Monochromatic triangles in two-colored plane

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Euclidean Ramsey theory

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X . . . a finite set of points in $\mathbb{E}^d$

c . . . a finite number of colors

Euclidean Ramsey theory

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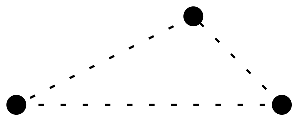
c ... a finite number of colors

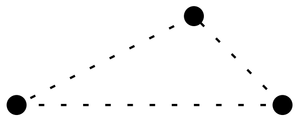
Question: Does every coloring of $\mathbb{E}^d$ with c colors contain a monochromatic copy of X?

copy of X = congruent copy of X = set obtained from X by translations and rotations

X' is monochromatic if all points of X' have the same color.



$\mathbb{E}^3$ 2 colors YES



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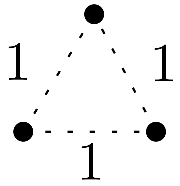
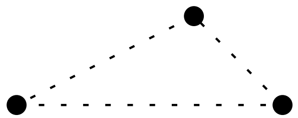


$\mathbb{E}^2$ 2 colors NO



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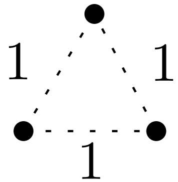
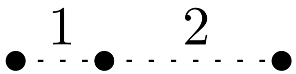
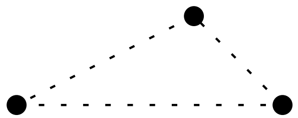
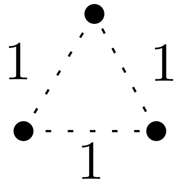
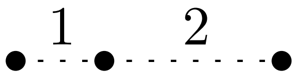
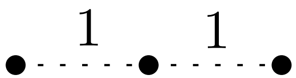
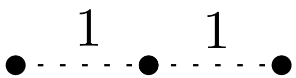


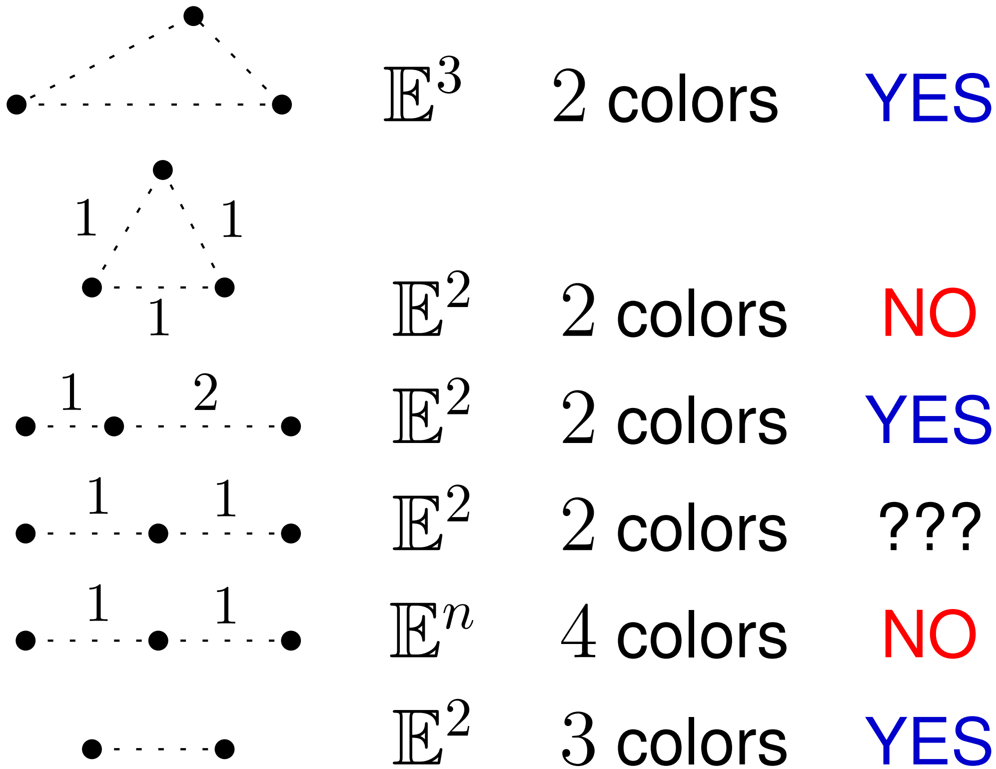
$\mathbb{E}^2$ 2 colors NO



$\mathbb{E}^2$ 2 colors YES

$\mathbb{E}^3$ 2 colors YES
1 1 1
$\mathbb{E}^2$ 2 colors NO
1 2
$\mathbb{E}^2$ 2 colors YES
1 1
$\mathbb{E}^2$ 2 colors ???

	$\mathbb{E}^3$	2 colors	YES
	$\mathbb{E}^2$	2 colors	NO
	$\mathbb{E}^2$	2 colors	YES
	$\mathbb{E}^2$	2 colors	???
	$\mathbb{E}^n$	4 colors	NO



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$\mathbb{E}^2$ 2 colors NO
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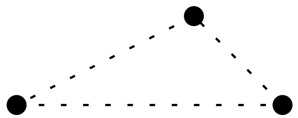
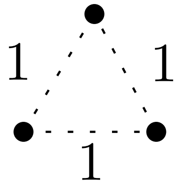
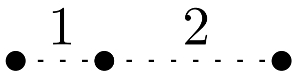
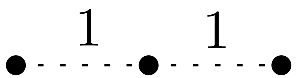
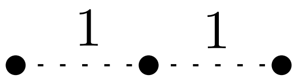
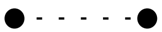
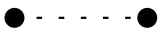
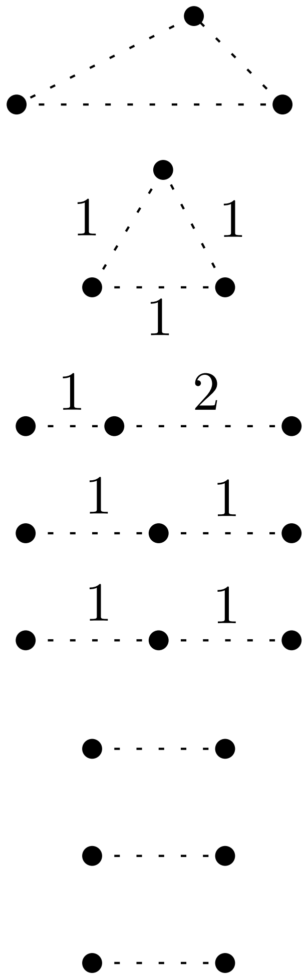
$\mathbb{E}^2$ 2 colors ???
1 1
$\mathbb{E}^n$ 4 colors NO
$\mathbb{E}^2$ 3 colors YES

	$\mathbb{E}^3$	2 colors	YES
	$\mathbb{E}^2$	2 colors	NO
	$\mathbb{E}^2$	2 colors	YES
	$\mathbb{E}^2$	2 colors	???
	$\mathbb{E}^n$	4 colors	NO
	$\mathbb{E}^2$	3 colors	YES
	$\mathbb{E}^2$	4 colors	???



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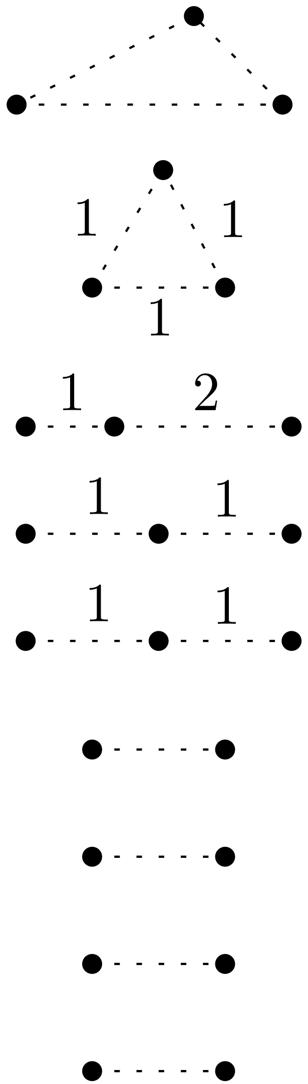

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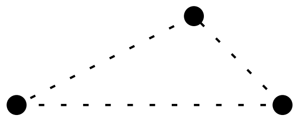
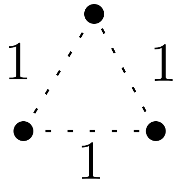
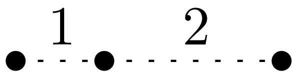
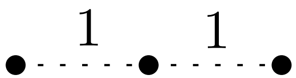
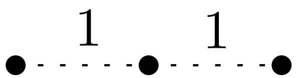
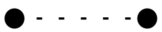
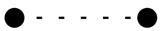
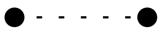
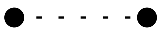
$\mathbb{E}^2$	5 colors	???
$\mathbb{E}^2$	6 colors	???

