Solved problems – Linear Algebra II

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Contents

1 Class

Hello, my name is Karel. Requirements for getting credit "zapocet". What are you interested in? How you should study...

0. Do you need me to recapitulate something? You should know: how to solve a system of linear equations using Gaussian elimination (Gauss-Jordan), how to multiply matrices, how to tell coordinates. We will focus on linear maps when we need to.

   Can you code? In a programming language of your choice and nothing complicated.

1. Let B be a basis of $\mathbb{R}^4$ with vectors:

   $(1/2, 1/2, 1/2, 1/2)^T, (1/2, -1/2, -1/2, 1/2)^T, (-1/2, 1/2, -1/2, 1/2)^T, (-1/2, -1/2, 1/2, 1/2)^T$

   Find the matrix corresponding to change of basis from the canonical basis to B (that is find the matrix ${}_B[id]_K$ such that $[u]_B = {}_B[id]_K[u]_K$. Find coordinates of the vector $(3, 1, 4, 1)^T$ in basis B. Did you noticed something about matrix ${}_B[id]_K$?

   Solution: When we say that a vector v has coordinates $[v]_B = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ with respecto to the basis B we mean that

   $$v = \alpha_1 \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix} + \alpha_3 \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} + \alpha_4 \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

   We may thus say that

   $$[v]_K = {}_K[id]_B[v]_B$$

   where the matrix of the change of basis from B to the canonical basis

   $$K = \left\{(1, 0, 0, 0)^T, (0, 1, 0, 0)^T, (0, 0, 1, 0)^T, (0, 0, 0, 1)^T\right\}$$

   where the matrix of change of basis

   $${}_K[id]_B = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}.$$

   From the equation

   $$[v]_K = {}_K[id]_B[v]_B$$

   and the fact that columns of ${}_K[id]_B$ are linearly independent and the matrix is square we have that

   $$({}_K[id]_B)^{-1}[v]_K = [v]_B$$

   and thus

   $${}_B[id]_K = ({}_K[id]_B)^{-1}$$

   We may notice that for this particular matrix with columns equal to vectors of the basis B we have that its transpose is equal to its inverse. This does not hold for all bases at all! But such bases are extremely important and we call such matrices that satisfy $A^{-1} = A^T$ orthonormal and such bases orthonormal.

2. Let V be a vector space over $\mathbb{R}$, we define a *dot product* (or a *scalar product*) as a binary operation $\langle \cdot \mid \cdot \rangle : V^2 \to \mathbb{R}$, such that for each $u, v, w \in V$ a $c \in \mathbb{R}$ we have:

(a) $\langle u \mid u \rangle \geq 0$ and equality holds only for $u = \vec{0}$

(b) $\langle u + v \mid w \rangle = \langle u \mid w \rangle + \langle v \mid w \rangle$

(c) $\langle cu \mid v \rangle = c\langle u \mid v \rangle$

(d) $\langle u \mid v \rangle = \langle v \mid u \rangle$ (respectively $\langle u \mid v \rangle = \overline{\langle v \mid u \rangle}$ for complex numbers).

We say that u, v are *orthogonal* if $\langle u \mid v \rangle = 0$.

We may define a *norm* using a dot product: $\|u\| = \sqrt{\langle u \mid u \rangle}$. Intuitively a norm gives you the length of a vector. Note that a norm can be defined in a more general way but this definition is extremely useful.

Geometric interpretation of the standard dot product in $\mathbb{R}^n$ is $\langle u \mid v \rangle = \|u\|\|v\| \cos(\varphi)$, where φ is the angle between vectors u, v (compare with the definition of orthogonality).

Moreover orthogonality of vectors implies linear independence.

3. Show that the following are dot products.

   (a) (Standard dot product) In $\mathbb{R}^n$ we define $\langle u \mid v \rangle = u^T v = \sum_{i=1}^{n} u_i v_i$

   Solution: Just check the definition and use properties of matrix operations.

   (b) In the space $\mathcal{C}_{[a,b]}$ of all continuous functions on the interval $[a, b]$ we define a dot product $\langle f \mid g \rangle = \int_a^b f(x)g(x)dx$.

   Solution: Just check the definition and use properties integrals.

4. Compute standard dot products of given vectors: $(1, 2, 3)^T, (0, 0, 1)^T, (1, -2, 1)^T$. Which ones are orthogonal? What is the length of the first vector? How far apart are the first and third vector?

   Solution:

   $\langle (1, 2, 3)^T | (0, 0, 1)^T \rangle = 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 1 = 3$ – these vectors are not orthogonal,

   $\langle (1, 2, 3)^T | (1, -2, 1)^T \rangle = 1 - 4 + 3 = 0$ – these vectors are orthogonal,

   $\langle (0, 0, 1)^T | (1, -2, 1)^T \rangle = 1$ – these vectors are not orthogonal.

