The Complexity of Several Realizability Problems for Abstract Topological Graphs

Jan Kynčl

Charles University, Prague

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- at most two edges can intersect in one crossing

simple: any two edges have at most one common point

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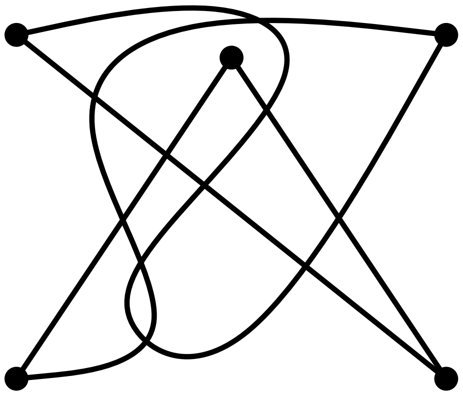

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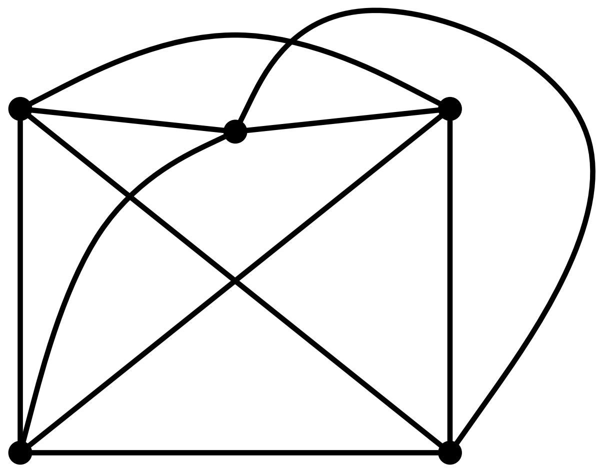

simple complete topological graph

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weakly realizable ... $R_T \subseteq R$

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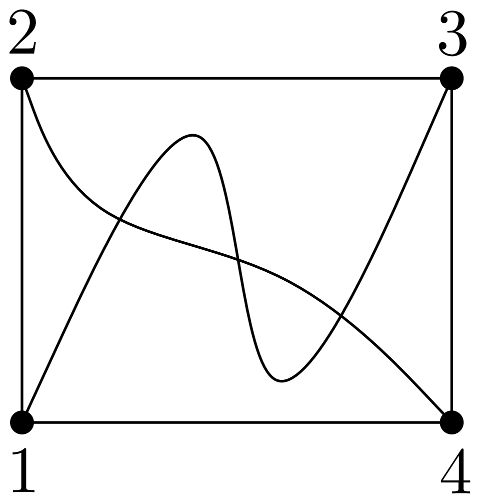
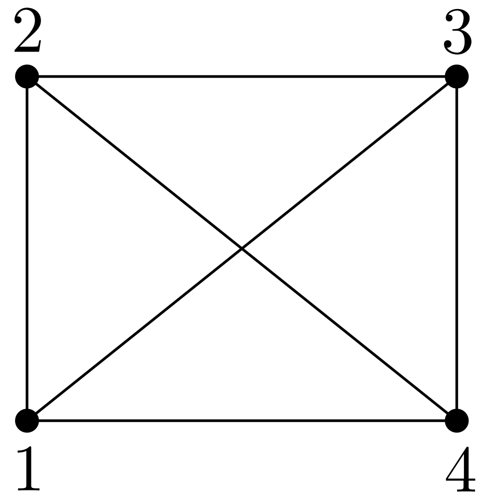
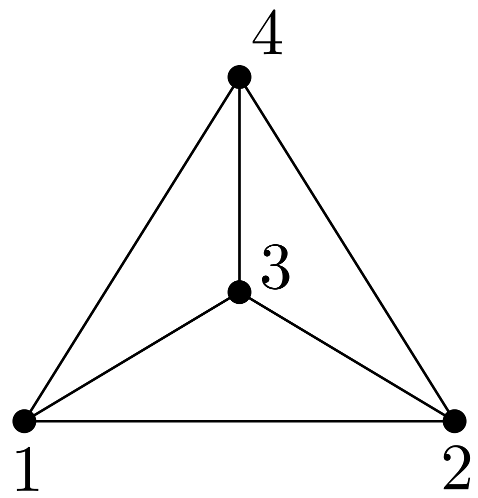