	$\mathbb{E}^3$	2 colors	YES
	$\mathbb{E}^2$	2 colors	NO
	$\mathbb{E}^2$	2 colors	YES
	$\mathbb{E}^2$	2 colors	???
	$\mathbb{E}^n$	4 colors	NO
	$\mathbb{E}^2$	3 colors	YES
	$\mathbb{E}^2$	4 colors	???
	$\mathbb{E}^2$	5 colors	???
	$\mathbb{E}^2$	6 colors	???
	$\mathbb{E}^2$	7 colors	NO

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degenerate triangle $\ldots$ a set of 3 collinear points

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Coloring χ contains a triangle T if there is a monochromatic copy of T, otherwise χ avoids T.

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Theorem [Erdös et al., 1973; Shader, 1979]

Every coloring contains every

- triangle with a 30°, 90° or 150° angle
- triangle with a ratio between two sides equal to $2\sin 15^\circ$, $2\sin 36^\circ$, $2\sin 45^\circ$, $2\sin 60^\circ$ or $2\sin 75^\circ$
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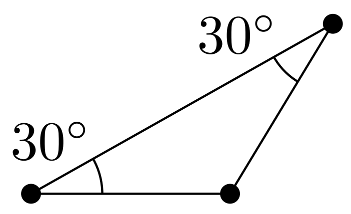
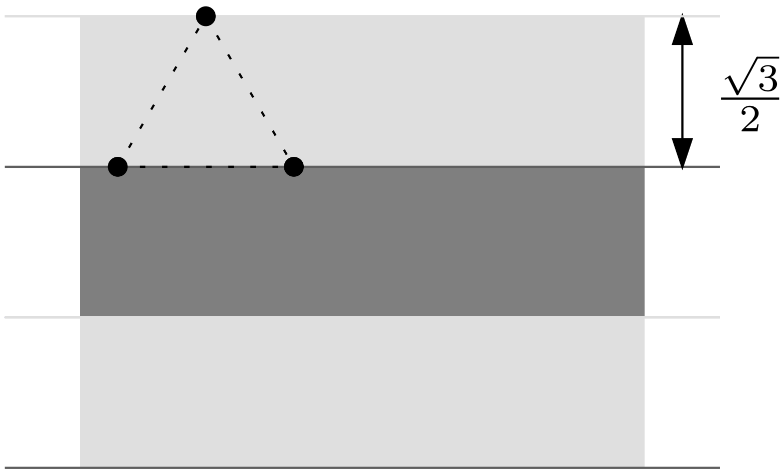


Strip coloring avoiding a unit triangle:



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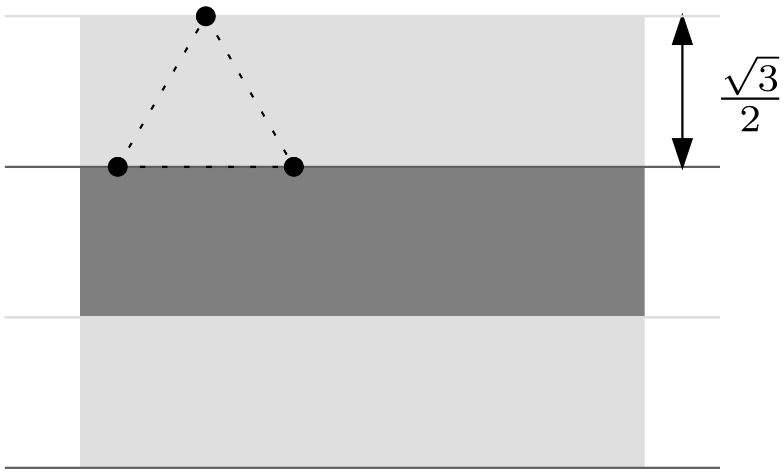

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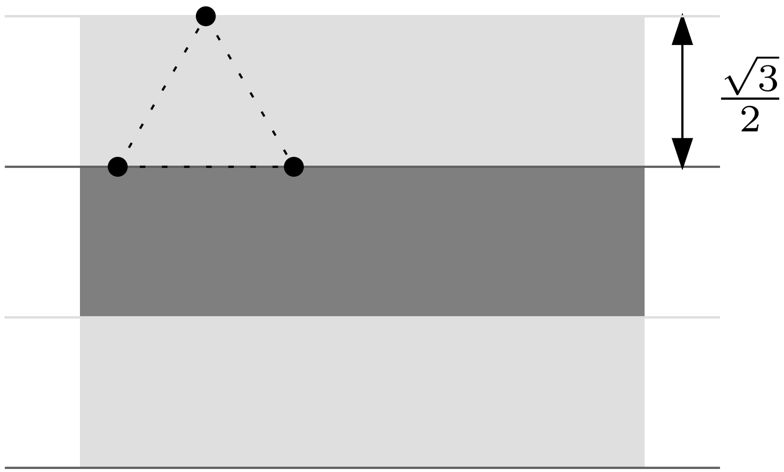

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Conjecture 2 [Erdös et al., 1973]

The strip coloring is the only coloring avoiding any triangle (up to scaling and modification of colors on the boundaries of the strips).

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- Characterization of all polygonal colorings avoiding an equilateral triangle
- Conjecture 2 is false.

Reduction to equilateral triangles

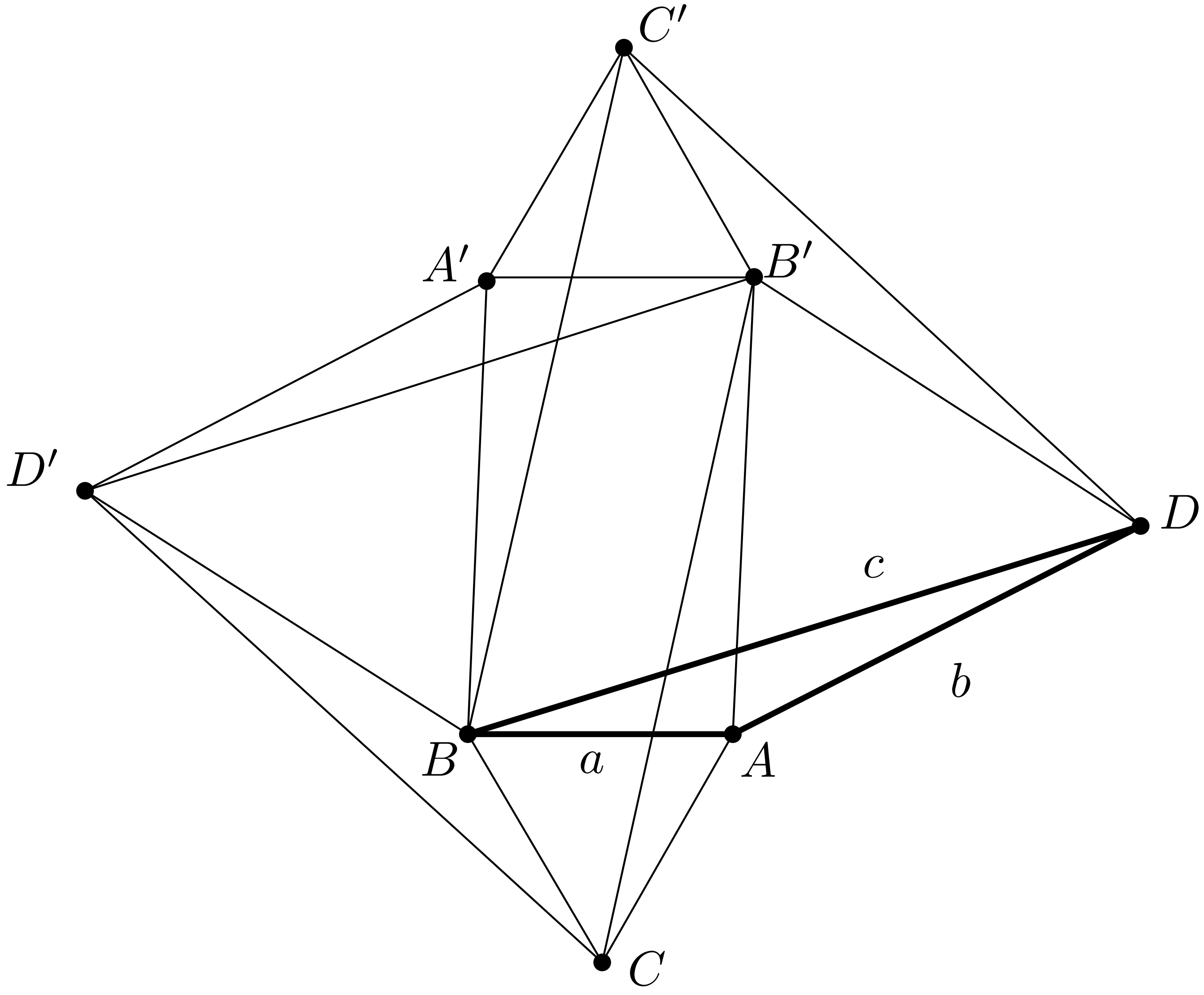
Lemma [Erdös et al., 1973]

Let χ be a coloring of the plane.