   The length of the first vector is $\|(1, 2, 3)^T\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

   The first and the third vectors are $\|(1, 2, 3)^T - (1, -2, 1)^T\| = \|(0, 4, 2)^T\| = \sqrt{20}$ apart.

5. Let us denote the rows of a matrix A as $v_1, \ldots, v_m$ and columns of a matrix B by $w_1, \ldots, w_p$. What are the entries of the matrix AB?

   Solution: $(AB)_{i,j} = \langle v_i | w_j \rangle$

   Prove that the row space of a matrix A and the kernel of the matrix A are orthogonal.

   Solution: The kernel is the vector space of all solutions of the homogenous system of linear equations. Thus for each vector $v \in \mathrm{Ker}(A)$ when we substitute v to a row we get zero (it is a solution where all right sides are equal to zero). In other words $\forall v \in \mathrm{Ker}(A)$: $Av = \vec{0}$.

2 Class

1. For the dot product $\langle f | g \rangle = \int_{-1}^{1} f(x)g(x)\, dx$ show that functions $3x^2 - 1$ a $5x^3 - 3x$ are orthogonal.

   Solution: $\int_{-1}^{1} (3x^2 - 1)(5x^3 - 3x)\, dx = \int_{-1}^{1} 15x^5 - 14x^3 + 3x\, dx = [\frac{15}{6}x^6 + \frac{7}{2}x^4 + \frac{3}{2}x^2]_{-1}^{1} = 0$

2. Let A be a symmetric real matrix such that $u^T A u > 0$ for each non-zero vector $u \in \mathbb{R}^n$ we call such matrices positive definite. Let us define a dot product as $\langle u|v\rangle = u^T A v$. Show that this is indeed a dot product if and only if A is positive definite.

   Solution: Check the definition.

   Given a dot product $\langle u|v\rangle$ as a black-box find a way how to find the corresponding positive definite matrix A which defines it.

   Solution: $A_{i,j} = \langle e_i \mid e_j\rangle$ – look at the dot products of vectors of the canonical basis and what is the term $e_i^T A e_j$.

   Show that a sum of two positive definite matrices is a positive definite matrix. Show that a positive multiple of a positive definite matrix is a positive definite matrix.

   Solution: Check the definition.

3. Show that vectors $v \in \mathbb{R}^3$ satisfying $\langle (1,0,-3)^T|v\rangle = 0$ form a vector space (subspace of $\mathbb{R}^3$). In other words $\left\{\vec{v} \in \mathbb{R}^3 | \langle (1,0,-3)^T|v\rangle = 0\right\}$ form a subspace of $\mathbb{R}^3$.

   Solution: Check the definition – what are the solutions of $Ax = 0$.

   Show that vectors in $\mathbb{R}^3$ satisfying $\langle (1,0,-3)^T|v\rangle = 2$ are an affine space.

   Solution: Check the definition – what are the solutions of $Ax = b$.

4. Let $(2,5)^T, (3,1)^T$ be two vectors in the real plane $\mathbb{R}^2$. What multiple of the first vector should we subtract from the second one so that the result is perpendicular to the first vector. What multiple of the second vector should we subtract from the first one so that the result is perpendicular to the second vector.

   Solution: **Simple algebraic solution.** We want to subtract c-times vector u from the vector v in such a way that $v - cu$ is orthogonal to u (we do not know the scalar c). We thus need

$$\langle v - cu \mid u\rangle = 0 \tag{1}$$
$$\langle v \mid u\rangle - c\langle u \mid u\rangle = 0 \tag{2}$$
$$\langle v \mid u\rangle = c\langle u \mid u\rangle \tag{3}$$
$$\frac{\langle v \mid u\rangle}{\langle u \mid u\rangle} = c \tag{4}$$

   Equation (2) is just using properties of inner products. There is no division by zero in equation 4 as the vektor u is of non-zero length.

   Geometric intuition: TODO translate

   Řešme druhou část, ilustrace viz obrázek 1. Délka vektoru $u = (2,5)$ je $\|(2,5)\| = \sqrt{2^2+5^2} = \sqrt{29}$. Délka vektoru $v = (3,1)$ je $\|(3,1)\| = \sqrt{3^2+1^2} = \sqrt{10}$. Vektor se stejným směrem jako v a jednotkovou délkou je $v/\|v\| = (3/\sqrt{10}, 1/\sqrt{10})$. Skalární součin $\langle u|v\rangle = 6 + 5 = 11$. Vzpomeňme si na středoškolskou goniometrii, vidíme že pokud by vektor v měl jednotkovou délku, promítl by se na $\frac{\cos(\varphi)}{\|u\|}u$. Využitím podobnosti trojúhelníků a vztahu $\langle x|y\rangle = \|x\|\|y\|\cos\varphi$ dostaneme první krok Gram-Schmidtovy ortogonalizace, tedy vyjádření kolmého vektoru $v - u\langle u|v\rangle/\langle u|u\rangle = (3-(22/29), 1-(55/29))$. Ověřme ještě ortogonalitu: $\langle (2,5)|(3-(22/29), 1-(55/29))\rangle = 6 - 44/29 + 5 - 275/29 = 0$.

