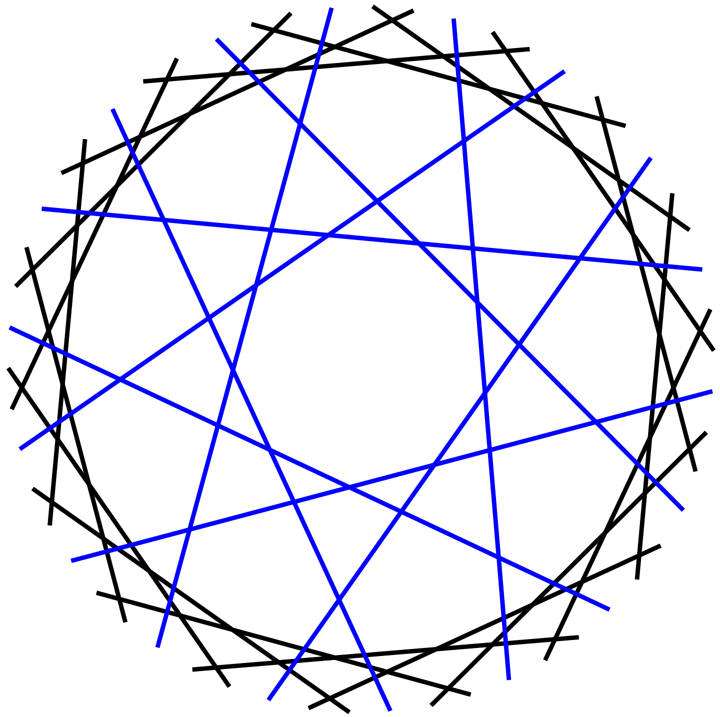
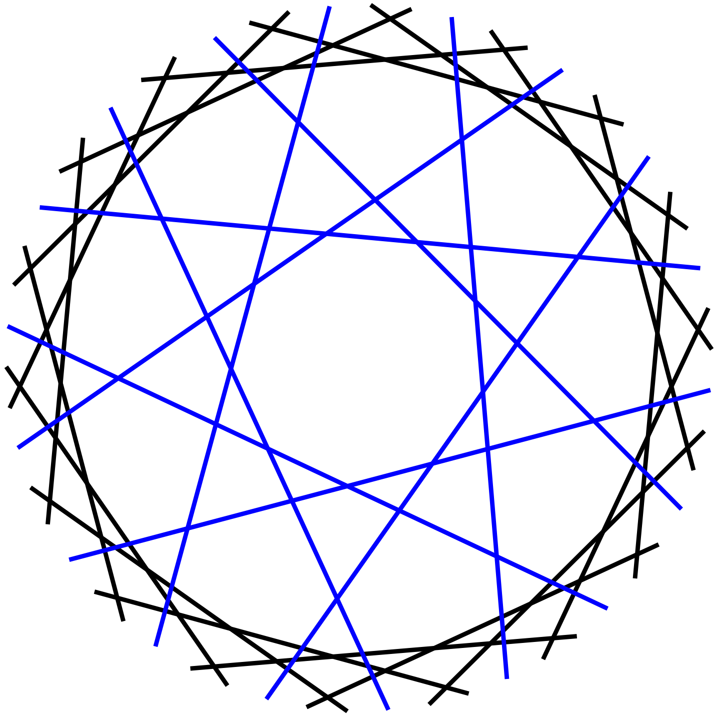


[Károlyi, Pach, Tóth, 1996]



27 segments, at most 4 pairwise crossing or pairwise disjoint

[Károlyi, Pach, Tóth, 1996]



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Lemma: Every **convex** arrangement can be flattened.

Limitations of convex arrangements

Theorem [Kostochka, 1988]

A **circle graph** G with $\alpha(G) \le k$ and $\omega(G) \le k$ has at most $(1 + o(1)) \cdot k^2 \log k$ vertices.

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A **circle graph** G with $\alpha(G) \leq k$ and $\omega(G) \leq k$ has at most $(1 + o(1)) \cdot k^2 \log k$ vertices.

$\Rightarrow$ large convex arrangements can not give better lower bound for $r(k)$.