realization



simple realization



weak realization

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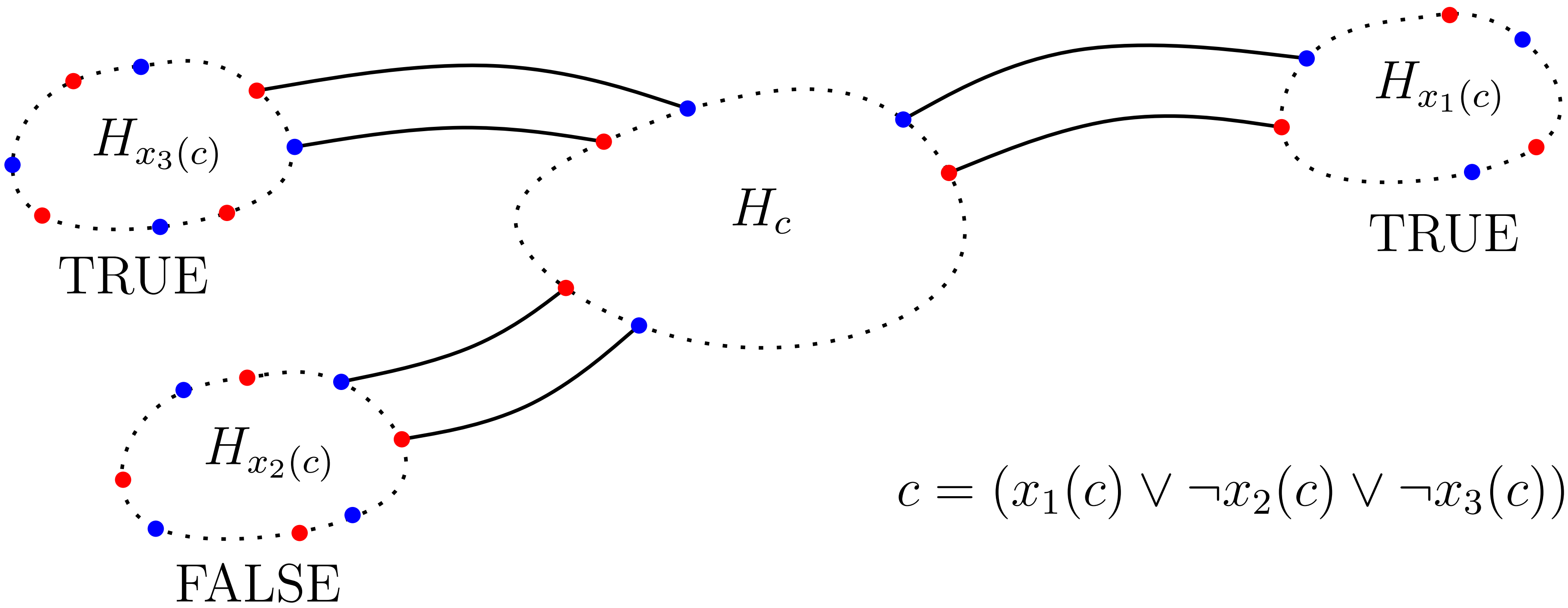
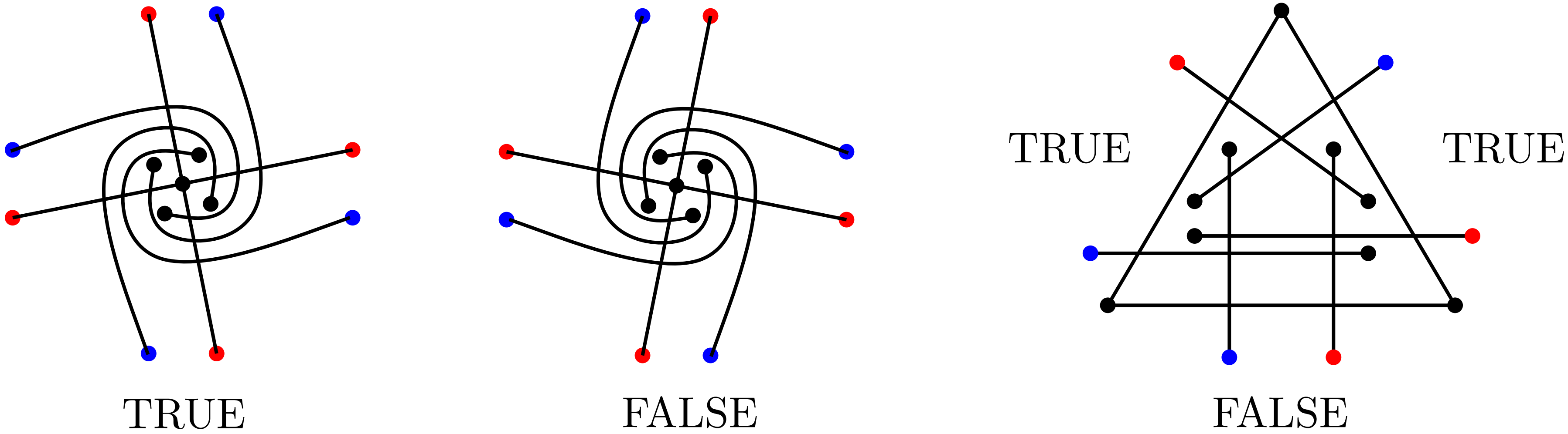


$$c = (x_1(c) \vee \neg x_2(c) \vee \neg x_3(c))$$

example of variable and clause gadgets for the simple realizability:



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(2) For every edge e of a simple complete topological graph T and for each pair of edges $f, f' \in E(G)$ that have a common end-point and cross e, the AT-graph of T uniquely determines the order of crossings of e with the edges f and f'.