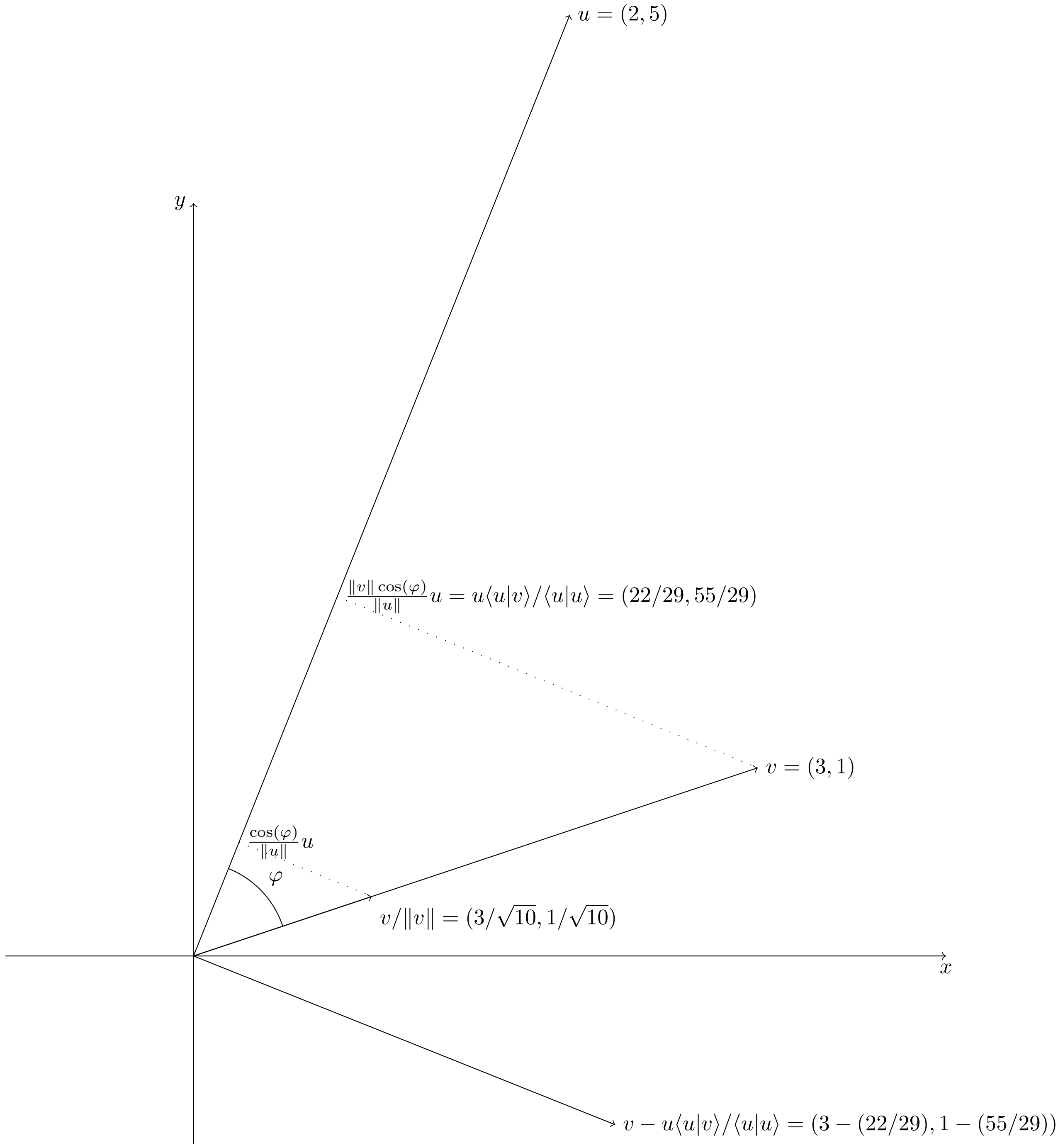


Figure 1: Počítání kolmé projekce vektoru v na vektor u.

5. Do Gram-Schmidt on the rows of the following matrices:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 & 2 \\ 4 & 3 & 2 & 4 \\ 6 & -5 & 3 & 6 \\ -4 & 2 & 4 & 2 \end{pmatrix}$$

6. Show that a norm defined by a dot product ($\|v\| = \sqrt{\langle v|v\rangle}$) satisfies the Parallelogram Law $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

   Solution: Expand the left side and use linearity of dot products.

   Can the norm $\|x\|_1 = \sum |x_i|$ or the norm $\|x\|_\infty = \max |x_i|$ be given by a dot product?

   Solution: No, use the first part of this problem.

Bonus Show that there are no four points in the real plane $\mathbb{R}^2$ such that the distance between each two of those is an odd number (these distances may or may not be the same).