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- Upper bound for convex case [Černý, 2008]:

convex (2, 4)-arrangement: at most 12 segments
convex (4, 2)-arrangement: at most 11 segments

A (2, 4)-arrangement

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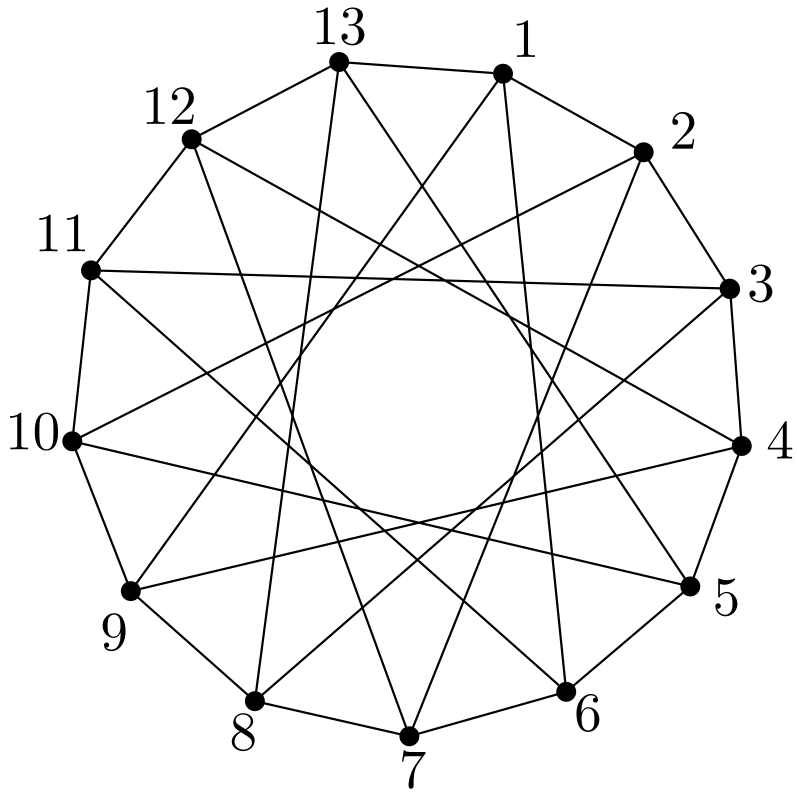
intersection graph: $\mathbf{Cay}(\mathbb{Z}_{13}; 1, 5)$



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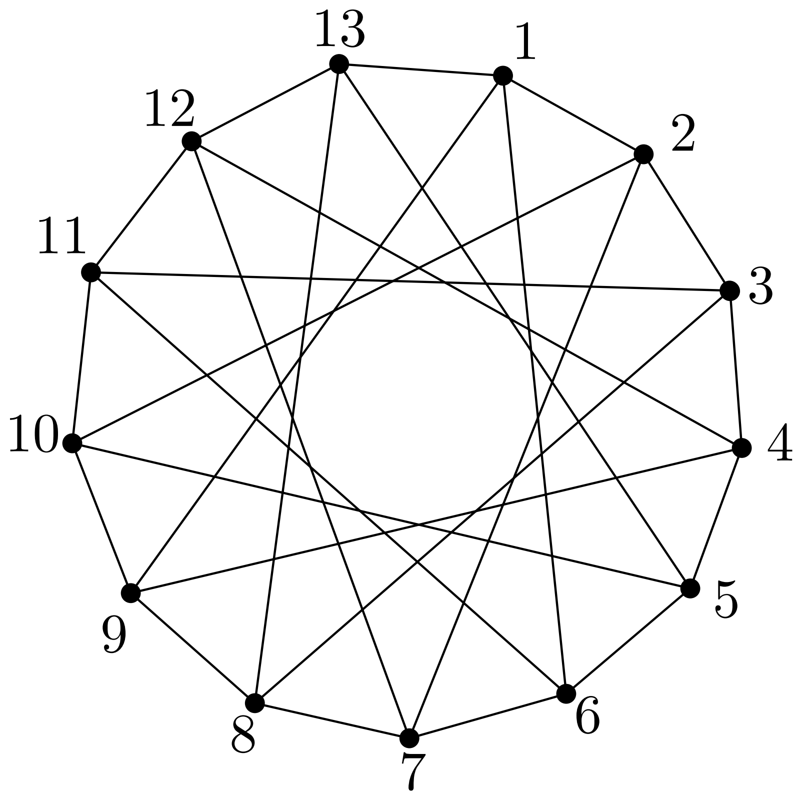