1. If χ contains an (a, a, a)-triangle for some $a > 0$, then χ contains any (a, b, c)-triangle, where $b, c > 0$ and a, b, c satisfy the (possibly degenerate) triangle inequality.

2. If χ contains an (a, b, c)-triangle, then χ contains an (a, a, a), (b, b, b), or (c, c, c)-triangle.

proof: (1)



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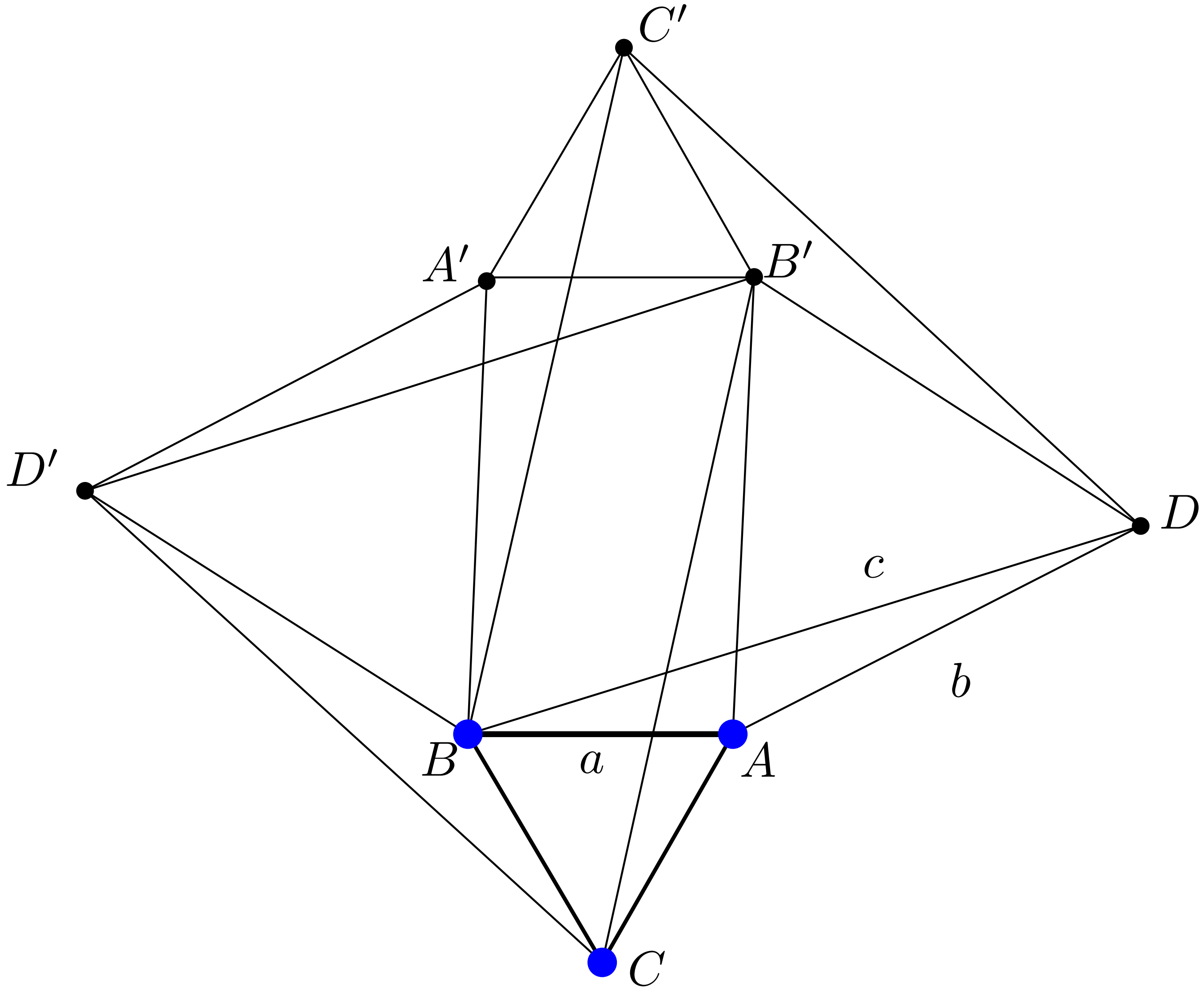

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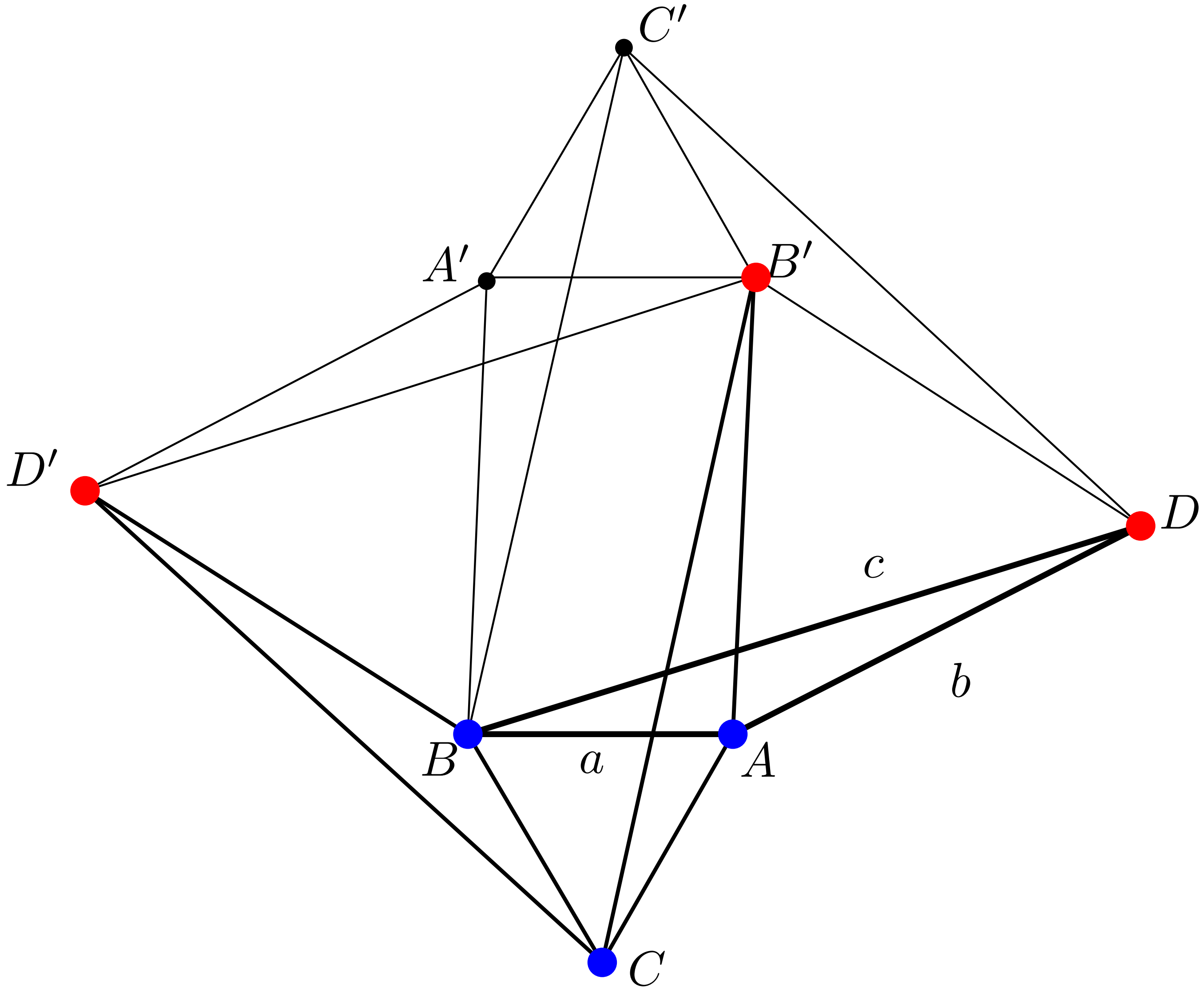

