Crossings in Topological Graphs

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- at most two edges can intersect in one crossing

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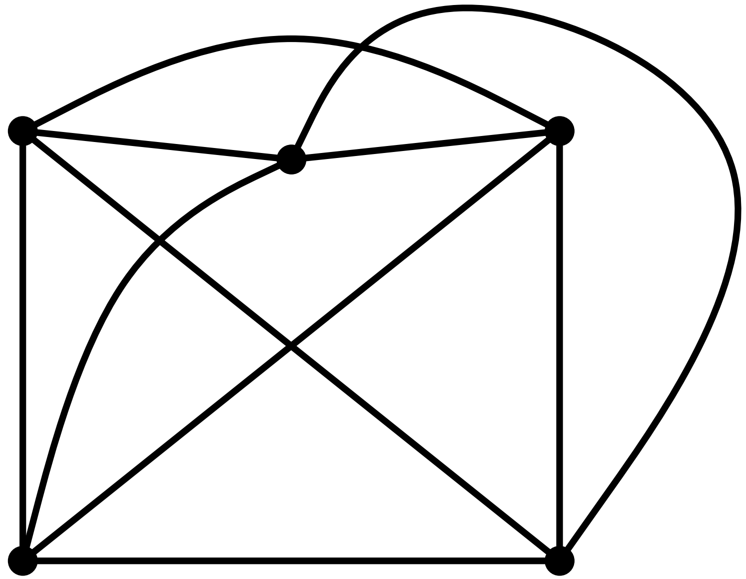

a simple complete topological graph

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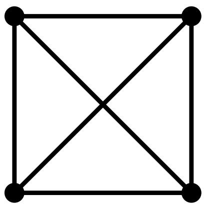

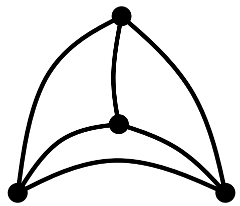

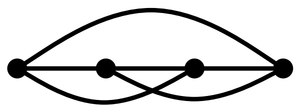

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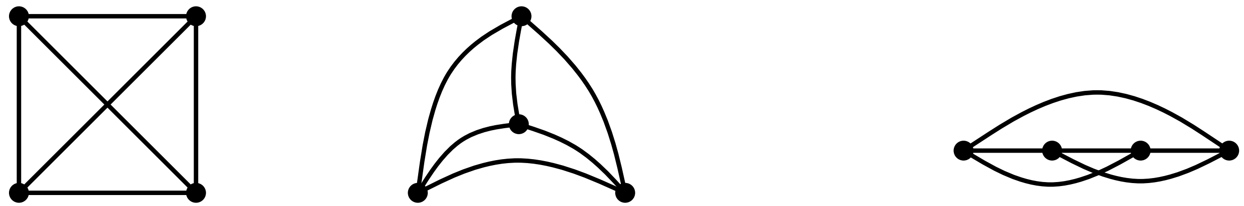

- $h(n) = 0$ for $n \leq 7$, $h(n) > 0$ for $n \geq 8$

  [H. Harborth, I. Mengersen, 1974]

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Conjecture: $h(n) = o(n^2)$

[P. Brass, W. Moser, J. Pach, 2005]

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Theorem: There exists a complete topological graph on n vertices, such that

- edges with common end-point do not cross
- every two edges have at most two common points
- every edge crosses $\Omega(n^2)$ other edges.

Theorem: On every connected compact surface except for the sphere there exists a simple complete topological graph with each edge having at least $\Omega(n^2)$ crossings.