- has no clique of size 3 and no independent set of size 5

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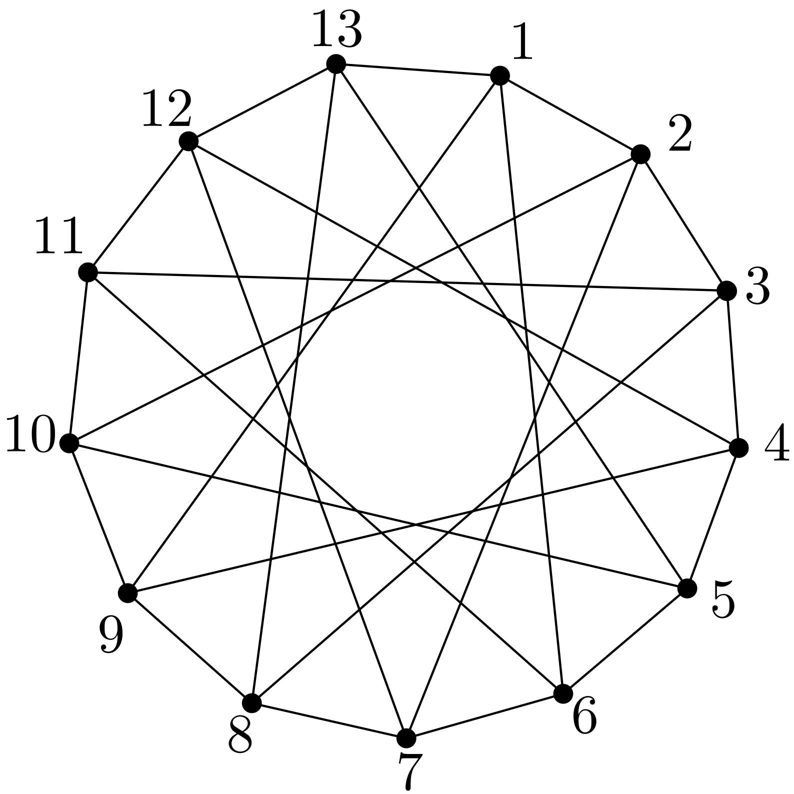