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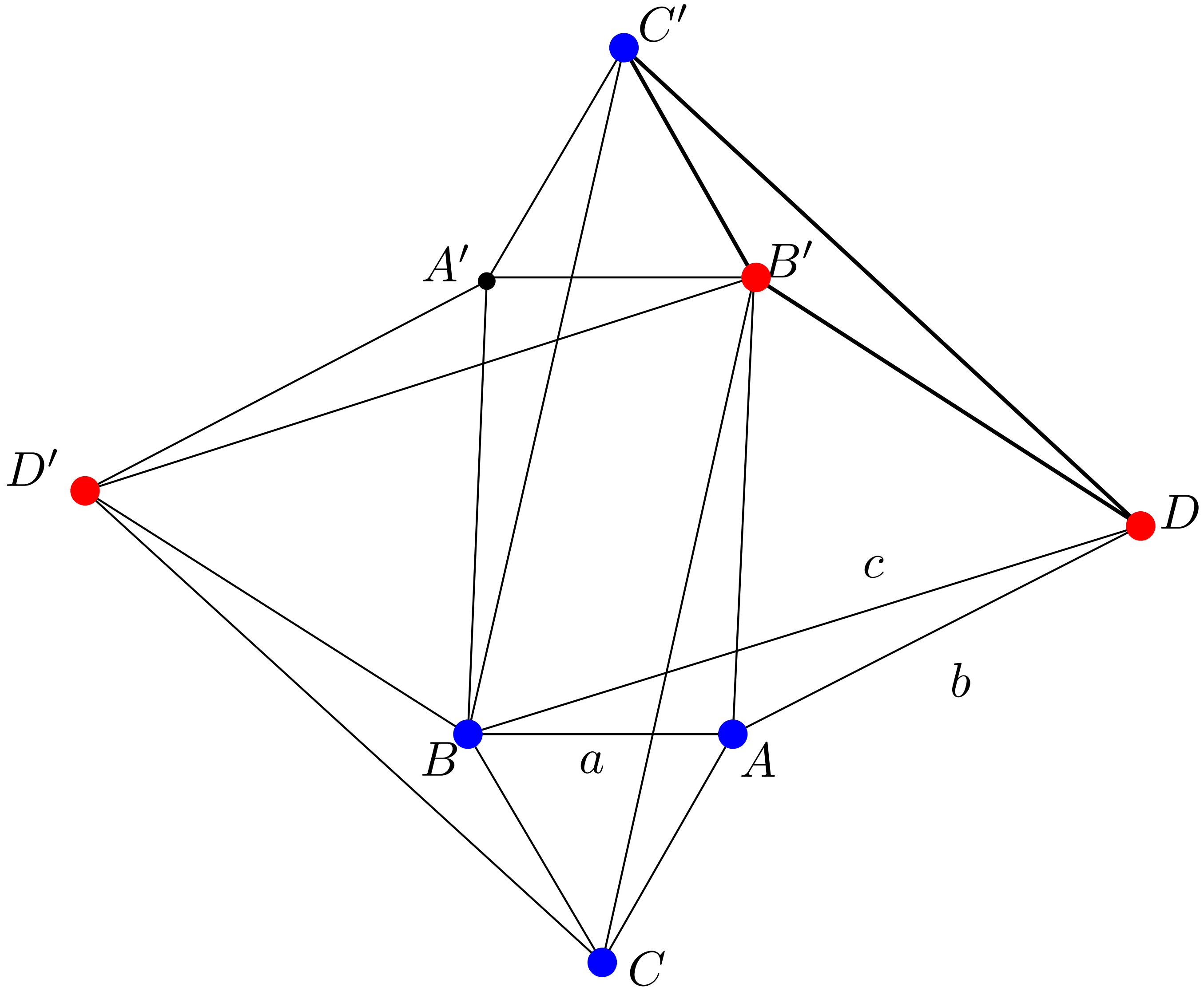

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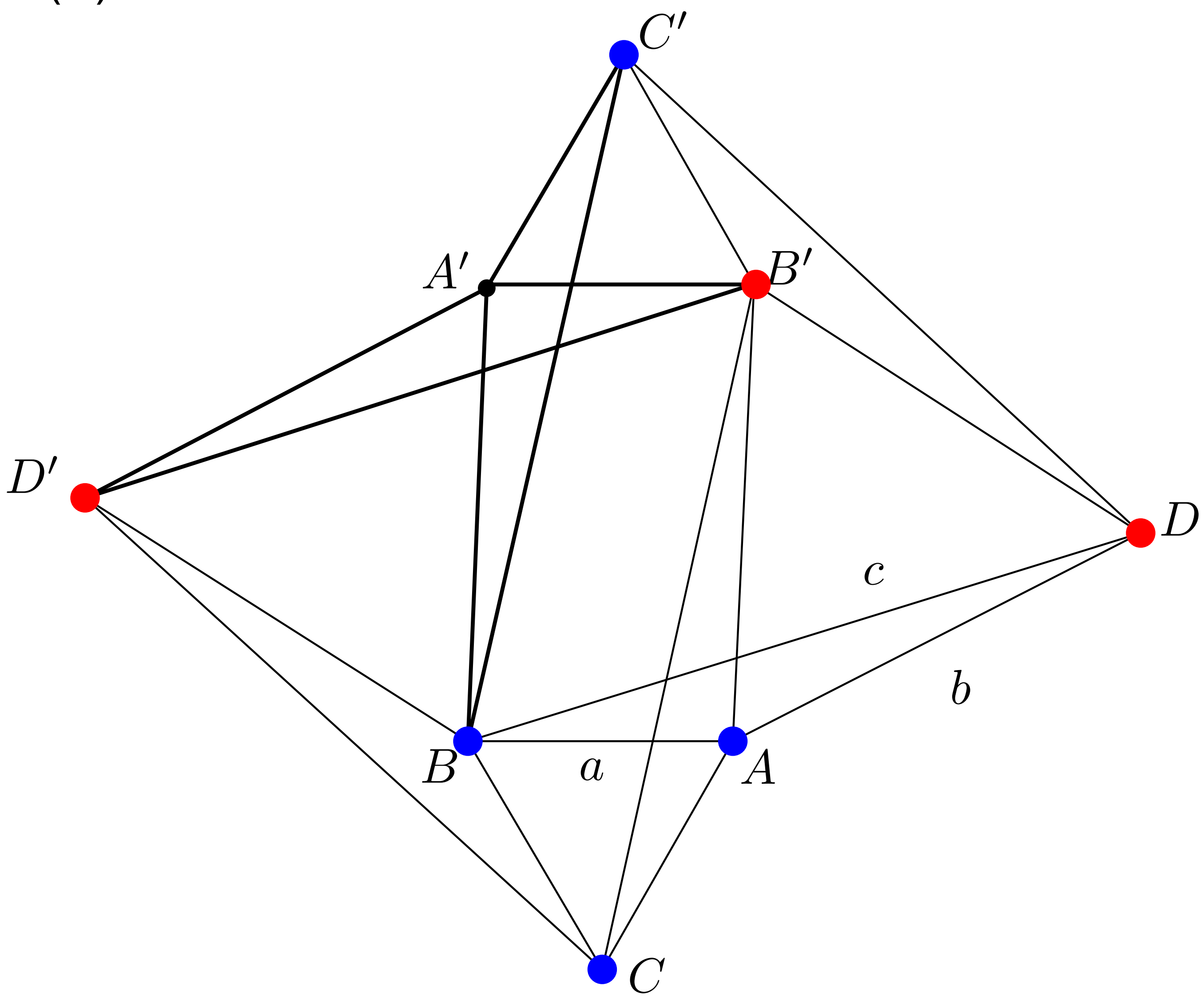
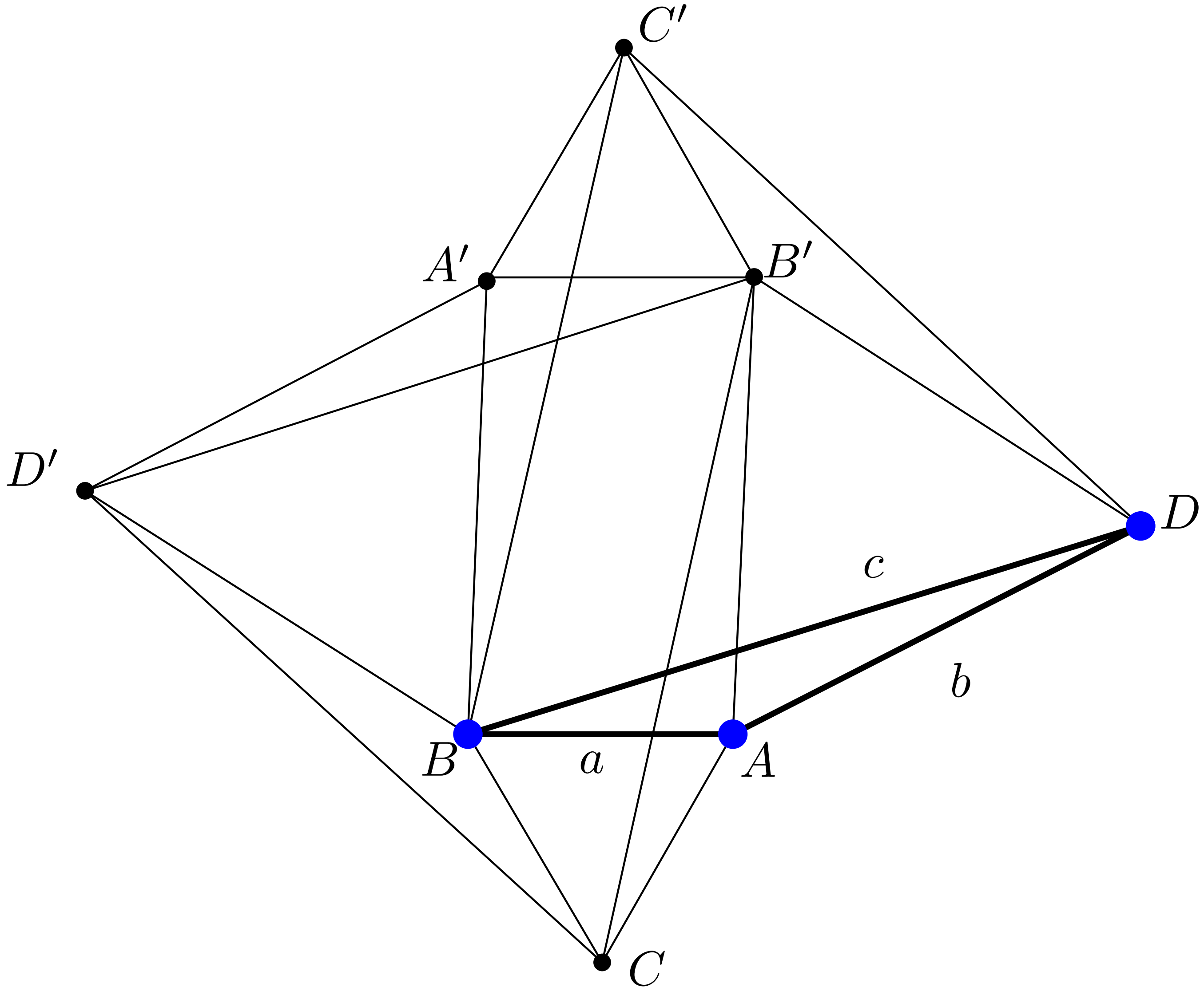


