→ 1) Lemma 6.5 [MR] & electrical nets
2) approximating permanent overview

$\boxed{L\,6.5\ [MR]}$ $\boxed{(134)}$ Let G be a (undirected) graph, $\forall (u,v) \in E(G)$ ((u,v) is an edge) & G is not bipartite, G is connected

$h_{uv} + h_{vu} \le 2m$

the random walk is a MC.
⇕
another M.C. states are oriented edges

states in the new MC. $= 2m = 2 \cdot |E(G)|$

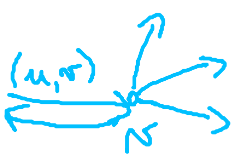
the transition matrix is doubly stochastic
(all columns & all rows sum to 1)

$$\boxed{P x_t = x_{t+1}}$$

$$P \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

a distribution ... $\Sigma = 1$

3rd column of P



state is (u,v) transition to (v,w) for $w \in \underbrace{\Gamma(v)}_{\text{neighbors of } v}$ with uniform prob.

$$\Pr[(u,v) \to (v,w)] = \frac{1}{\deg(v)}$$
where $(v,w) \in E(G)$

$$P_{(u,v)(v,w)} = \Pr[(u,v) \to (v,w)]$$

$\pi_{(u,v)}$ (u,v) is a state $(u,v) \in E(G)$ is (u,v)

$$\boxed{P} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \left.\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\} 2m$$

$$\sum_{u \in \Gamma(v)} P_{(u,v)(v,w)} = \sum_{u \in \Gamma(v)} \frac{1}{\deg v} = 1$$

doubly stochastic ⇒ uniform distribution is stationary

$$P\pi = \pi \qquad \pi = \begin{pmatrix} 1/2m \\ 1/2m \\ \vdots \end{pmatrix}$$

$$\mathbb{E}[\text{returning to } (u,v)] = \frac{1}{\pi_{(u,v)}} = 2m$$

Thm 6.5 (3)

HOBIT

$$\boxed{h_{uv} + h_{vu}} \le 2m$$

Thm 6.2 (Fundamental Thm of M.C.)

Any irreducible, finite, aperiodic M.C. has the following properties:

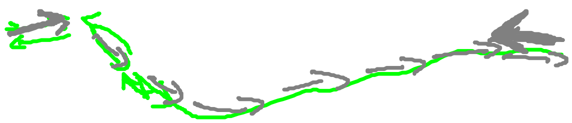
1) all states are ergodic
2) $\exists !$ stationary distribution with $\pi_i > 0$
3) $f_{ii} = 1$ = Pr[that when starting in i we return]

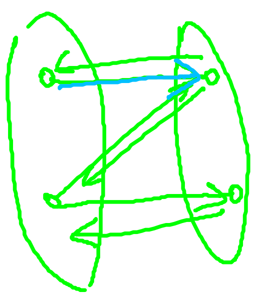
$\Rightarrow$ $h_{ii} = \frac{1}{\pi_i}$

4) $N(i,t)$ = # of visits of i in t steps

$$\forall i \quad \lim_{t \to \infty} \frac{N(i,t)}{t} = \pi_i$$

irreducible if the original undirected graph G is connected

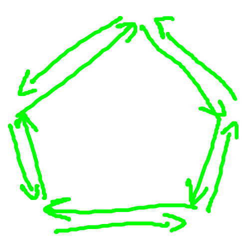




the directed M.C. is aperiodic?

period 2

when G is bipartite



odd cycle in our orig. graph

(G is not bipartite)

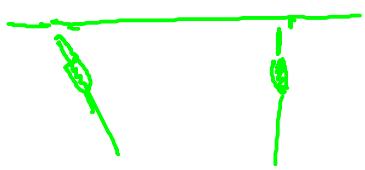
$\Downarrow$ aperiodic

Electrical networks & random walks

can be stated without
but it is very nice

current I [Amp] "how many electrons are moving across an intersection of a wire"

voltage V [Volt] ... energy to move a particle electron from one place to another



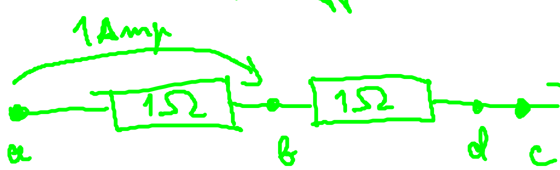
V = difference of potentials
we measure difference of potentials

resistance R [Ω] "how hard it is to move for an electron"

Ohm law $I = \frac{V}{R}$ $\qquad R = I \cdot V$

Kirchhoff laws what goes in goes out

1)



$\longrightarrow I = 1\,\text{Amp}$

$V_{dc} = 0$ (ideal wire)

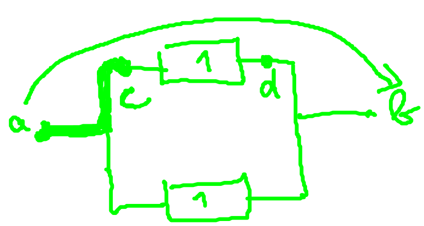
I always mess the sign, hopefully consistently

$$V_{ac} = \Phi_a - \Phi_c = \Phi_a - \Phi_b - \Phi_c + \Phi_b = V_{ab} + V_{bc}$$

$V_{ab} = 1 \cdot 1 = 1$ Volt $\qquad V_{bc} = 1$ Volt $\qquad V_{ac} = 2$ Volts

$R = 1 \cdot 2 = 2\,\Omega$ "in series $R = R_1 + R_2$"

2)