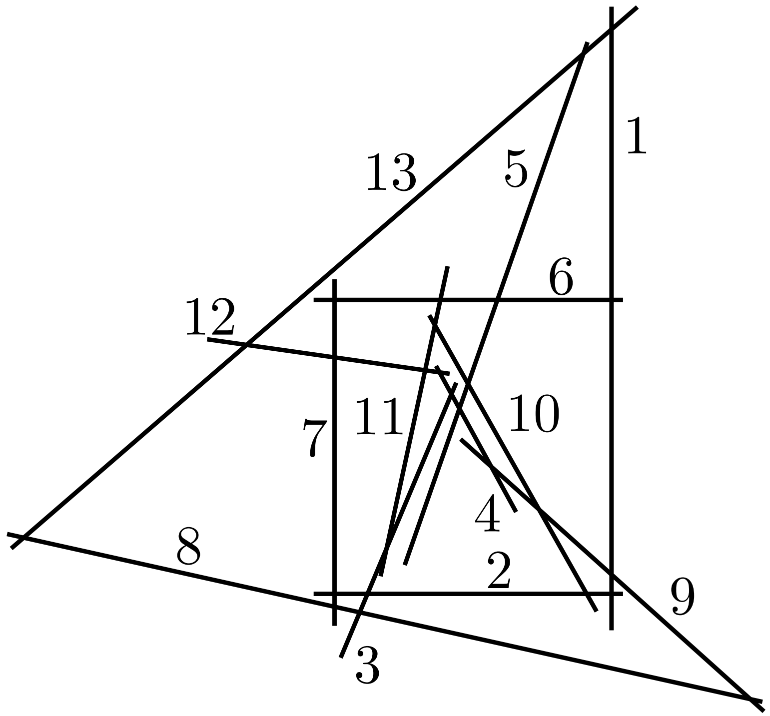

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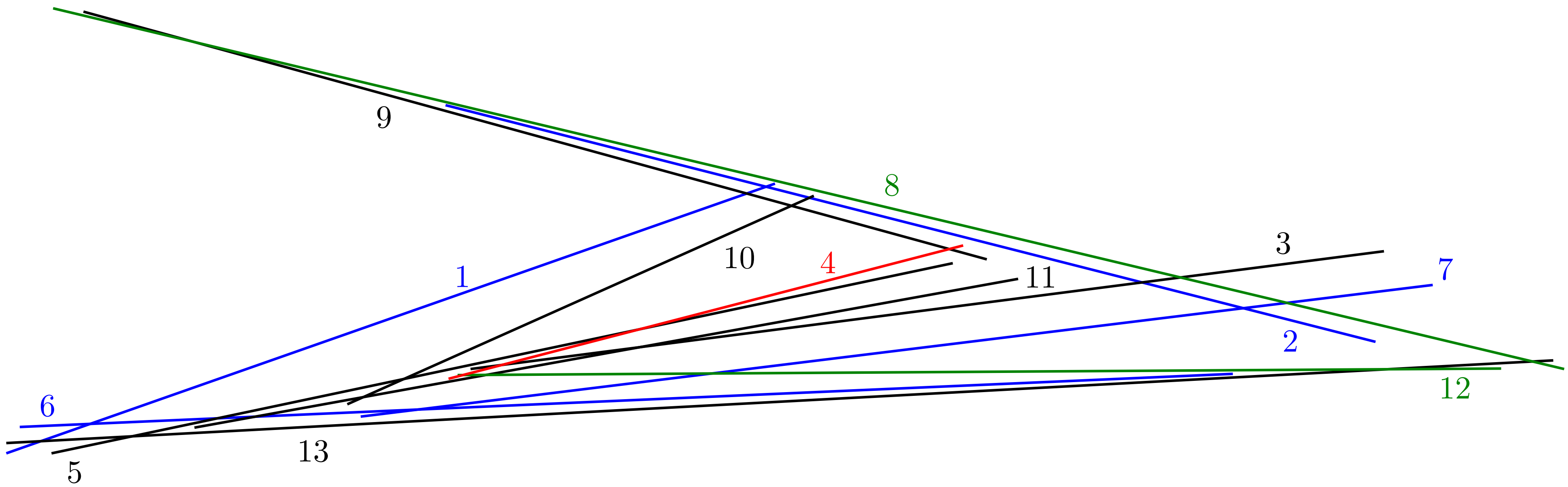
A partially flattened (2, 4)-arrangement



A flattened (2, 4)-arrangement

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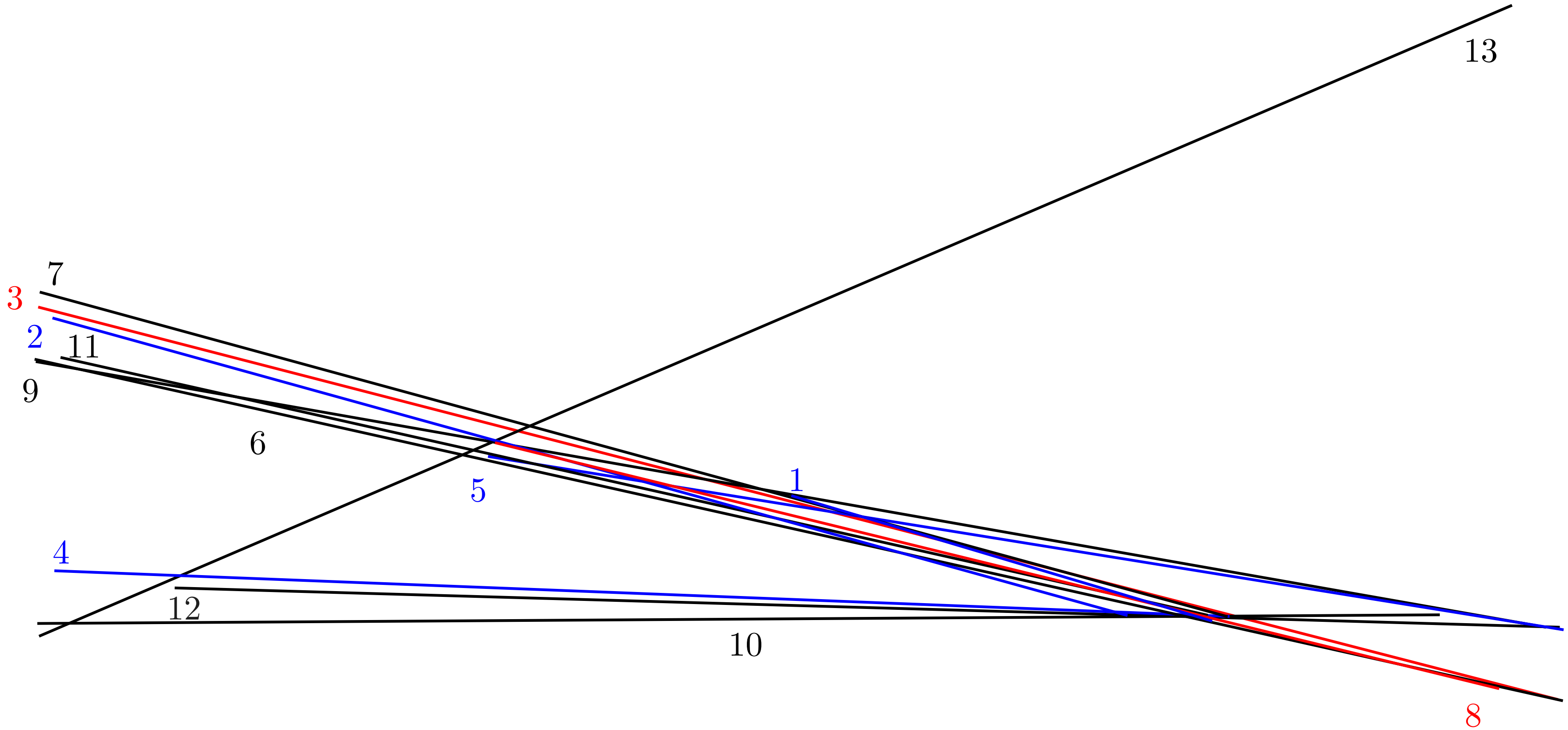
	left x	left y	right x	right y
1	$-\varepsilon$	0	$1-2\varepsilon$	$2\varepsilon^2+2\varepsilon^6$
2	ε^2	$\varepsilon-\varepsilon^3$	$1-\varepsilon^2$	ε^3
3	0	$\varepsilon^4+\varepsilon^6$	1	$\varepsilon^3+3\varepsilon^4$
4	0	$\varepsilon^4-\varepsilon^6$	$1-2\varepsilon$	$2\varepsilon^2-\varepsilon^6$
5	$-\varepsilon+\varepsilon^2$	0	$1-2\varepsilon^2$	$2\varepsilon^3-2\varepsilon^4$
6	$-\varepsilon$	$2\varepsilon^6$	$1-\varepsilon$	$2\varepsilon^6$
7	0	ε^6	1	$\varepsilon^3+2\varepsilon^4$
8	0	ε	$1+\varepsilon^3$	0
9	0	ε	$1-2\varepsilon^2$	$2\varepsilon^3-\varepsilon^4$
10	$-\varepsilon^2+3\varepsilon^3$	$3\varepsilon^6$	$1-2\varepsilon$	$2\varepsilon^2+\varepsilon^6$
11	$-\varepsilon^2$	ε^6	$1-2\varepsilon^2$	$2\varepsilon^3-3\varepsilon^4$
12	0	ε^4	1	0
13	$-\varepsilon$	0	$1+\varepsilon$	0