proof: (2)



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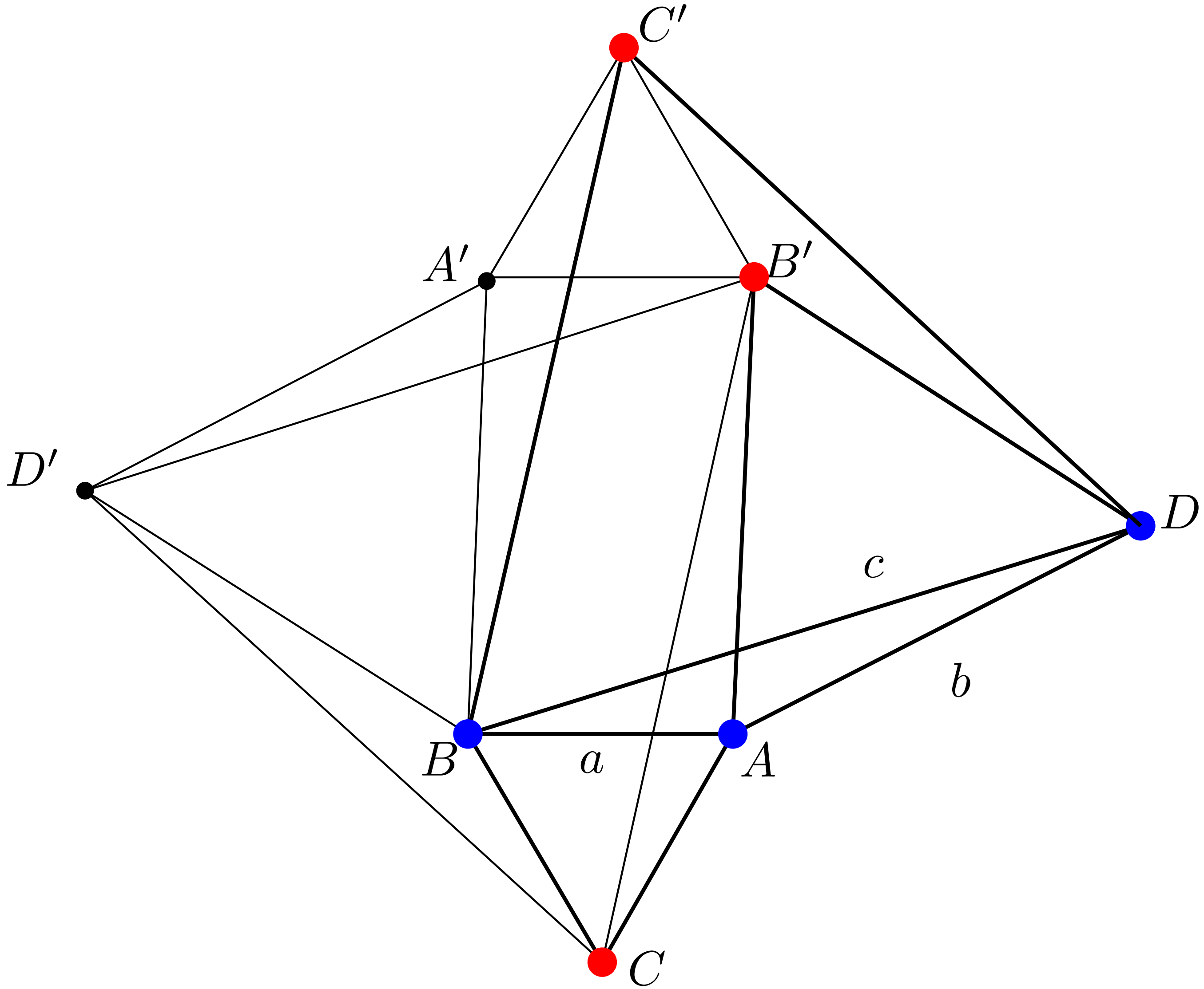

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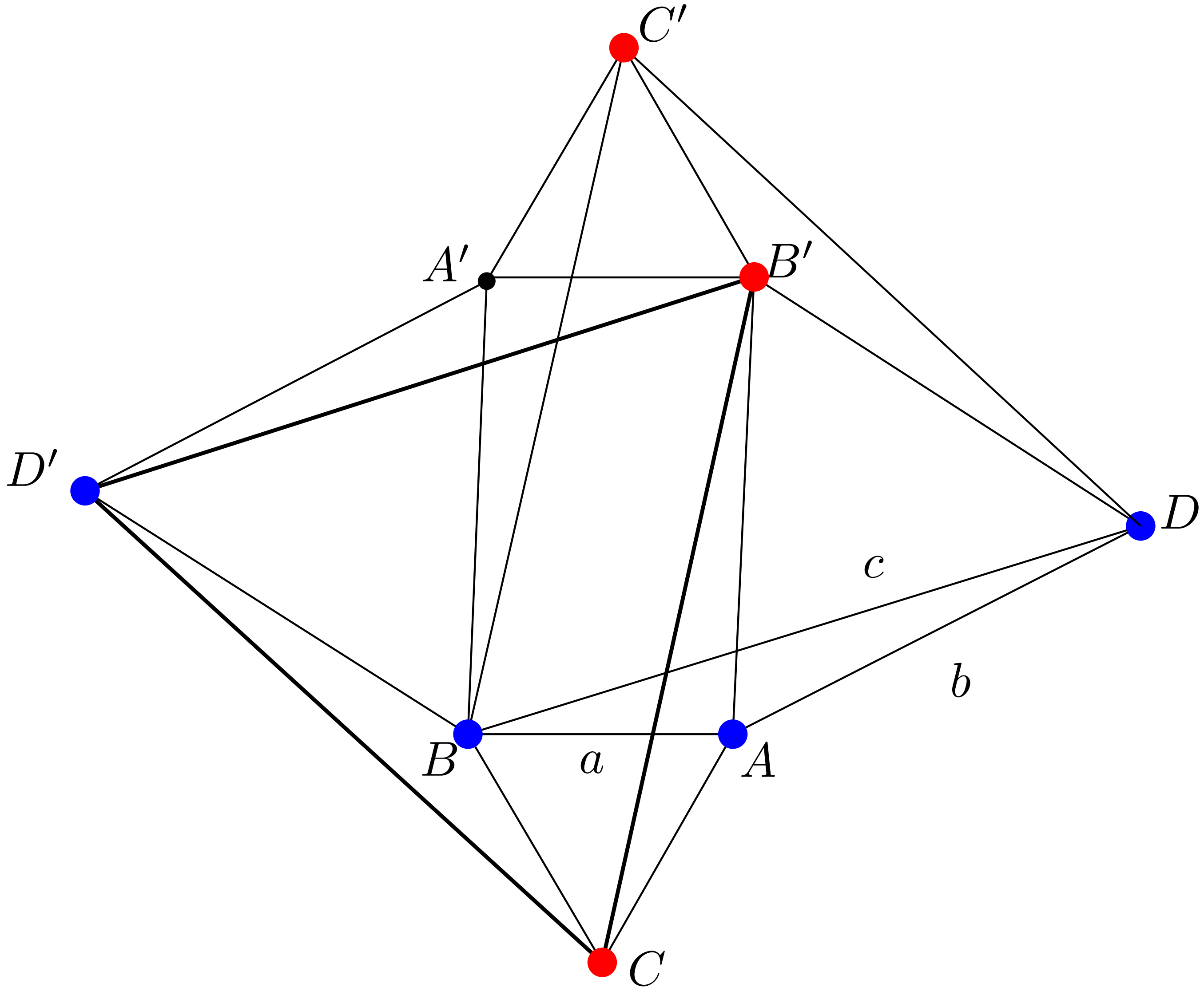

