Enumeration of Simple Complete Topological Graphs

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- any two edges have only finitely many common points
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- at most two edges can intersect in one crossing

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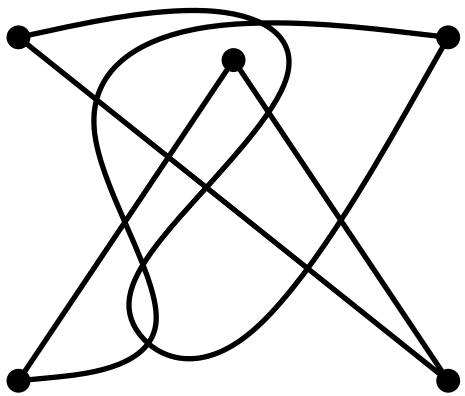
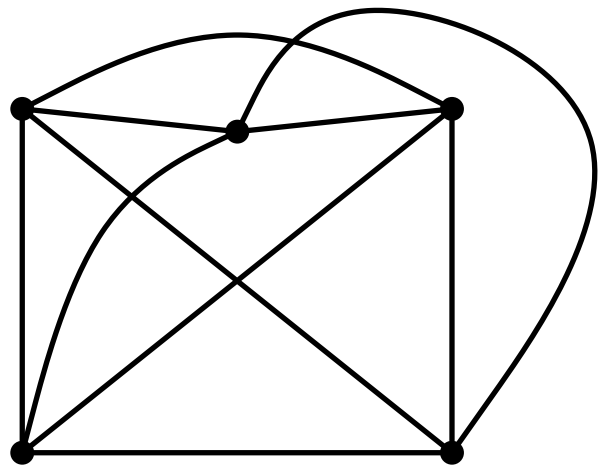


topological graph



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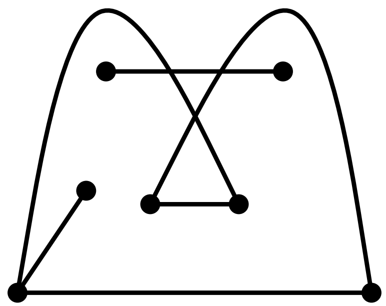

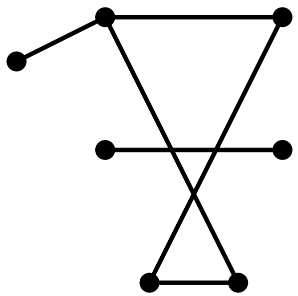

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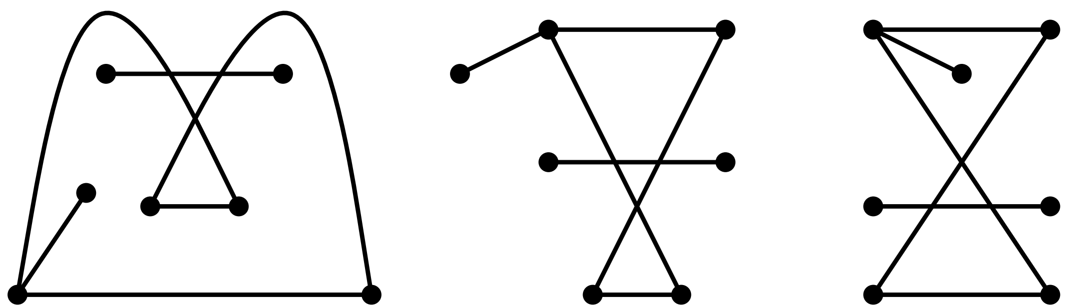

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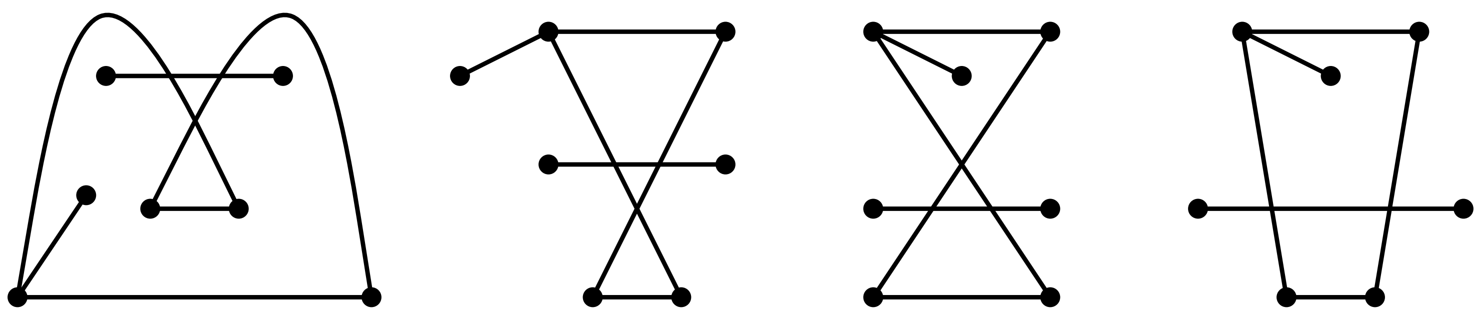