Solution: 33 miniatures by prof. Matoušek
`https://kam.mff.cuni.cz/~matousek/stml-53-matousek-1.pdf`

3 Class

1. Which pairs of following vectors are perpendicular with respect to the standard scalar product? $(1,2,3)$, $(5,2,-3)$, and $(-2,-1,-4)$

   Which properties of the relation of perpendicularity hold: reflexivita, symmetry, tranzitivity?

   Solution: $(1,2,3) \perp (5,2,-3)$, $(5,2,-3) \perp (-2,-1,-4)$, ale $(1,2,3) \not\perp (-2,-1,-4)$.

   The relation is just symmetric. It is not transitive or reflexive.

2. Do Gram-Schmidt on the rows of the following matrix: $\begin{pmatrix} 0 & 3 & 4 & 0 \\ 0 & 0 & 5 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}$

3. How far is the point $(1,2,0,1)^T$ from the plane spanned by vectors $(1,1,0,0)^T, (2,-1,0,0)^T$?

   Solution: We orthonormalize the vectors determining the plane and we get: $(1/\sqrt{2}, 1/\sqrt{2}, 0, 0)^T$, $(1/\sqrt{2}, -1/\sqrt{2}, 0, 0)^T$. After subtracting the projection of $(1,2,0,1)^T$ to the orthonormal basis $(1/\sqrt{2}, 1/\sqrt{2}, 0, 0)^T$, $(1/\sqrt{2}, -1/\sqrt{2}, 0, 0)^T$ we get the vector $(0,0,0,1)^T$ and its length is equal to one (and is equal to the distance we were supposed to find). We just did the Gram-Schmidt and the result was the length of $\vec{y_3}$.

   One can even see that the vectors $(1,1,0,0)^T, (2,-1,0,0)^T$ span everything in the first two coordinates thus the distance from this plane is equal to the length of $(0,0,0,1)^T$.

4. Using projection find the best solution of the following system of equations: $Ax = b$ where $A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \\ 2 & -4 & -1 \\ 1 & -2 & 2 \end{pmatrix}$, $b = (10,5,13,9)^T$

   Notice that the columns of A are perpendicular. How bad is your solution (i.e. compute $b - Ax$)?

   The least squares method is often used when the errors are small – but it is hard to compute with such systems with pen and paper. Is the solution the same as the solution of the system $A^T Ax = A^T b$?

5. Using Gram-Schmidt find an orthonormal basis of the row-space of the following matrix and expand it to an orthonormal basis of $\mathbb{R}^4$. $\begin{pmatrix} 2 & 4 & 2 & 1 \\ -1 & -2 & -2 & -1 \\ 1 & 2 & 4 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

6. Show that a norm defined by a dot product ($\|v\| = \sqrt{\langle v|v\rangle}$) satisfies the Parallelogram Law $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

   Solution: Expand the left side and use linearity of dot products.

   Can the norm $\|x\|_1 = \sum |x_i|$ or the norm $\|x\|_\infty = \max |x_i|$ be given by a dot product?

   Solution: No, use the first part of this problem.

7. Show that columns of Hadamard matrices $H_m \in \mathbb{R}^{2^m \times 2^m}$ defined as

$$H_0 = \begin{pmatrix} 1 \end{pmatrix},$$

$$H_m = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{pmatrix},$$

   are orthonormal.

Bonus Controlling matrix multiplication: someone is selling you a program that can multiply two matrices fast. Can you control that it returns correct results? It is enough to check if $Cx = A(Bx)$ where C is the output of the program and x is uniformly random $\{0,1\}$ vector of the right length. Show that if $C \neq AB$ then with big probability $Cx = ABx$ does not hold.

4 Class

1. Using projection find the best solution of the following system of equations: $Ax = b$ where $A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \\ 2 & -4 & -1 \\ 1 & -2 & 2 \end{pmatrix}$, $b = (10, 5, 13, 9)^T$

   Notice that the columns of A are perpendicular. How bad is your solution (i.e. compute $b - Ax$)?

   The least squares method is often used when the errors are small – but it is hard to compute with such systems with pen and paper. Is the solution the same as the solution of the system $A^T A x = A^T b$?

2. Determine a basis of the orthogonal complement of the row space of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 4 & 1 \end{pmatrix}$.

   Solution: It is the kernel of A (dot products are zero – we are solving a homogenous system of linear equations).

3. Intro to determinants – geometric intuition, why is there the sign needed, definition.

   Computing determinants of 2×2 matrices and the influence of row operations. Geometric intuition in the plane.

4. Compute determinants of following real matrices:

   $\begin{pmatrix} 4 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ *Solution:* Determinant is equal to -9.

   $\begin{pmatrix} 3 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$ *Solution:* Determinant is equal to 30.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$ *Solution:* Determinant is equal to 2.

$$\begin{pmatrix} 18 & 11 & 11 \\ 11 & 11 & 11 \\ 11 & 11 & 24 \end{pmatrix}$$ *Solution:* Use row operations first. Determinant is equal to 1001.