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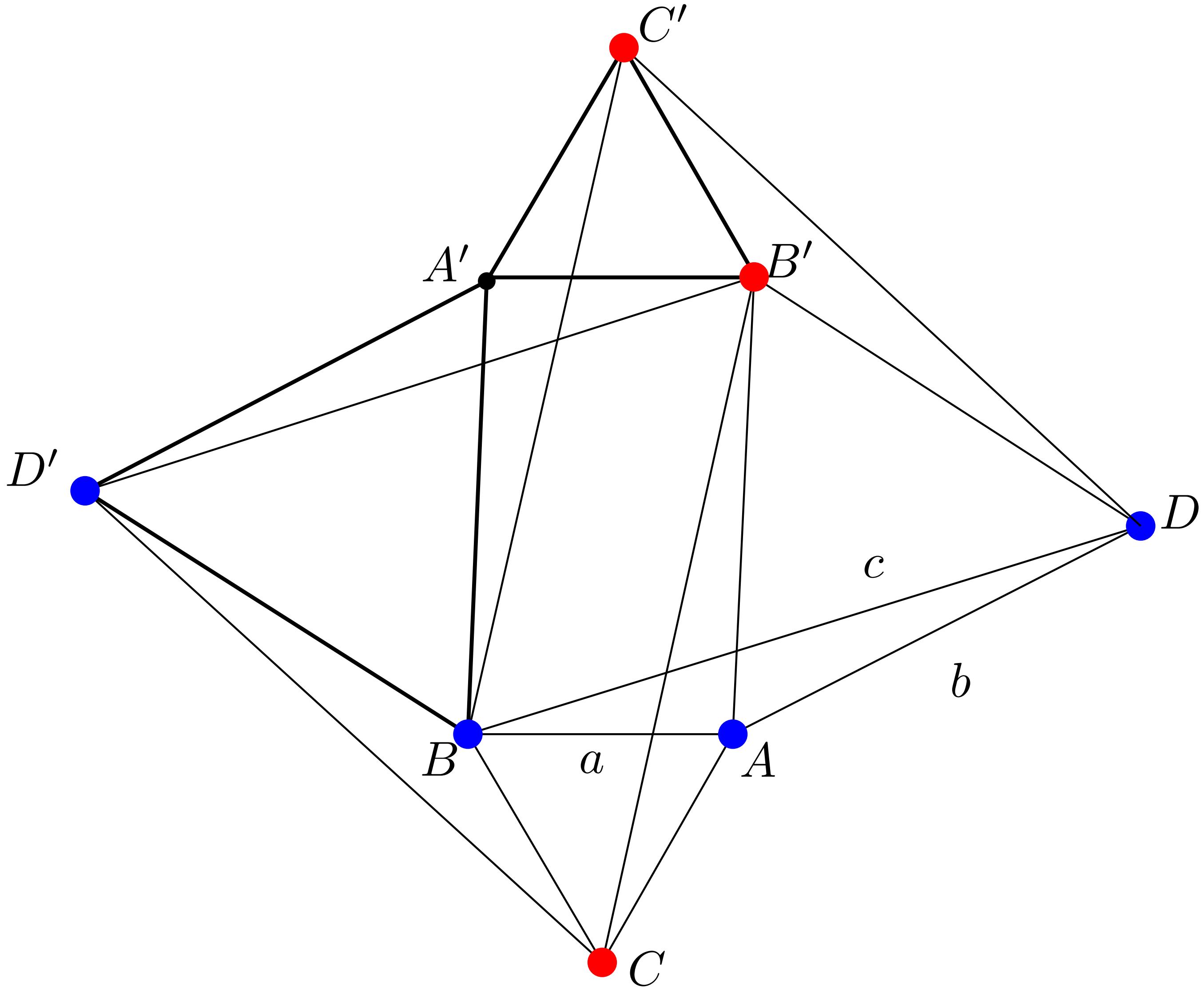

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3. χ contains an (a, b, c)-triangle if and only if χ contains a (b, a, c)-triangle.

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(proof of Theorem 1)

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Proposition Let $Q(3) = \mathcal{B} \cup \mathcal{R}$ be an arbitrary coloring of the square $Q(3)$ avoiding the unit triangle. Then for every $\varepsilon > 0$ both $\mathcal{B}$ and $\mathcal{R}$ contain an ε-almost unit triangle.

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(if $\mathcal{B}$ is closed, then $\mathcal{B}^3$ is a compact set containing a sequence of $\frac{1}{n}$-almost unit triangles...)

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- $K(\alpha)$ and $K(\alpha + \frac{\pi}{3})$ must have different color

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- in $Q(1)$, find a red point R and a blue point S, such that $|R - S| < \varepsilon$

- construct a circle $\mathcal{C}$ with the center S and radius 1

- denote $K(\alpha) = S + (\cos\alpha, \sin\alpha)$

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$\Rightarrow$ whole $\mathcal{C}$ is blue, a contradiction. $\square$

Polygonal colorings

A coloring $\chi = (\mathcal{B}, \mathcal{W})$ is polygonal, if

- each of the two sets $\mathcal{B}$ and $\mathcal{W}$ is contained in the closure of its interior
- The boundary of χ (a common boundary of $\mathcal{B}$ and $\mathcal{W}$), is a union of straight line segments (called boundary segments), which can intersect only at their endpoints (boundary vertices) .
- Every bounded region of the plane is intersected by only finitely many boundary segments.

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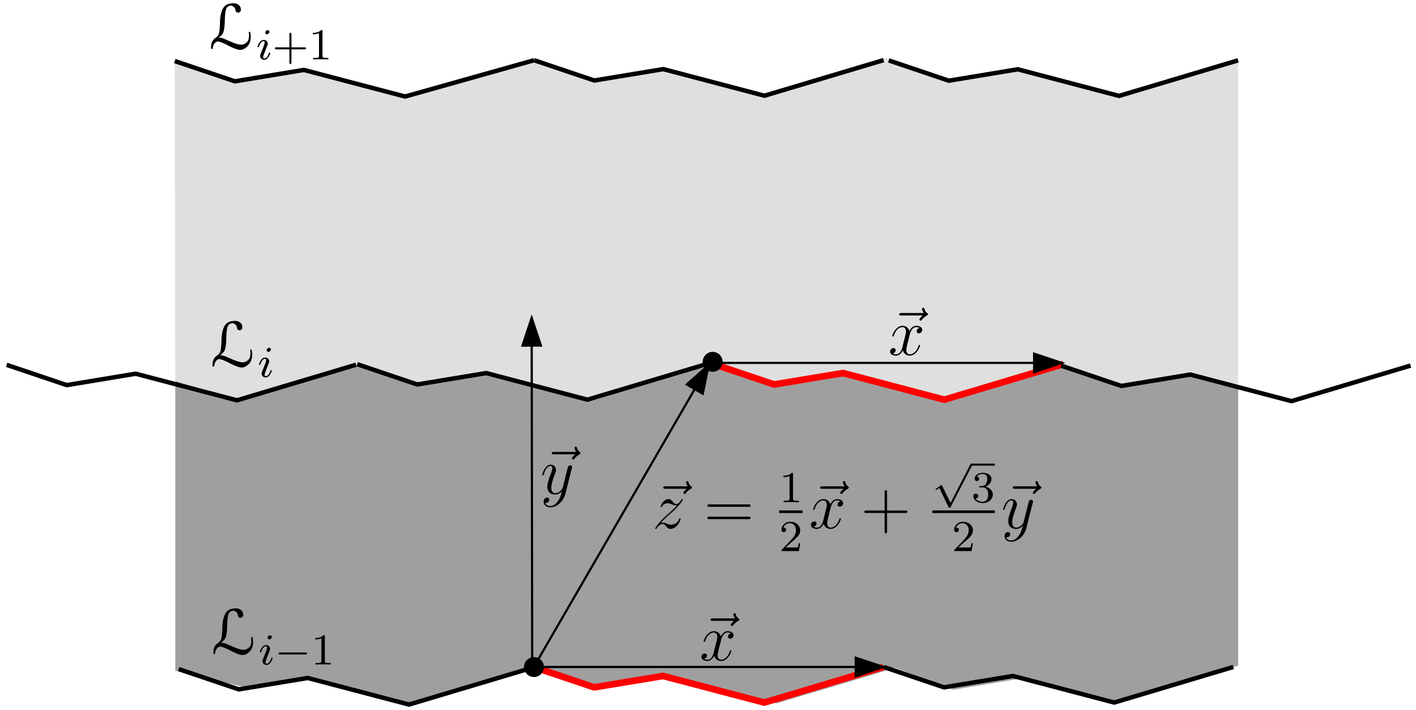