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Remark: The number of weak isomorphism classes of complete **geometric** graphs on n vertices is $2^{O(n \log n)}$

Graphs with maximum number of crossings

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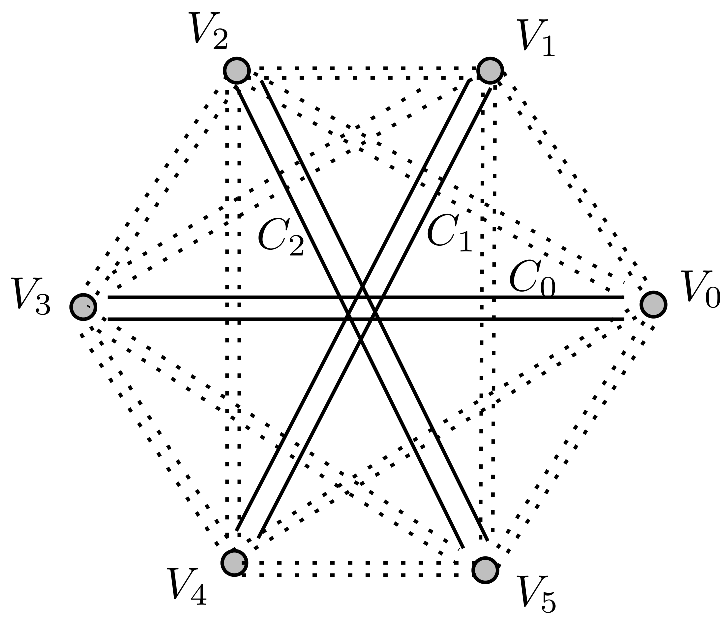

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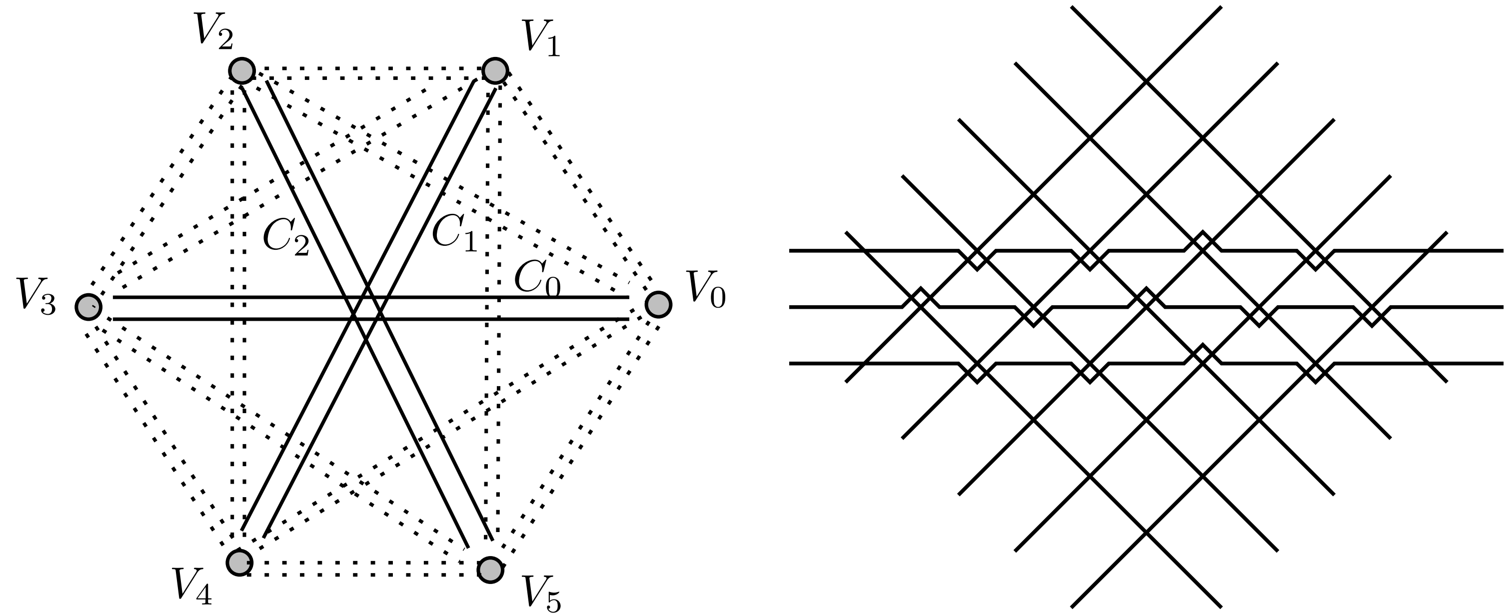