5 Class

1. I need you to remember the following string: JFMAMJJASOND.
   - How should you remember this? There is JASOND in the end, so that is easier.
   - Why should you know this? Is it enough to learn this string by heart or should you be able to use it? What does it mean to use it?

     What does it mean to use knowledge in programming? Is it useful to know by hear how to write a for loop or is it better to know how to use it to output first 99 numbers?
2. Talking about Test 2. How does the previous problem apply to learning linear algebra?
3. Fibonacci – how to compute F_n fast?
4. What is the matrix of the linear transformation where $f((1,2)^T) = (3,4)^T$ and $f((1,1)^T) = (4,3)^T$?
5. What is the matrix of first derivatives of polynomial with degree at most five?
6. Are there linear maps f_0, f_1, f_2, f_3 such that:
   - f_0 is injective and surjective,
   - f_1 is not injective but is surjective,
   - f_2 is injective but not surjective,
   - f_3 is not injective and not surjective?

6 Class

1. Intro to determinants – geometric intuition, why is there the sign needed, definition.

   Computing determinants of 2×2 matrices and the influence of row operations. Geometric intuition in the plane.
2. Compute determinants of following real matrices:

   $$\begin{pmatrix} 4 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$ *Solution:* Determinant is equal to -9.

   $$\begin{pmatrix} 3 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$ *Solution:* Determinant is equal to 30.

   $$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$ *Solution:* Determinant is equal to 2.

$\begin{pmatrix} 18 & 11 & 11 \\ 11 & 11 & 11 \\ 11 & 11 & 24 \end{pmatrix}$ *Solution:* Use row operations first. Determinant is equal to 1001.

7 Class

1. Compute the inverse matrix using the adjugate matrix: $\begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. *Solution:* TODO, `https://en.wikipedia.org/wiki/Adjugate_matrix`

2. Compute the determinant of the real matrix: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ -1 & 0 & 2 & 3 & 4 & \dots & n-1 \\ -1 & -2 & 0 & 3 & 4 & \dots & n-1 \\ -1 & -2 & -3 & 0 & 4 & \dots & n-1 \\ -1 & -2 & -3 & -4 & 0 & \dots & n-1 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ -1 & -2 & -3 & -4 & \dots & 1-n & 0 \end{pmatrix}$.

   Solution: Add the first row to all others. Determinant is equal to $n!$.

3. Compute determinants of following matrices:

   $\begin{pmatrix} a_1+x & a_2 & a_3 & \dots & a_n \\ a_1 & a_2+x & a_3 & \dots & a_n \\ a_1 & a_2 & a_3+x & \dots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_n+x \end{pmatrix}$, *Solution:*

   Subtract the last row from all others (a remains just on the last row), then add all columns to the last one.

   Determinant is equal to $(a_1 + \dots + a_n + x)x^{n-1}$.

   $\begin{pmatrix} x & -1 & 0 & \dots & 0 \\ 0 & x & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x & -1 \\ a_0 & a_1 & \dots & a_{n-1} & a_n \end{pmatrix}$, *Solution:*

   Eleminate entries on the diagonal using adding x times the next column to the previous one (starting from the penultimate column). There will be -1 above the diagonal and the bottom row will become $a_0 + xa_1 + x^2a_2 + \dots + x^na_n \dots, a_{n-2} + xa_{n-1} + x^2a_n, a_{n-1} + xa_n, a_n$. There is only one permutation that contributes to the determinant in the sum from the definition.

   Alternatively do an expansion using the first column.

   Determinant is equal to $\sum_{i=0}^{n} x^i a_i$.

   $\begin{pmatrix} a+1 & a & 0 & \dots & 0 \\ 1 & a+1 & a & \ddots & \vdots \\ 0 & 1 & a+1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a \\ 0 & \dots & 0 & 1 & a+1 \end{pmatrix}$ *Solution:*

Let us denote this matrix A_n we have that $|A_1| = a+1$, $|A_2| = a^2+a+1$ and using expansion using the first column we get the recurrence $|A_n| = (a+1)|A_{n-1}| - a|A_{n-2}|$. This recurrence has a unique solution.

Determinant is equal to $a^n + a^{n-1} + \cdots + a + 1$.

4. Compute determinants of the following matrices:

$\begin{pmatrix} \sin x & \cos x & 1 \\ \sin y & \cos y & 1 \\ \sin z & \cos z & 1 \end{pmatrix}$, *Solution:*

Use expansion of the last column and the formula $\sin(x-y) = \sin x \cos y - \cos x \sin y$.

Determinant is equal to $\sin(x-y) + \sin(y-z) + \sin(z-x)$.

$\begin{pmatrix} \cos x & \sin x \cos y & \sin x \sin y \\ -\sin x & \cos x \cos y & \cos x \sin y \\ 0 & -\sin y & \cos y \end{pmatrix}$, *Solution:*

Using Sarrus rule $\cos^2 x \cos^2 y + \sin^2 x \sin^2 y + \sin^2 x \cos^2 y + \cos^2 x \sin^2 y = 1$.