$$|\vec{x}| = |\vec{y}| = 1, \vec{x} \perp \vec{y}$$

$$L_i = L_i + \vec{x}$$

$$L_{i+1} = L_i + \vec{z}$$

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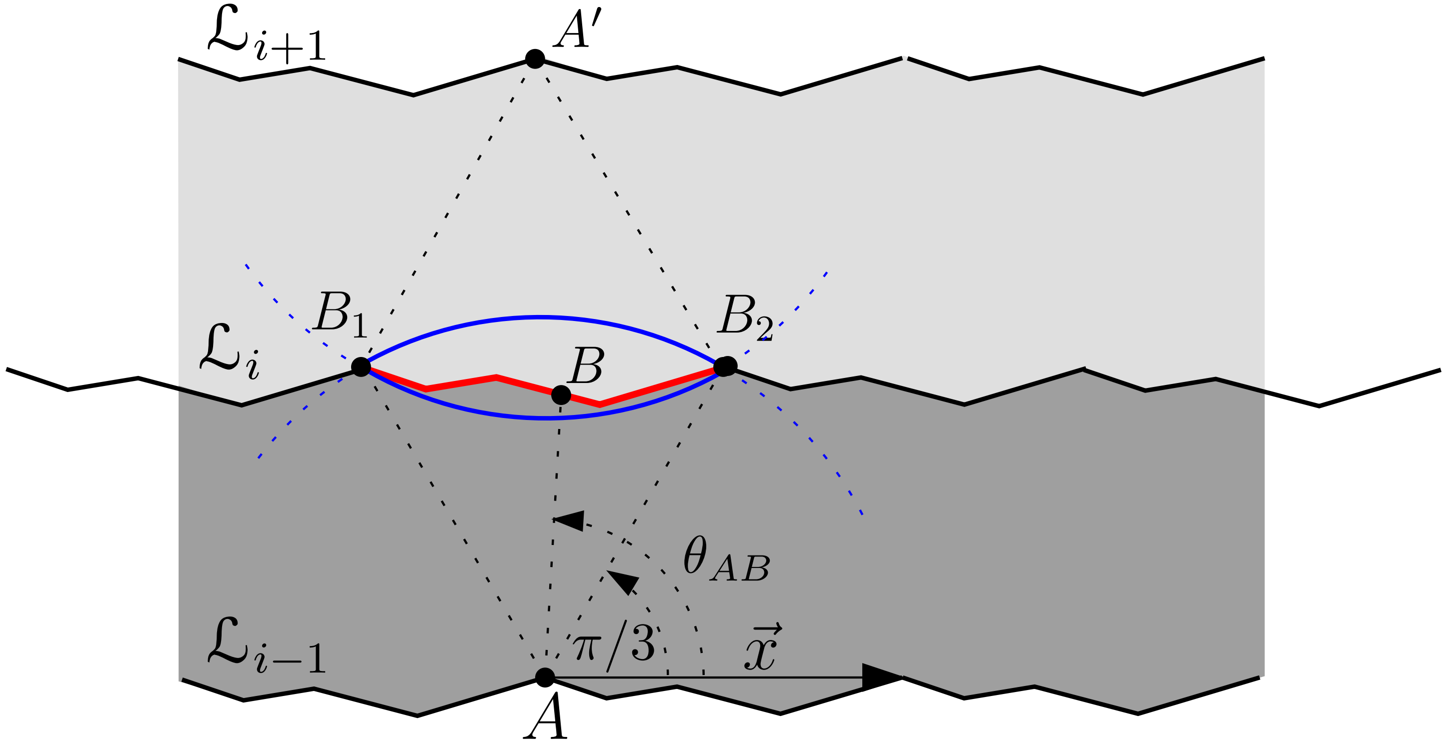

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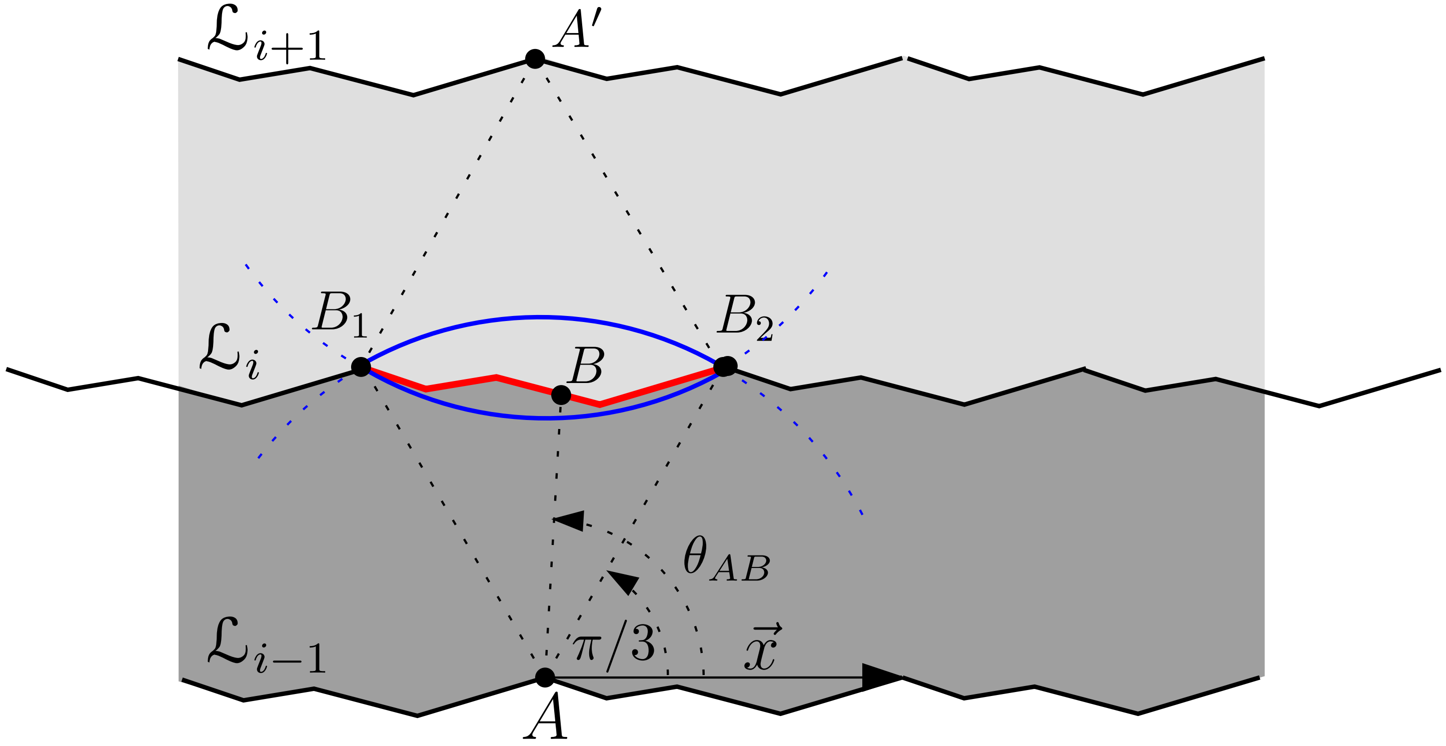

$$|AB| < 1 \Leftrightarrow \theta_{AB} \in (\pi/3, 2\pi/3)$$

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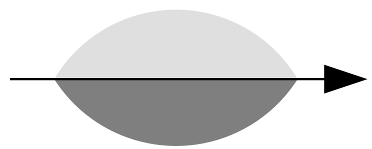
- orientation of boundary segments (white region on the left)



Local properties of χ

Lemma: Let s be a (horizontal) boundary segment containing a feasible point A, let $P(\alpha)$ denote the point $A + (\cos\alpha, \sin\alpha)$ on $\mathcal{C}(A)$. Let $B = P(\beta) \in \Delta$ and let t be a segment passing through B. Then

