Combinatorial problems in geometry

Jan Kynčl

Supervisor: Doc. RNDr. Pavel Valtr, Dr.

1. J. Kynčl,
Improved enumeration of simple topological graphs,
submitted.
2. J. Kynčl,
Ramsey-type constructions for arrangements of segments,
European Journal of Combinatorics **33**(3) (2012), 336–339.
3. J. Kynčl and T. Vyskočil,
Logspace reduction of directed reachability for bounded genus graphs to the planar case,
ACM Transactions on Computation Theory **1**(3) (2010), 1–11.
4. J. Černý, J. Kynčl and G. Tóth,
Improvement on the decay of crossing numbers,
to appear in *Graphs and Combinatorics*.

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- at most two edges can intersect in one crossing

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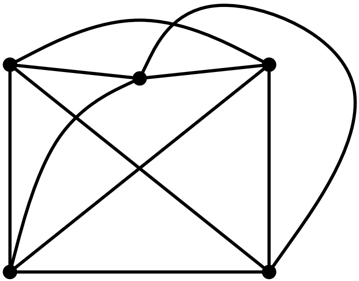

a simple complete topological graph

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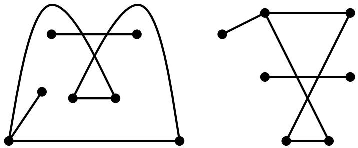

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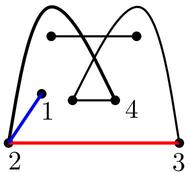

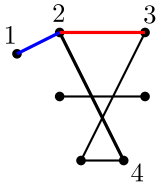

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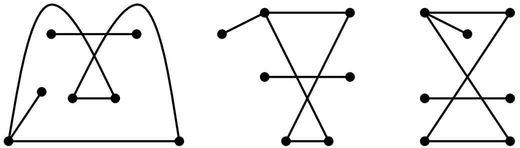

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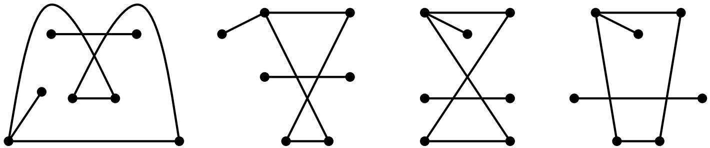

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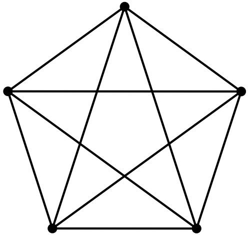
  Every simple complete topological graph with 4^{30^4} vertices contains one of the following subgraphs:



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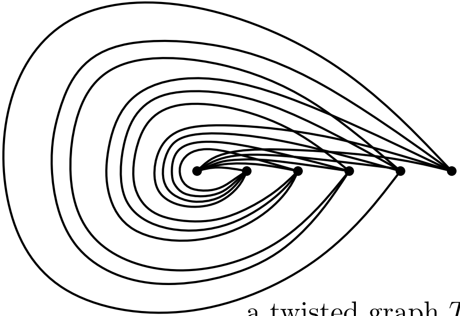

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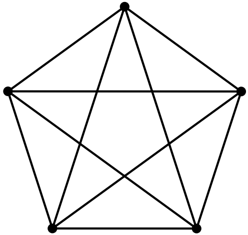

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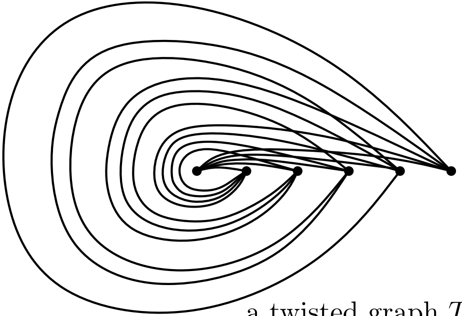

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- an upper bound on the size of a set of permutations with bounded VC-dimension (J. Cibulka and JK, 2012)

General graphs

Main Theorem 2: Let G be a graph with n vertices and m edges. Then

$$T_{\mathrm{w}}(G) \leq 2^{O(n^2 \log(m/n))}.$$

If $m < n^{3/2}$, then

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Let $\varepsilon > 0$. If G is a graph with no isolated vertices and at least one of the conditions $m > (1+\varepsilon)n$ or $\Delta(G) < (1-\varepsilon)n$ is satisfied, then

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Corollary: There are at most $2^{O(n^{3/2} \log n)}$ intersection graphs of n pseudosegments.

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Theorem: (JK, 2009)

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$$T(G) \leq 2^{m^2+2mn(1+3\log_2 3)+O(n\log n)} \leq 2^{23.118m^2} + o(1),$$ and

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"very sparse" graphs $\rightarrow$ rooted connected planar loopless maps (T.R.S. Walsh and A. B. Lehman, 1975)

$$T(G) \leq 2^{(\log_2(256/27)+o(1))m^2} \leq 2^{3.246m^2} + o(1)$$

4. Continuous decay of the crossing number

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Theorem: (J. Fox and Cs. D Tóth, 2008)
For every $\varepsilon > 0$, there is an n_ε such that every graph G with $n(G) \geq n_\varepsilon$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with

$$m(G') \leq \left(1 - \frac{\varepsilon}{24}\right) m(G)$$

and

$$\mathrm{CR}(G') \geq \left(\frac{1}{28} - o(1)\right) \mathrm{CR}(G).$$

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tools:

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$k^{\log 27/\log 4} > k^{2.3774}$ (G. Károlyi, J. Pach and G. Tóth, 1997)

3. Reachability in graphs on surfaces

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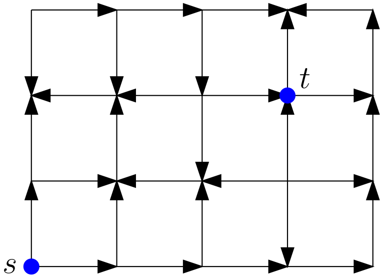
Directed reachability:



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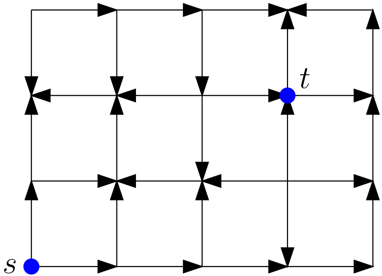

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Theorem: For each fixed connected compact surface S, the reachability problem for graphs embedded in S is logspace-reducible to planar reachability.