Determinant is equal to 1.

$\begin{pmatrix} 1 & \log_b a & \log_c a \\ \log_a b & 1 & \log_c b \\ \log_a c & \log_b c & 1 \end{pmatrix}$ *Solution:*

Use the formula $\log_a b = \ln b / \ln a$ and Sarrus rule or Gaussian elimination.

Determinant is equal to 0.

5. Numbers 697, 476, and 969 are divisible by 17. Without computing the determinant show that the determinant of the following matrix is divisible by 17. $\begin{pmatrix} 6 & 9 & 7 \\ 4 & 7 & 6 \\ 9 & 6 & 9 \end{pmatrix}$

Solution: We know that determinant is linear with respect to each row. When we add 100 times the first column and 10 times the second column to the third column.

Formally: $\begin{vmatrix} 6 & 9 & 7 \\ 4 & 7 & 6 \\ 9 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 6 & 9 & 697 \\ 4 & 7 & 476 \\ 9 & 6 & 969 \end{vmatrix} = 17 \cdot \begin{vmatrix} 6 & 9 & \frac{697}{17} \\ 4 & 7 & \frac{476}{17} \\ 9 & 6 & \frac{969}{17} \end{vmatrix}$

The last matrix consists of integers and thus its determinant is an integer.

6. Compute the volume of a parallelogram determined by vectors $a^T = (3,1,1)$, $b^T = (2,1,1)$, and $c^T = (2,3,2)$. (A parallelogram in $\mathbb{R}^3$ consists of points which can be written as a linear combination $\alpha a + \beta b + \gamma c$, where $\alpha, \beta, \gamma \in \langle 0, 1 \rangle$.)

Solution: The volume is equal to the absolute value of determinant with columns equal to vectors a, b, c in other words $V = \left| \det \begin{pmatrix} 3 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} \right| = |-1| = 1$.

The volume is equal to 1.

7. Let f be a linear map such that $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ maps vectors $a^T = (1,3,1), b^T = (1,0,3), c^T = (1,1,1)$ to vectors $f(a)^T = (3,1,0), f(b)^T = (1,0,2), f(c)^T = (4,1,5)$.

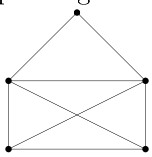
Determine the volume of the ellipsoid $f(B_3)$ which is the image of a unit ball B_3 (a real ball of unit radius) with respect to the map f.

Solution: The linear map is given by the matrix such that $f(u) = [f]_{KK} u$ where

$$[f]_{KK} = BA^{-1} = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 0 & 1 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}^{-1}.$$

Volumes of bodies transformed by linear maps are multiplied by the coefficient $|\det([f]_{KK})|$, that means $V(f(B_3)) = |\det([f]_{KK})| \cdot \frac{4}{3}\pi = \frac{|\det(B)|}{|\det(A)|} \cdot \frac{4}{3}\pi = \pi$

The volume of the ellipsoid is equal to π.

8. Compute the number of spanning trees in a graph drawn on the blackboard (a tree and a  more complicated graph).

   Solution: Laplace matrix of the graph is

$$L = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

   Using the theorem counting the number of spanning trees (for a proof see `https://ocw.mit.edu/courses/mathematics/18-314-combinatorial-analysis-fall-2014/readings/MIT18_314F14_mt.pdf`, or the book Invitation to discrete mathematics by Matoušek and Nešetřil), there is even an outline of the proof on `https://en.wikipedia.org/wiki/Kirchhoff%27s_theorem`) is equal to

$$\kappa(G) = \det(L^{1,1}) = \begin{vmatrix} 4 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 3 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 3 & 0 \\ 7 & -2 & -2 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 3 & -1 \\ -1 & -1 & 3 \\ 7 & -2 & -2 \end{vmatrix}$$
$$= \begin{vmatrix} -1 & 3 & -1 \\ 0 & -4 & 4 \\ 0 & 19 & -9 \end{vmatrix} = -4 \begin{vmatrix} -1 & 1 \\ 19 & -9 \end{vmatrix} = -4(9 - 19) = 40.$$

   We can make sure that the graph has 40 spanning trees combinatorially: K_4 has 16 spanning trees and for each of these there are two ways how to attach the upper vertex (from the left or from the right) — in total 32 spanning trees.

   Otherwise the spanning tree has to contain the roof and we have four spanning trees that contain the bottom most edge (the base of the house) and four spanning trees that do not contain it.

   The graf has 40 spanning trees.

8 Class

1. The following matrices represent a linear transformations in the plane $\mathbb{R}^2$. Determine eigenvalues and the associated eigenvectors, and interpret these in geometric terms.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

   Solution: The mapping is the scaling with factor 2. It has a single eigenvalue with multiplicity two: $\lambda_1 = \lambda_2 = 2$. Any vector is an eigenvector, the mapping scales it.