A partially flattened (4, 2)-arrangement



A flattened (4, 2)-arrangement

	left x	left y	right x	right y
1	ε	$\varepsilon^2 - \varepsilon^3 + \varepsilon^4 - 2\varepsilon^5$	$1 + \varepsilon^2$	$-\varepsilon^4 + \varepsilon^6$
2	0	$\varepsilon^2 + 3\varepsilon^5$	$1 - \varepsilon^3$	ε^7
3	0	$\varepsilon^2 + 4\varepsilon^5$	$1 + \varepsilon$	$-\varepsilon^3$
4	0	$2\varepsilon^3$	$1 + 3\varepsilon^4$	$-\varepsilon^8$
5	$\varepsilon - \varepsilon^2 + \varepsilon^3$	$\varepsilon^2 - \varepsilon^3 + \varepsilon^4 - \varepsilon^8$	$1 + \varepsilon$	$-\varepsilon^4$
6	0	$\varepsilon^2 + \varepsilon^5$	$1 + \varepsilon$	$-\varepsilon^3$
7	0	$\varepsilon^2 + 5\varepsilon^5$	$1 + 3\varepsilon^4$	$-3\varepsilon^7$
8	$\varepsilon - \varepsilon^2 + \varepsilon^3 + \varepsilon^4 + 2\varepsilon^5$	$\varepsilon^2 - \varepsilon^3 + \varepsilon^4 + \varepsilon^5 + \varepsilon^6$	$1 + \varepsilon - \varepsilon^4$	$-\varepsilon^3$
9	0	ε^2	$1 + \varepsilon$	$-\varepsilon^4$
10	0	0	$1 + 5\varepsilon^3$	0
11	0	$\varepsilon^2 + 2\varepsilon^5$	$1 + 3\varepsilon^4 - 2\varepsilon^5$	ε^8
12	$\varepsilon - \varepsilon^3$	$\varepsilon^3 - \varepsilon^4$	$1 + \varepsilon$	$-\varepsilon^4$
13	0	0	1	ε