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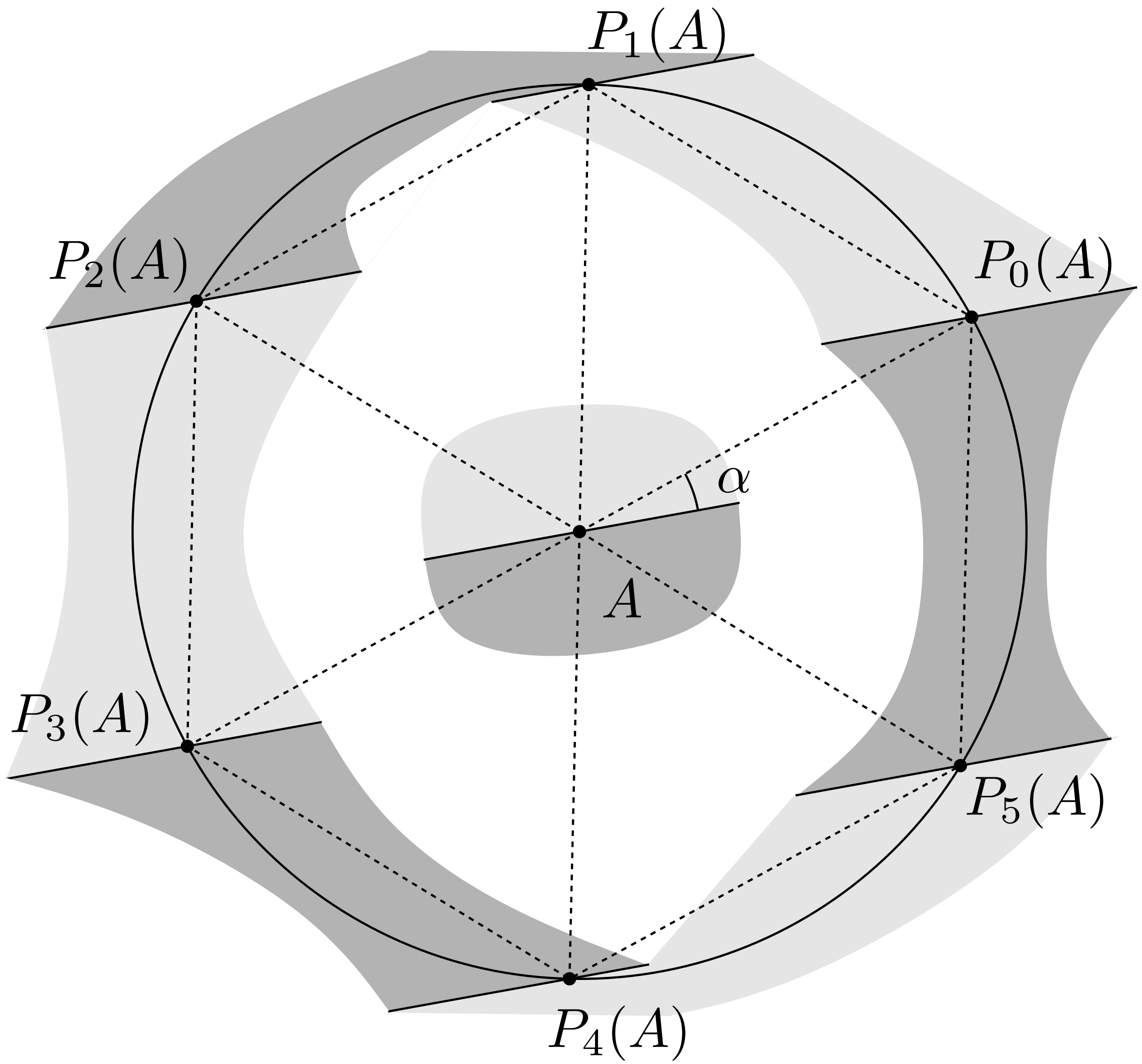
- For every θ there is exactly one value of $\alpha \in [\theta, \theta + \frac{\pi}{3})$ such that $P(\alpha) \in \Delta$.



$P_1(A)$
$P_2(A)$
$P_0(A)$
α
A
$P_3(A)$
$P_5(A)$
$P_4(A)$

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Let $A \in \Delta$. For $t \in \mathbb{R}$ let $A(t)$ be a point on the same boundary component as A, such that the directed length of the boundary curve between A and $A(t)$ is t.

Let $p_i(t) = P_i(A(t))$ (for feasible $A(t)$).

Clearly, $A(t)$ is a continuous function of t.

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Lemma: Infeasible boundary points A have similar local properties as feasible points (the circle $\mathcal{C}(A)$ can touch the boundary at points different from $P_i(A)$).

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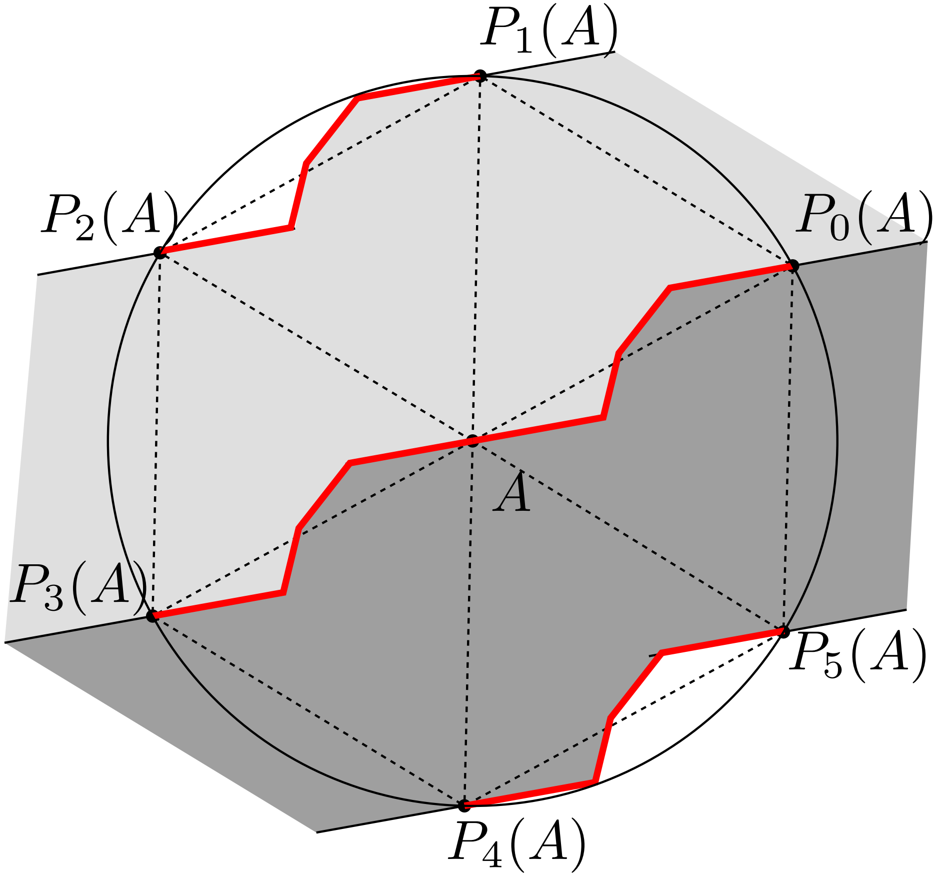

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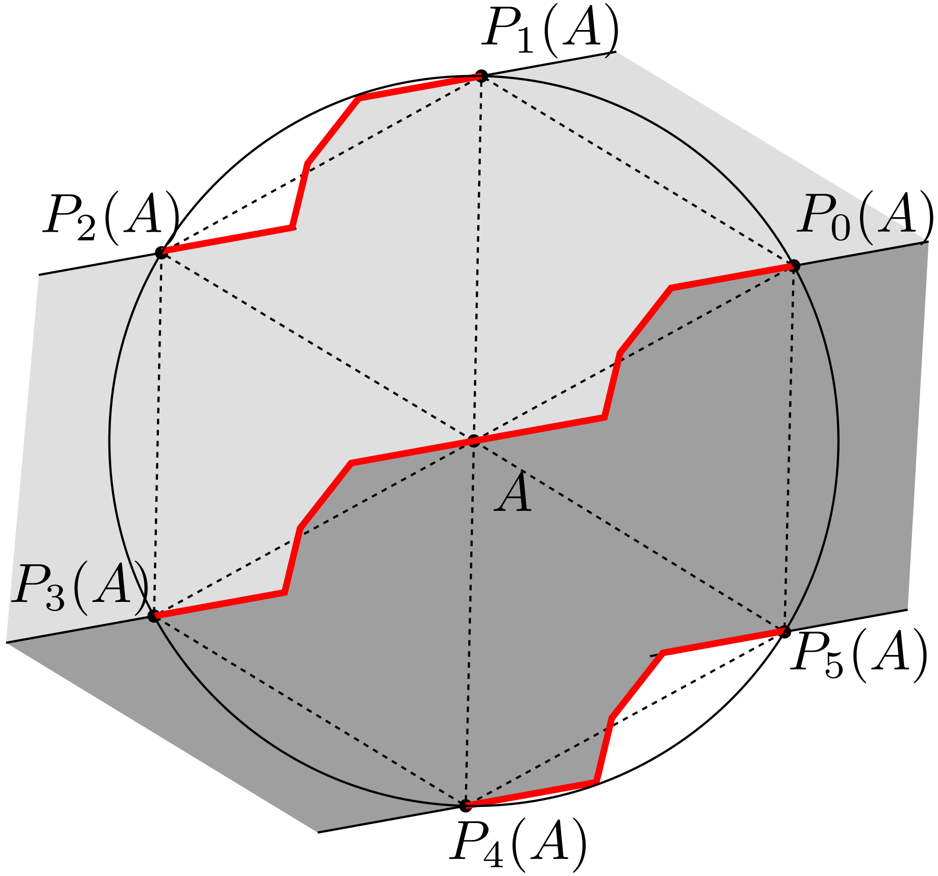

$\Rightarrow \chi$ **is zebra-like!**

Open problems

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