$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

Solution: The mapping is the scaling in the direction of the x-axis. The vectors on the axes are the eigenvectors and correspond to the eigenvalues 1 and 2. All vectors outside the axes change their direction.

$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

Solution: The mappping corresponds to a scaling and a skew. It has a single eigenvalue of (algebraic) multiplicity 2, but the eigenvectors are only on the x-axis. Their geometric multiplicity is 1.

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$,

Solution: This is the axix symmetry along the axis of the 2nd and the 4th quadrant. It has eigenvalues 1 and -1, that corresponds vectors on ths axis (stay on the place) and on the line orthogonal to this axis (these change the direction, but not the length).

$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Solution: The map is the orthogonal projection on the axis of the 1st and the 3rd quadrant. Eigenvalues are 1 and 0. They correspond to vectors on the axis (do not change) and to those perpendicular to the axis (are mapped to the origin).

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Solution: These corespond to the rotation by 90°. It has no real eigenvalues, but could be interpreted over $\mathbb{C}$.

$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

Solution: This is a rotation by angle α, has no real eigenvalues.

2. Determine eigenvalues and the corresponding eigenvectors for the following matrix over the field $\mathbb{C}$:

   $\begin{pmatrix} 2 & 6 \\ 6 & -3 \end{pmatrix}$ *Solution:* Eigenvalues λ_i of the matrix A are the root of the charakteristic polynomial $p_A(t) = |A - tI|$.

   Eigenvectors corersponding to λ_i satisfy $A\vec{x} = \lambda_i\vec{x}$, i.e. the are the solutions of the homogeneous system $(A - \lambda_i I)\vec{x} = \vec{0}$.

   $\begin{vmatrix} 2-t & 6 \\ 6 & -3-t \end{vmatrix} = t^2 + t - 42 = (t-6)(t+7)$ hence $\lambda_1 = 6$, $\lambda_2 = -7$

   Eigenvectors are calculated from the associated systems of linear equations

   $$\begin{aligned} -4x_1 + 6x_2 &= 0 \\ 6x_1 - 9x_2 &= 0 \end{aligned}$$

   with a solution $\vec{x} = (x_1, x_2)^T = c \cdot (3,2)^T$.

   For the second eigenvalue -7

   $$\begin{aligned} 9x_1 + 6x_2 &= 0 \\ 6x_1 + 4x_2 &= 0 \end{aligned}$$

we get $\vec{x} = c \cdot (2, -3)^T$.

Matrices of these systems are always singular. Observe that on matrices of order two it means that the second row is a scalar multiple of the first one.

$\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$

Solution: $\lambda_1 = 1 + i$, $\vec{x} = c \cdot (1, 1 + i)^T$;
$\lambda_2 = 1 - i$, $\vec{x} = c \cdot (1, 1 - i)^T$;

$\begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$

Solution: $\lambda_1 = 6$, $\vec{x} = c \cdot (1, 1)^T$;
$\lambda_2 = -1$, $\vec{x} = c \cdot (-5, 2)^T$;

$\begin{pmatrix} 5 & 10 \\ 4 & -1 \end{pmatrix}$

Solution: $\lambda_1 = 9$, $\vec{x} = c \cdot (5, 2)^T$;
$\lambda_2 = -5$, $\vec{x} = c \cdot (-1, 1)^T$;

3. Determine eigenvalues and the corresponding eigenvectors for the following matrix over the field $\mathbb{C}$. Decide whether this matrix is diagonalizable: $\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$

   Solution: A matrix is diagonalizable if and only if the spaces of eigenvectors have dimension equal to the algebraic multiplicity of the associated eigenvalue.

   This is satisfied e.g. when the matrix is of order n and has n distinct eigenvalues.

   $\lambda_1 = \lambda_2 = \lambda_3 = -1$, $\vec{x} = c \cdot (1, 1, -1)^T$.

   The matrix is not diagonalizable. $\begin{pmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

   Solution: $\lambda_1 = 2$, $\vec{x} = c \cdot (1, 0, 0)^T$; $\lambda_2 = 1$, $\vec{x} = c \cdot (1, 0, 1)^T$; $\lambda_3 = -1$, $\vec{x} = c \cdot (0, 1, -1)^T$;

   The matrix is diagonalizable.

   $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -4 \\ -1 & 0 & 4 \end{pmatrix}$

   Solution: $\lambda_1 = \lambda_2 = 3$, $\vec{x} = c \cdot (1, -2, 1)^T$; $\lambda_3 = 0$, $\vec{x} = c \cdot (4, 4, 1)^T$;

   The matrix is not diagonalizable.