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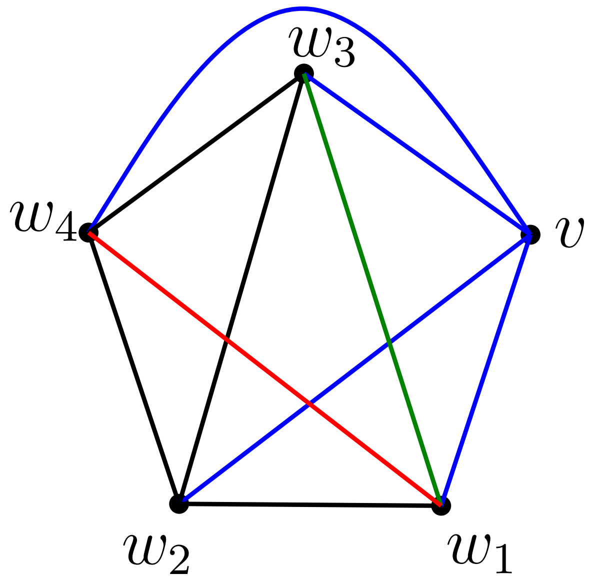

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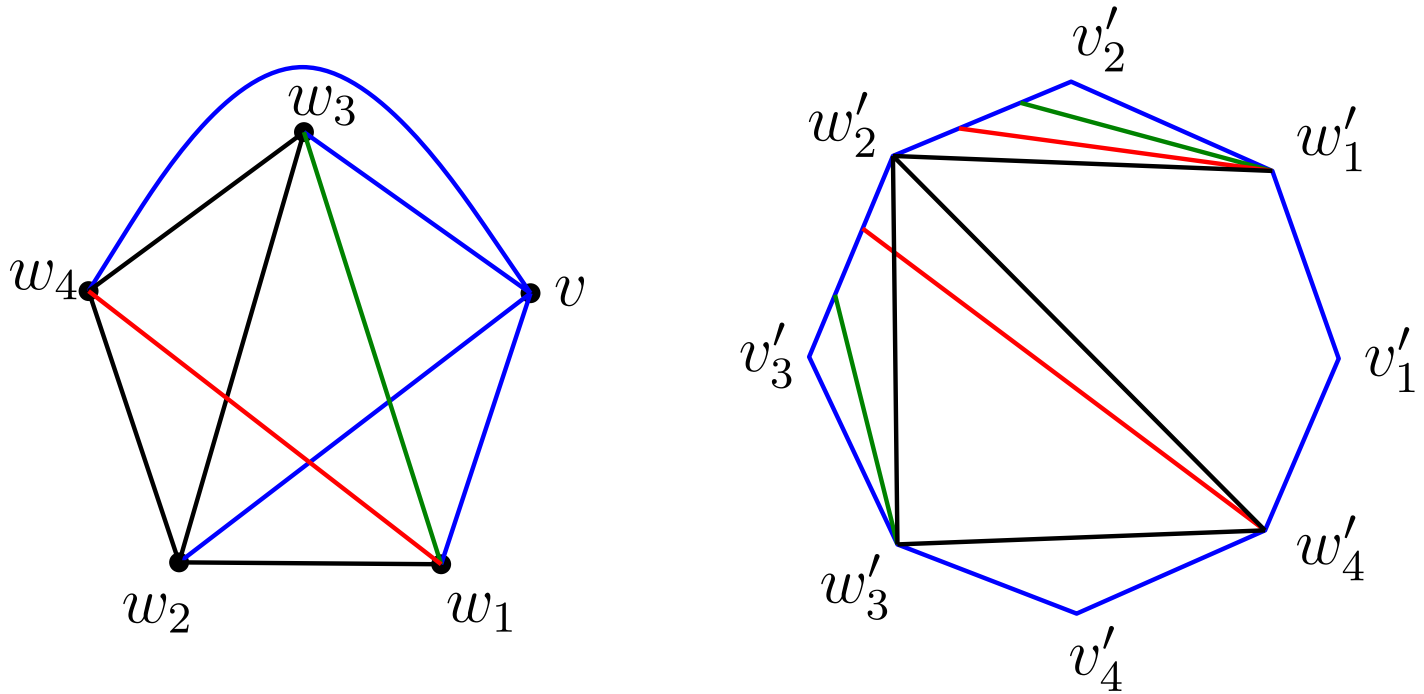

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