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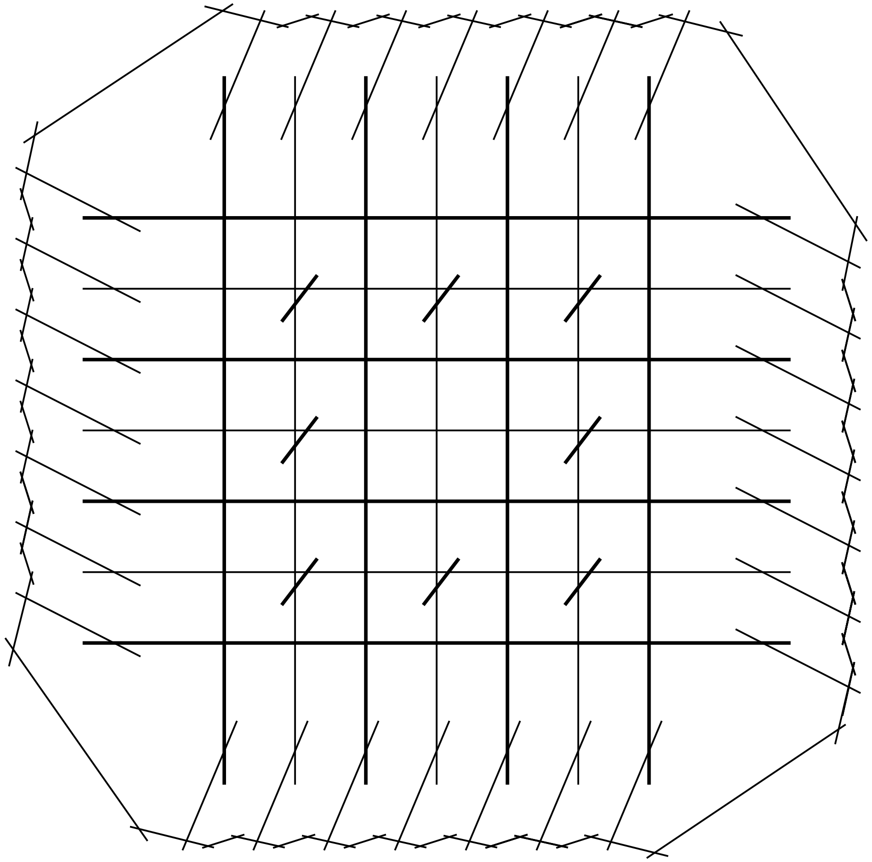

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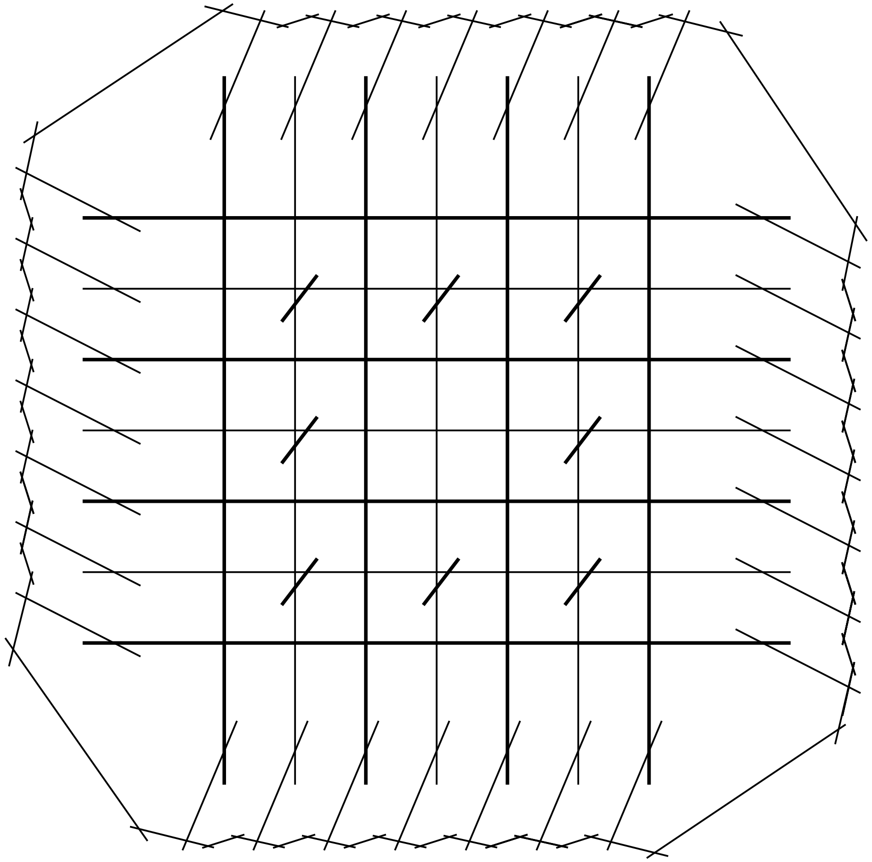

(not even topologically)

Theorem There exists a non-flattenable arrangement of segments such that all segments cross a common line (so it can be flattened topologically).