4. Determine eigenvalues and the corresponding eigenvectors for the following matrix over the field $\mathbb{Z}_5$. Decide whether this matrix is diagonalizable:

   $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix}$

   Solution: $\lambda_1 = 2$, $\vec{x}_1 = c \cdot (0, 1, 3)^T$; $\lambda_2 = \lambda_3 = 1$, $\vec{x}_2 = c \cdot (1, 0, 1)^T$, $\vec{x}_3 = c \cdot (1, 4, 0)^T$;

   It is diagonalizable. The diagonal form is e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

9 Class

1. The matrix
$$\begin{pmatrix} 10 & 0 & 7 & -7 \\ 4 & 5 & 2 & -2 \\ 16 & 4 & 15 & -8 \\ 30 & 4 & 26 & -19 \end{pmatrix}$$
has three eigenvalues $3, -4$ and 5. Determine the remaining eigenvalue.

   Solution: The most tedious way: factorize the characteristic polynomial by the monomials of the known eigenvalues.

   A bit simpler, but still complicated way: use the fact that the product of all eigenvalues is the determinant of the matrix. (Could be derived by substititing 0 to the characteristic polynomial.)

   The most simple approach: use the fact that the sum of all eigenvalues is equal to the sum of the elements on the diagonal. (Could be derived from the coefficient by t^{n-1} in the characteristic polynomial.)

   The missing eigenvalue is 7.

2. Factorize the following matrix as RJR^{-1}, where R is regular and J is in the Jordan normal form.

   $\begin{pmatrix} -11 & 30 \\ -10 & 24 \end{pmatrix}$ *Solution:* When the multiplicity of each eigenvalue is one, the Jordan's factorization RJR^{-1} can be constructed from a diagonal matrix J with eigenvalues on the diagonal and R with the eigenvectors as columns.

   $\begin{vmatrix} -11-t & 30 \\ -10 & 24-t \end{vmatrix} = t^2 - 13t + 36 = (t-9)(t-4)$ hence $\lambda_1 = 9$, $\lambda_2 = 4$.

   For $\lambda_1 = 9$ we have $-20x_1 + 30x_2 = 0$ with a solution $x = (x_1, x_2)^T = (3, 2)^T$.

   For $\lambda_2 = 4$ we have $-15x_1 + 30x_2 = 0$ with a solution $x = (2, 1)^T$.

   $R = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$, $R^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$

   $\begin{pmatrix} 0 & 2 & -2 \\ 1 & -1 & 5 \\ 2 & -4 & 8 \end{pmatrix}$ *Solution:* $R = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $R^{-1} = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -2 \\ 1 & -1 & 1 \end{pmatrix}$

   $\begin{pmatrix} 2 & 0 & 0 \\ -4 & 1 & 3 \\ -4 & 0 & 4 \end{pmatrix}$ *Solution:* $R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $R^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

   $\begin{pmatrix} 4 & -2 & 0 \\ 0 & 2 & 0 \\ 6 & -5 & 1 \end{pmatrix}$ *Solution:* $R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $R^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

3. Transform the following matrix into Jordan normal form and determine eigenvectors, and if necessary also generalized eigenvectors. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ *Solution:* The generalized eigenvector x_i can be obtained from the system$(A - \lambda I)x_i = x_{i-1}$.

Characteristic polynomial $p_A(t) = \begin{vmatrix} 1-t & 1 & 1 \\ 0 & 1-t & 0 \\ -1 & 0 & 3-t \end{vmatrix} = (1-t)(2-t)^2.$

The system $(A-2I)x^1 = \vec{0}$ has a solution $x^1 = p(1,0,1)^T$.

The eigenvalue $\lambda = 2$ has geometric multiplicity 1 and algebraic 2, we shall find a generalized eigenvector. In the sequel we choose $p = 1$, i.e. $x^1 = (1,0,1)^T$.

The generalized eigenvector x^2 we get from $(A-2I)x^2 = x^1$. It has a solution $x^2 = q(1,0,1)^T + (-1,0,0)^T$.

The system $(A-1I)x = \vec{0}$ has solution $x^3 = r(2,-1,1)^T$.

By a suitable choice of parameters $q = 1$ and $r = 1$ we get the desired matrix R. We also calculate its inverse R^{-1}.

$$R = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \qquad R^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The given matrix can be factorized into Jordan normal form as

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} = RJR^{-1}$$

4. Use Jordan normal form and calculate the third power and a square root of the following matrix. (By a square root consider a matrix whose second power is the given matrix.)

$\begin{pmatrix} -11 & 30 \\ -10 & 24 \end{pmatrix}$ *Solution:* The third power of $A = RJR^{-1}$ is straightforwardly $A^3 = RJ^3R^{-1}$.

Analogously, the square root is $RJ^{\frac{1}{2}}R^{-1}$, where $J^{\frac{1}{2}}$ has on the diagonal square roots of the eigenvalues.

(In total 2^n solutions on matrices of order n.)

$$\begin{pmatrix} -11 & 30 \\ -10 & 24 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 729 & 0 \\ 0 & 64 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} -1931 & 3990 \\ -1330 & 2724 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ -2 & 6 \end{pmatrix}$$