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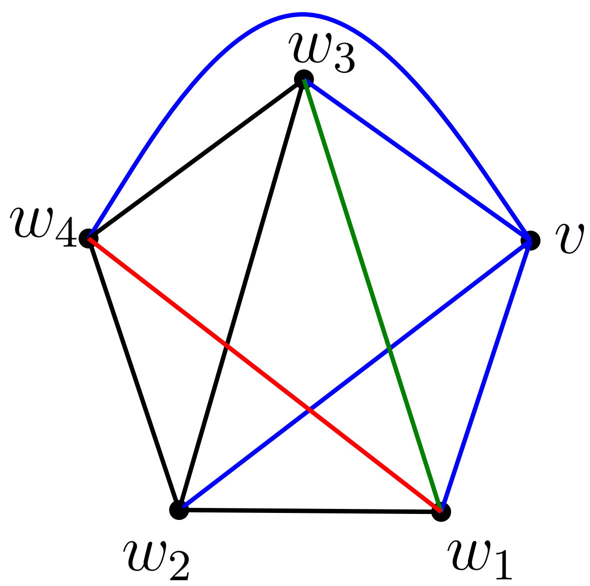

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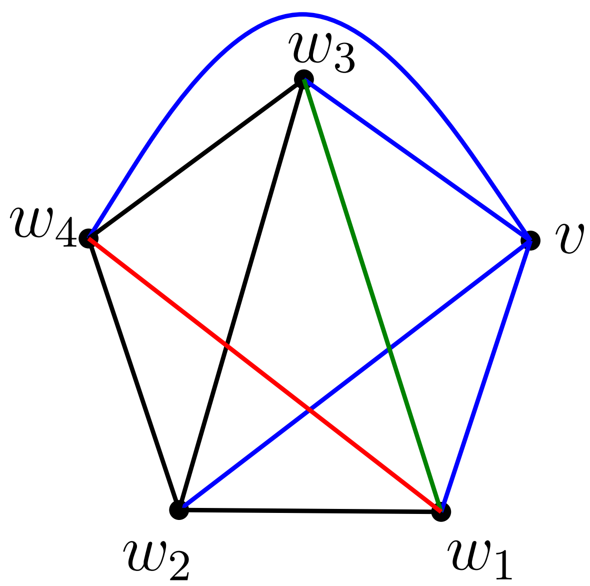

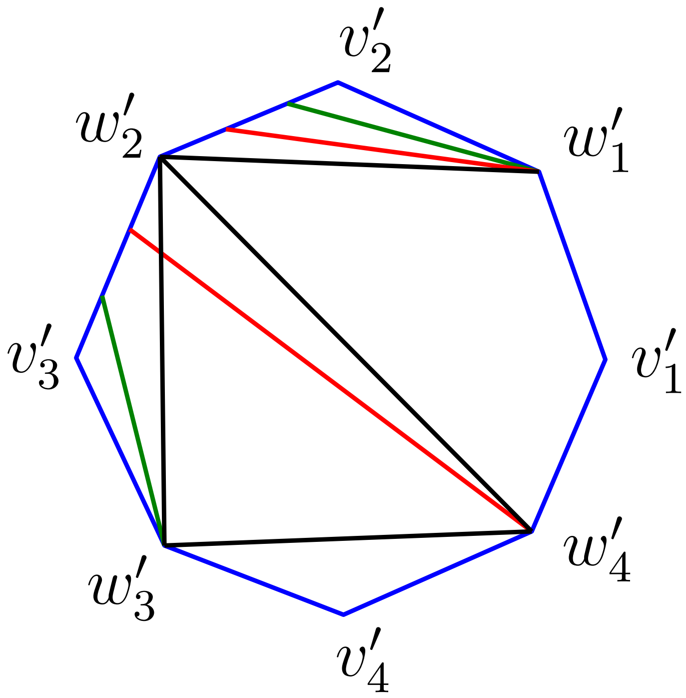

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- the order of the end-points of the pseudochords on the perimeter minimizing the total number of crossings
- the order of crossings of pseudochords with other pseudochords