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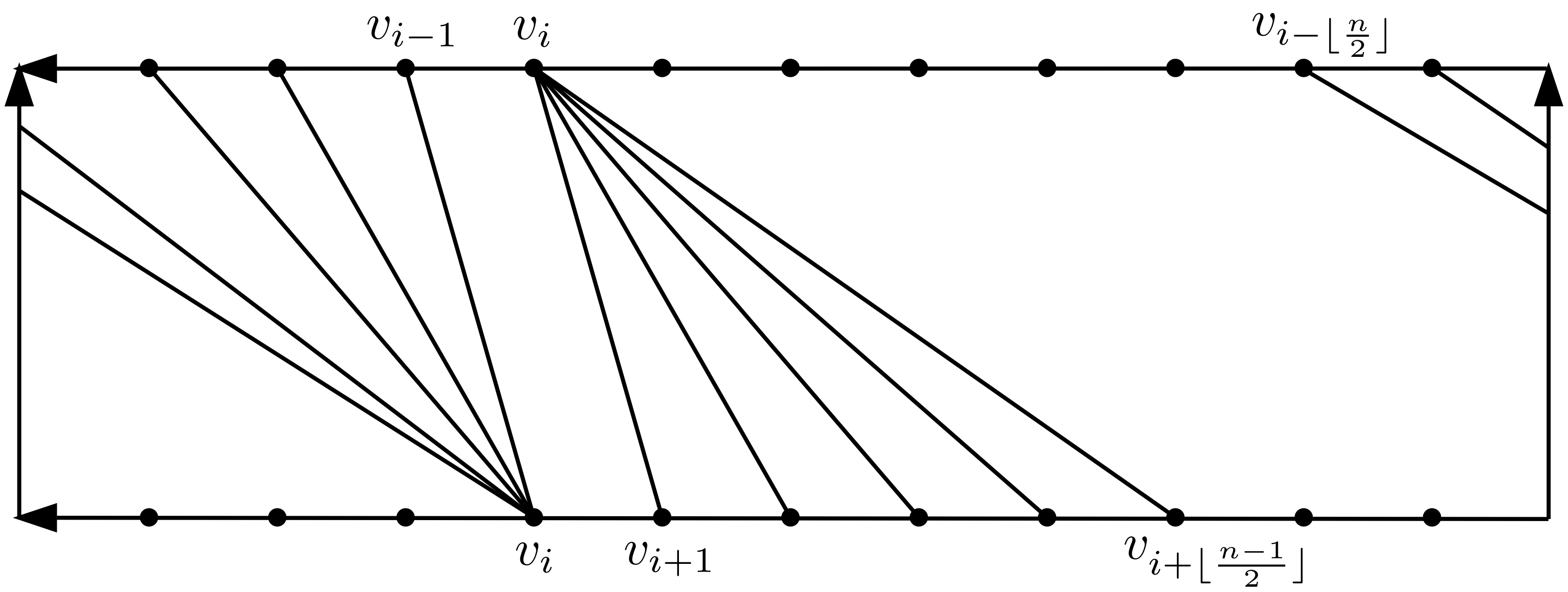
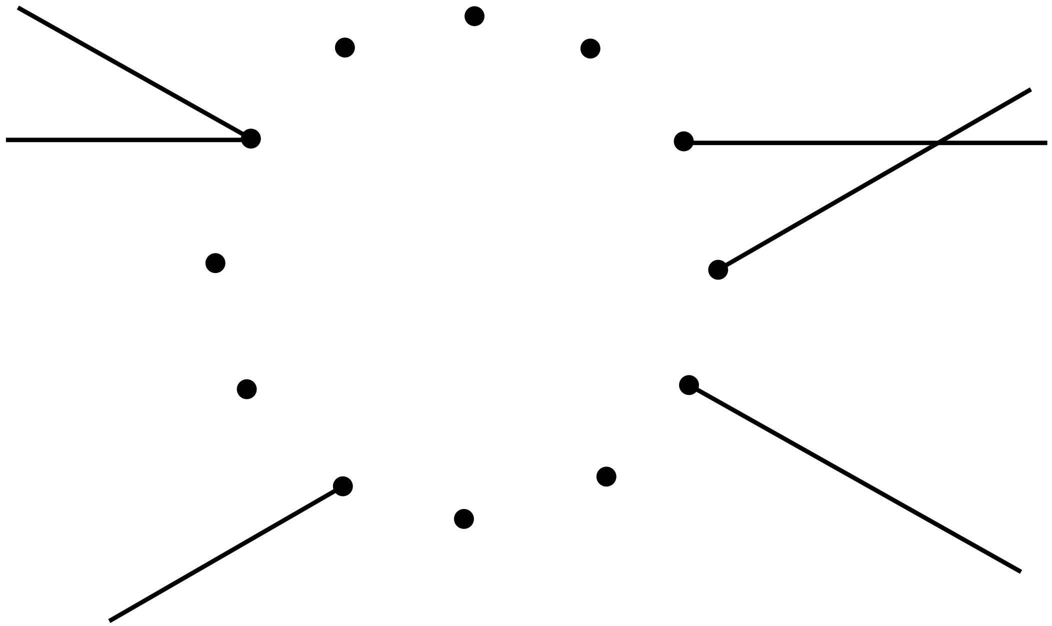
torus:



projective plane: [A. Pór]