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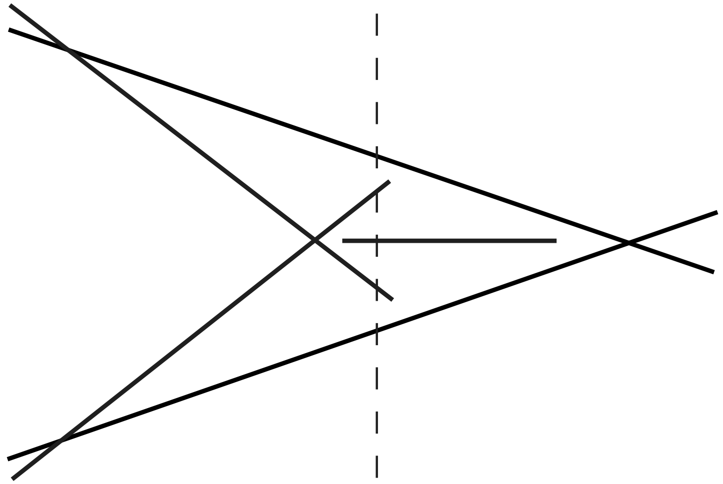
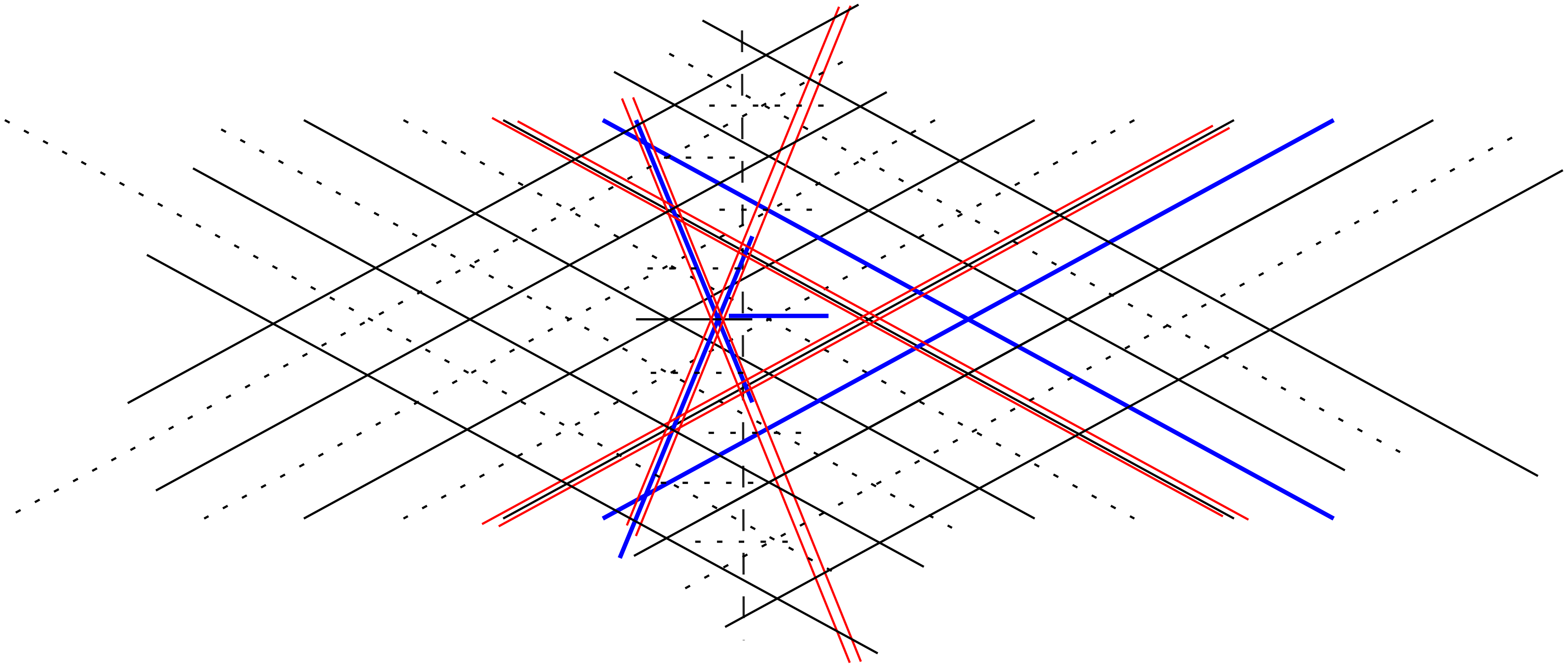
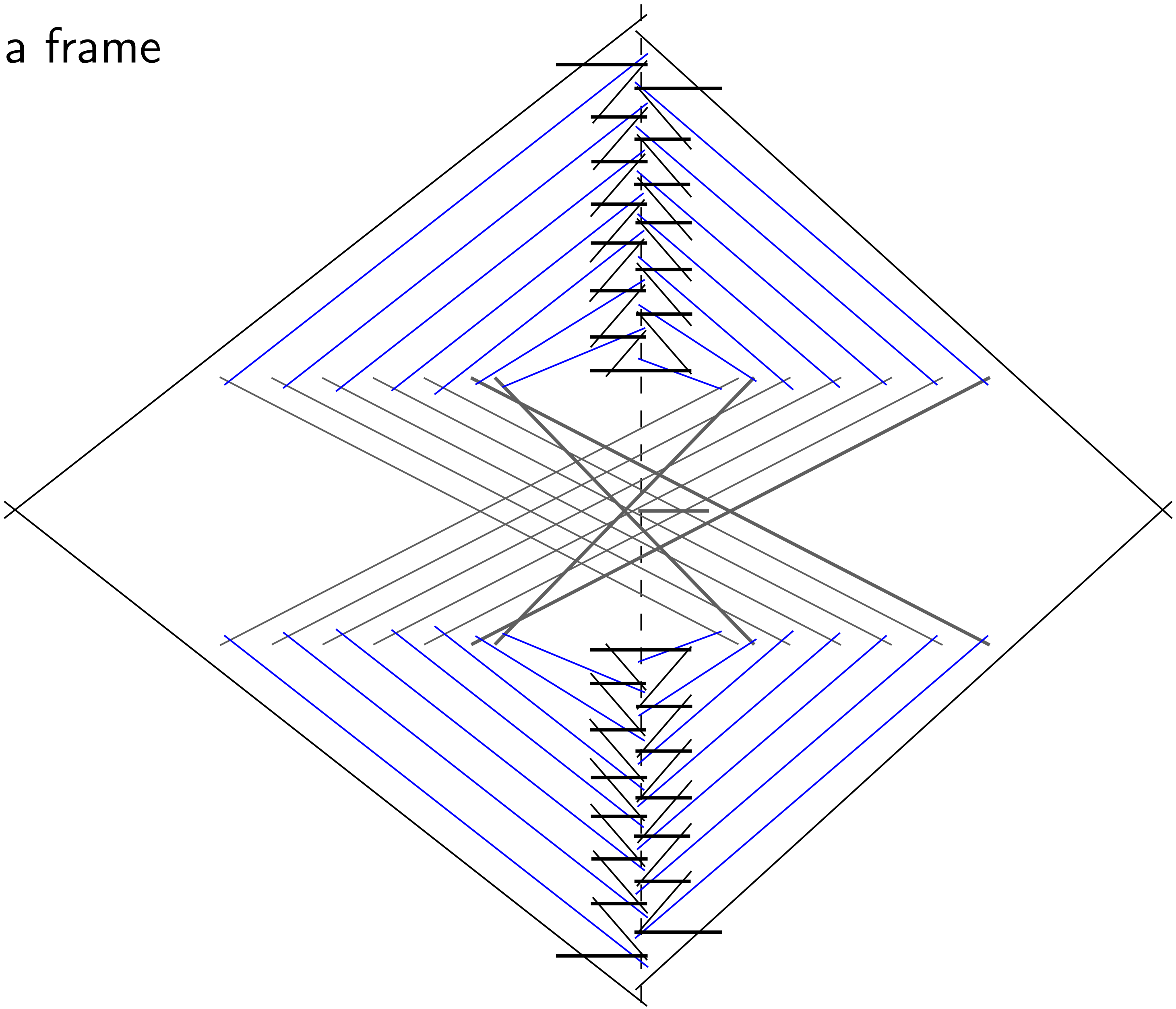
core:



a grid of supporting segments



a frame



Open problems

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- upper bound for pseudosegments

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- better upper and lower bound for $r(k)$
- upper bound for pseudosegments
- upper bound for curves (string graphs)