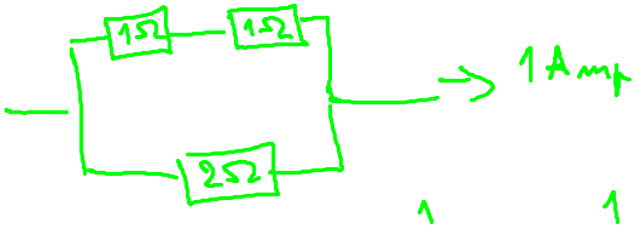
$1\,\text{Amp} = I_1 + I_2 = I$

$$\underbrace{\frac{V}{R}}_{I} = \frac{V}{R_1} + \frac{V}{R_2} \quad /:V \quad V \neq 0 \qquad I = \frac{V}{R}$$

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}}$$

"in parallel"

3)



$R = ?$

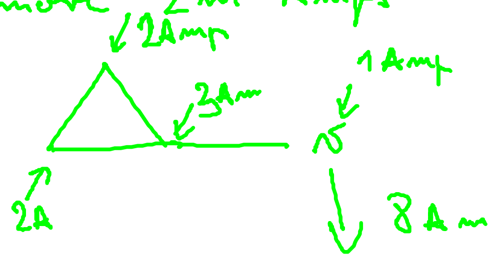
$$\frac{1}{R} = \frac{1}{1+1} + \frac{1}{2} = 1\ \Omega$$

Thm G not bipartite & connected $\forall u,v \in V(G)$

$$C_{u,v} = h_{uv} + h_{vu} = 2m\, R_{uv}$$

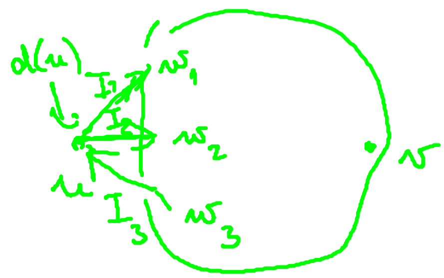
$\forall w \in V$ inject $d(w)$ Amp

from v remove $2m$ Amps



want $h_{uv} = V_{uv}$

$$\boxed{d(u) = \sum_{w \in \Gamma(u)} (V_{uv} - V_{wv})}$$



$d(u) = I_{uv} = \sum_{w\in\Gamma(u)} I_{uw} = (*)$

we are pushing $d(u)$ A

all edges $R_e = 1$

$$(*) = \sum_{w \in \Gamma(u)} V_{uw} = \sum_{w\in\Gamma(u)} (V_{uv} - V_{wv})$$

$h_{uv} = V_{uv}$

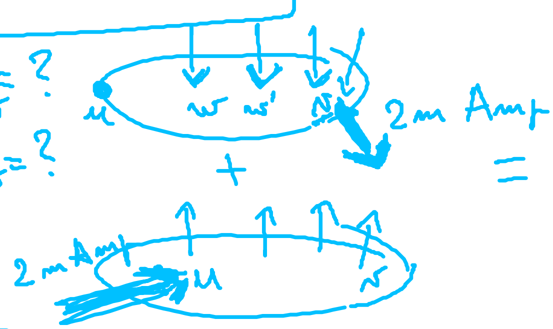
$$\boxed{h_{uv} = \sum_{w \in \Gamma(u)} \frac{1}{d(u)} (1 + h_{wv})}$$

$R_{u,v}$

$V_{uv} + V_{vu}$

$V_{uv} = ?$

$I_{uv} = ?$



=

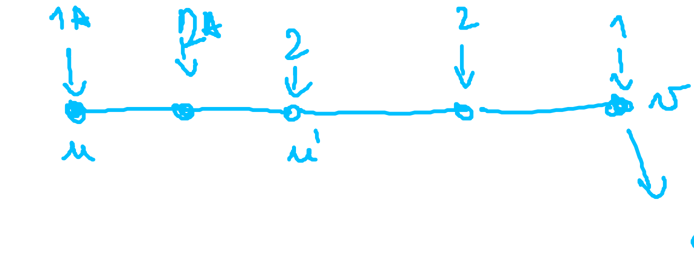
current we push to u & pull from v

no other vertices receive anything

$$\boxed{V_{uv} + V_{vu} = 2m \cdot R_{uv}}$$

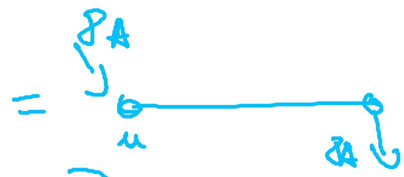
$Ax = \vec{0}$ $\quad A(x+y) = \vec{0}$

$Ay = \vec{0}$

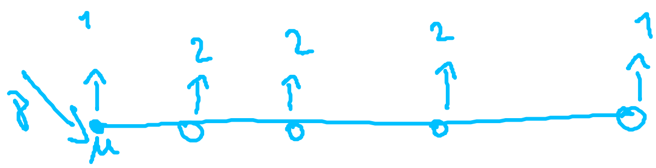


$$V_{uv} = \frac{I\ ?}{R_{uv}}$$

$=$

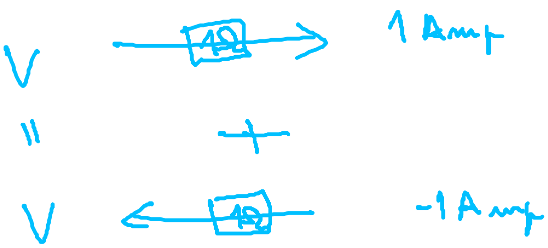


$+$



$$V_{uv} = \frac{I = 2m^{v}}{R_{uv}}$$

? what happens with voltages when we sum two networks?



$=$ $V = 0$

$V = I \cdot R$

the tricky part is to realize V_{uv} and V_{vu} are the same (up to signs)