Recognition of simply realizable complete abstract topological graphs

Abstract topological graph (AT-graph):

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Remark: Other similar problems are NP-hard

[J. Kratochvíl, 1991; J.K., 2006]

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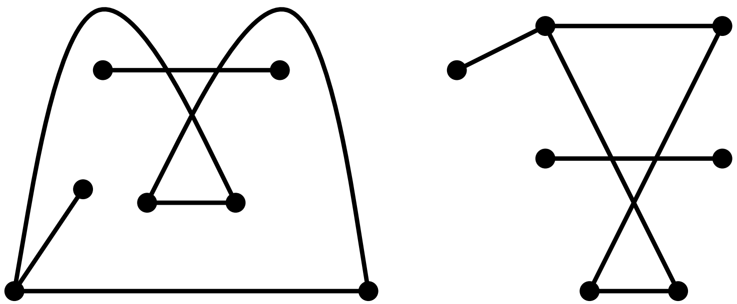

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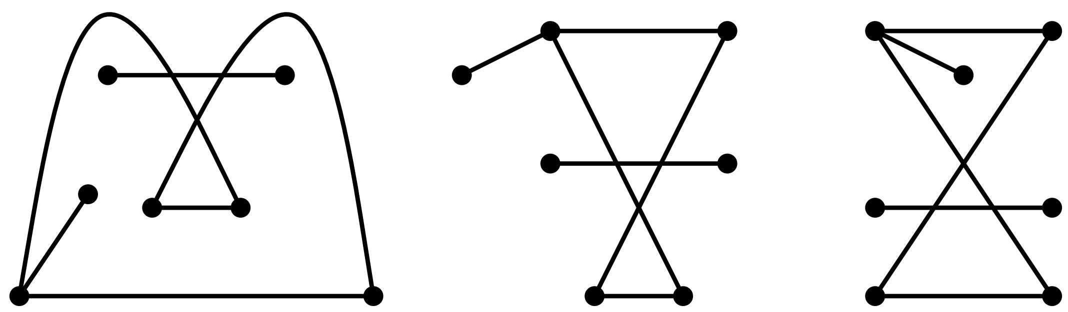

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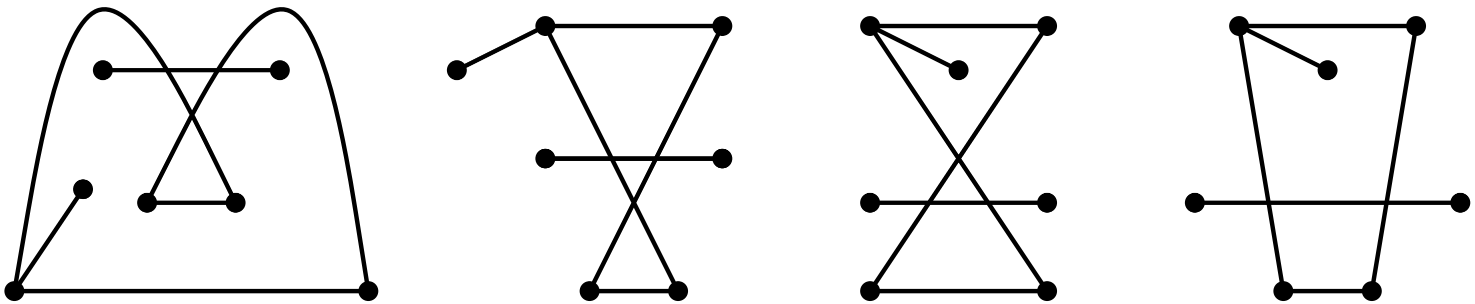

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Theorem:

$$2^{\Omega(n^2)} \leq T_w(n) \leq 2^{O(n^2 \log n)}$$

[J. Pach, G. Tóth, 2004]

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Remark: The number of weakly non-isomorphic complete geometric graphs on n vertices is $2^{O(n \log n)}$